Lie Theory Through Examples, Winter 09, J. Baez, 7-1

**Lie Group Coincidences**

- \( B_1 \): \( \text{Spin}(3) \cong \text{SU}(2) \) \( A_1 \)
- \( D_2 \): \( \text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2) \) \( A_1 \times A_1 \)
- \( B_2 \): \( \text{Spin}(5) \cong \text{Sp}(2) \) \( C_2 \)
- \( D_3 \): \( \text{Spin}(6) \cong \text{SU}(4) \) \( A_3 \)

Last time, we saw \( \text{SU}(4) \) double covers \( \text{SO}(6) \), so \( \text{SU}(4) \cong \text{Spin}(6) \). Now let's use this to show \( \text{Sp}(2) \) double covers \( \text{SO}(5) \), so \( \text{Sp}(2) \cong \text{Spin}(5) \).

We'll see that \( \text{Sp}(2) \) is a subgroup of \( \text{SU}(4) \), and obviously \( \text{SO}(5) \) is a subgroup of \( \text{SO}(6) \), so indeed we have:

\[
\begin{align*}
\text{Sp}(2) & \xrightarrow{2-1} \text{SO}(5) \\
\downarrow & \downarrow \\
\text{SU}(4) & \xrightarrow{2-1} \text{SO}(6)
\end{align*}
\]
In fact, \( \text{Sp}(n) \) is a subgroup of \( \text{SU}(2n) \).

Why? \( \text{Sp}(n) \) is all \( HH \)-linear transformations of \( HH^n \) preserving the quaternion-valued inner product:

\[
(v, w) = \sum_{i=1}^{n} v_i \overline{w_i}, \quad v_i, w_i \in HH
\]

\( \text{U}(2n) \) is all \( C \)-linear transformations of \( C^{2n} \) preserving the complex inner product:

\[
\langle v, w \rangle = \sum_{i=1}^{2n} v_i \overline{w_i}, \quad v_i, w_i \in C
\]

To think of \( C^{2n} \) as being \( HH^n \), we need to equip it with extra structure; i.e., the subgroup of \( \text{U}(2n) \) preserving this extra structure is \( \text{Sp}(n) \).
In fact, this \( SP(n) \) lies in \( SU(2n) \), but we won't prove that.

To make a \( 2n \)-dimensional complex vector space \( V \) into an \( n \)-dimensional quaternionic vector space, we need to equip it with an operator

\[
j : V \rightarrow V
\]

with \( j^2 = -1, \quad i \neq j \), \( ij = -ji \). (So \( j \) is a real-linear but not complex-linear operator. Instead, it's conjugate linear!)

We then define \( K = ij \), & check:

\[
k^2 = -1
\]

\( ij = K = -ji \) & cyclic

\( ijk = -1 \).
So a quaternionic Stiefel vector space is a conjugate-linear $J$ w/ $J^2 = -1$.

If our complex vector space had an inner product $\langle \cdot, \cdot \rangle$ on it, and we're trying to make it into a quaternionic inner product space w/ inner product $(\cdot, \cdot)$, then we also need

$$\langle jv, jw \rangle = \langle v, w \rangle$$

Then we can define:

$$(v, w) = \langle v, w \rangle - j \langle jv, w \rangle \in \mathbb{H}$$

(Or something like it!), and we get a quaternionic inner product space. This needs to be checked!
Back to showing

\[ \text{SU}(4) \rightarrow \text{SO}(6) \]

\[ \text{Sp}(2) \rightarrow \text{SO}(5) \]

Last time we got the double cover \( \text{SU}(4) \rightarrow \text{SO}(6) \) as follows. We saw \( \text{SU}(4) \) acts on \( C^4 \) and thus on the exterior algebra \( \Lambda C^4 \), preserving the Hodge star operator:

\[ \ast : \Lambda^p C^4 \rightarrow \Lambda^{4-p} C^4 \]

So that

\[ \omega \ast \nu = \langle \omega, \nu \rangle \text{ vol} \]

where \( \text{vol} = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \), \( \langle \cdot, \cdot \rangle \) on \( \Lambda^p C^4 \) comes from \( \langle \cdot, \cdot \rangle \) on \( C^4 \).

We saw that \( \Lambda^2 C^4 \) is \( \binom{4}{2} = 6 \) dimensional complex inner product space, on which \( \text{SU}(4) \) acts.
We also saw that \( \ast : \Lambda^2 \mathbb{C}^4 \to \Lambda^2 \mathbb{C}^4 \) is conjugate-linear and \( \ast^2 = 1 \).

A conjugate-linear operator \( \ast : V \to V \) with \( \ast^2 = 1 \) is called a real structure, since it allows us to define real vector spaces.

\[
\begin{align*}
\text{Re } V &= \{ v \in V : \ast v = v \} \\
\text{Im } V &= \{ v \in V : \ast v = -v \}
\end{align*}
\]

\[3.7.\]

\[\text{Re } V \oplus \text{Im } V = V\]

As real vector spaces, \( V \cong \mathbb{C} \otimes_{\mathbb{R}} \text{Re } V \).
Since $SU(4)$ acts on $\Lambda^2 \mathbb{C}^4$ preserving $*$, it acts on the subspace $\text{Re}(\Lambda^2 \mathbb{C}^4)$, which is a 6-dim. real vector space. We saw this gives a homomorphism

$$SU(4) \rightarrow SO(6)$$

which is 2-1 and onto.

Now we want $Sp(2) \rightarrow SO(5)$. For this, take $\mathbb{C}^4$ and equip it with $j : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ making it into a quaternion-like inner product space. $Sp(2)$ will be the subgroup of $SU(4)$ preserving this extra structure.
So $\text{Sp}(2)$ also acts on $\text{Re}(\Lambda^2 \mathbb{C}^4)$, so we get:

$$
\begin{array}{ccc}
\text{Sp}(2) & \rightarrow & \text{SO}(6) \\
& \downarrow & \\
& \text{SU}(4) & 
\end{array}
$$

In fact, the image of $\text{Sp}(2)$ lies in $\text{SO}(5)$; why?

It's because $\Lambda^2 \mathbb{C}^4$ gets a special element in it, coming from $i: \mathbb{C}^4 \rightarrow \mathbb{C}^4$. $\text{Sp}(2)$ preserves this special element $w \epsilon \Lambda^2 \mathbb{C}^4$, so it preserves the 5-dim. complex inner product space $w^\perp \subseteq \mathbb{C}^4$. In fact, we're $\text{Re}(\Lambda^2 \mathbb{C}^4)$ so $\text{Sp}(2)$ preserves the 5-dim real space.
\[ \omega^1 \in \text{Re} (\Lambda^2 \mathbb{C}^4). \]

This gives a homomorphism

\[ \text{Sp}(2) \to \text{SO}(5) \]

which is 2-1 and onto.

To see this, we need to check:

1. \( \text{AP}(2) \to \text{AO}(5) \) is 1-1.

2. Therefore it is onto if the dimensions agree:

\[ \dim \text{AO}(5) = 1 + 2 + 3 + 4 = 10 \]

\[ \dim \text{AP}(2) = 3 + 3 + 4 = 10 \]

3. Thus \( \text{Sp}(2) \to \text{SO}(5) \) is defined if \( \text{Sp}(2) \) is simply-connected, onto if \( \text{SO}(5) \) is connected.
Thus has discrete kernel, which turns out to be \( \mathbb{Z}_2 \).

How do we get \( \Lambda^2 \mathbb{C}^4 \)? We get it from an \( \text{elt} \) of \( \Lambda^2 \mathbb{C}^{4^*} \), which we can identify with \( \Lambda^2 \mathbb{C}^4 \). An \( \text{elt} \) of \( \Lambda^2 \mathbb{C}^{4^*} \) is a skew symmetric bilinear map

\[
\omega : \mathbb{C}^4 \times \mathbb{C}^4 \rightarrow \mathbb{C}
\]

We build this from the inner product:

\[
\langle \cdot, \cdot \rangle : \mathbb{C}^4 \times \mathbb{C}^4 \rightarrow \mathbb{C}
\]

Which also is conjugate-linear in the second slot.

\[
j : \mathbb{C}^4 \rightarrow \mathbb{C}^4
\]
which is conjugate-linear, so:

\[ \omega(v, w) = \langle v, jw \rangle \]

This is bilinear, and also skew-symmetric:

\[
\begin{align*}
\omega(v, w) &= \langle v, jw \rangle \\
&= \overline{\langle jv, j^2w \rangle} \\
&= -\langle jv, w \rangle \\
&= -\langle w, jv \rangle \\
&= -\omega(v, w)
\end{align*}
\]