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Solving nonlinear inverse problems by evolution equations based on Gauss–Newton methods

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In this article, we solve nonlinear inverse problems by an evolution equation method which can be viewed as the continuous analogue of the Gauss–Newton method. Under certain conditions we prove the convergence and derive the rate of convergence when the discrepancy principle is coupled.

Keywords: nonlinear inverse problems; evolution equation based on Gauss–Newton method; the discrepancy principle; convergence; rates of convergence

AMS Subject Classifications: 65J15; 65J20; 47H17

1. Introduction

We consider the ill-posed equations

\[ F(x) = y, \]

arising from nonlinear inverse problems, where \( F: D(F) \subseteq X \rightarrow Y \) is a Fréchet differentiable nonlinear operator between two Hilbert spaces \( X \) and \( Y \) whose norms and inner products are denoted as \( \| \cdot \| \) and \( \langle \cdot , \cdot \rangle \). The Fréchet derivative of \( F \) at \( x \in D(F) \) is denoted by \( F'(x) \) and the adjoint of \( F'(x) \) is denoted by \( F'(x)^* \). We assume that (1.1) has a solution \( x^\dagger \) in the domain \( D(F) \) of \( F \). Let \( y^\delta \) be the only available approximate data to \( y \) satisfying

\[ \| y^\delta - y \| \leq \delta \]

with a given small noise level \( \delta > 0 \). Due to the ill-posedness of (1.1), how to use \( y^\delta \) to construct a stable approximate solution to \( x^\dagger \) becomes an important topic. Various regularization methods have been considered in the literature, see [1–5] and the references therein.

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In this article, we will solve the nonlinear inverse problems (1.1) by considering the evolution equation

\[
\begin{align*}
\frac{d}{dt} x^\delta(t) &= \Theta^\delta(x^\delta(t), t), \
x^\delta(0) &= x_0,
\end{align*}
\]

(1.3)

where \(x_0 \in D(F)\) is an initial guess of \(x^\delta\) and

\[\Theta^\delta(x, t) := -(\alpha(t) I + F'(x)^* F'(x))^{-1} F'(x)^* (F(x) - y^\delta) + \alpha(t) (x - x_0)\]

with \(\alpha(t)\) being a positive continuous function defined on \([0, \infty)\). In order to find a reasonable approximation to \(x^\delta\), we define \(t_\delta \geq 0\) to be the first number satisfying

\[\|F(x^\delta(t_\delta)) - y^\delta\| \leq \tau \delta\]

(1.4)

with a given number \(\tau > 1\). We will consider the approximation property of \(x^\delta(t_\delta)\) to \(x^\delta\) as \(\delta \to 0\).

According to the local existence theory of evolution equations, if \(F'(x)\) is locally Lipschitz in \(D(F)\), then (1.3) has a unique solution \(x^\delta(t)\) defined on some interval \([0, t_\delta)\) with \(t_\delta > 0\). In order to guarantee the method to be well-defined, we need to impose certain conditions on \(\alpha(t)\) and \(F\).

We will assume that \(\alpha(t)\) is a \(C^1\) function defined on \([0, \infty)\) such that

\[\lim_{t \to \infty} \alpha(t) = 0, \quad \alpha(t) > 0 \quad \text{and} \quad 0 \leq - \frac{\alpha'(t)}{\alpha(t)} \leq c_{\alpha} \quad \forall t \geq 0\]

(1.5)

for some constant \(c_{\alpha} > 0\). There are many choices of \(\alpha(t)\), for instance, we may choose \(\alpha(t) = ae^{-bt}\) or \(\alpha(t) = (a + bt)^{-\mu}\) for given positive numbers \(a, b\) and \(\mu\). In the convergence analysis the constant \(c_{\alpha}\) in (1.5) needs to be suitably small, say \(c_{\alpha} \leq 1/2\). We will specify this later.

For the nonlinear operator \(F\), we will assume that there is a ball \(B_\rho(x^\dagger)\) around \(x^\dagger\) of radius \(\rho > 0\) with \(B_\rho(x^\dagger) \subset D(F)\) and that there exist constants \(K_0\) and \(K_1\) such that

\[\|(F'(x) - F'(z))w\| \leq K_0 \|x - z\| \|F'(z)w\| + K_1 \|F'(z)(x - z)\| \|w\|\]

(1.6)

for any \(x, z \in B_\rho(x^\dagger)\) and \(w \in X\). This condition clearly implies that \(F'(x)\) is uniformly bounded and Lipschitz on \(B_\rho(x^\dagger)\). Thus, (1.6) guarantees the local existence of a unique solution for the initial value problem (1.3) for any initial guess \(x_0 \in B_\rho(x^\dagger)\).

In this article we will show, under the conditions (1.5) and (1.6), that the evolution equation (1.3) has a unique solution defined on a longer time interval so that the discrepancy principle (1.4) is well-defined. We will further show that \(x^\delta(t_\delta) \to x^\dagger\) as \(\delta \to 0\). Moreover, we will derive the rate of convergence when \(x_0 - x^\dagger\) satisfies certain source-wise representation. For simplicity of presentation, throughout this article we will assume that the operator is properly scaled so that

\[\|F'(x)\| \leq \sqrt{\alpha(0)}, \quad x \in B_\rho(x^\dagger)\]

(1.7)

The evolution equation (1.3) together with the discrepancy principle (1.4) has been considered in [6] under the condition

\[F'(x) = R(x, z) F'(z) \quad \text{and} \quad \|I - R(x, z)\| \leq K_2 \|x - z\|\]
for all $x, z \in B_{\delta}(x^\dagger)$. This condition is much stronger than (1.6). Moreover, the analysis in [6] requires $\tau$ to be sufficiently large. Therefore, our result significantly improves the counterpart in [6].

We remark that if the Euler scheme is used to solve the evolution equation (1.3), it results in the iteratively regularized Gauss–Newton method of Bakushinskii [7] which has been studied extensively, see [4] and the references therein. Therefore, (1.3) can be viewed as a continuous analogue of the iteratively regularized Gauss–Newton method. However, other iterative methods could be generated if different schemes are used to solve (1.3).

This article is organized as follows. In Section 2, we consider the evolution equation (1.3) with $z$ replaced by $y$ and show that the solution exists for all $t$ and converges to $x^\dagger$ as $t \to \infty$. In Section 3, we consider (1.3) with noisy data and show that the solution actually exists on an interval larger enough so that the discrepancy principle (1.4) is well-defined. In addition, we obtain the stability estimates. In Section 4, we show that $x^\dagger(t_0)$ converges to $x^\dagger$ as $t_0 \to 0$ and derive the rate of convergence when $x_0 - x^\dagger$ satisfies certain source-wise representation. Finally, in Section 5, we present an example from the identification problem in partial differential equations to illustrate the verification of condition (1.6).

For ease of exposition, throughout this article we will always use $C$ to denote a positive constant depending only on $\tau$. For two quantities $\Phi_1$ and $\Phi_2$, we will use $\Phi_1 \leq \Phi_2$ to mean that $\Phi_1 \leq C\Phi_2$. When we say $(K_0 + K_1)||x_0 - x^\dagger||$ is suitably small, we will mean that $(K_0 + K_1)||x_0 - x^\dagger|| \leq \eta$ for some constant $\eta > 0$ depending only on $\tau$.

2. Convergence analysis on the noise-free case

In this section we will consider the function $x(t)$ defined by (1.3) with $y^\delta$ replaced by $y$, i.e.

$$\begin{cases} \frac{dx(t)}{dt} = \Theta(x(t), t), & t \geq 0, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where

$$\Theta(x, t) := -(\alpha(t)I + F'(x)^*F'(x))^{-1}(F'(x)^*(F(x) - y) + \alpha(t)(x - x_0)).$$

When $F$ and $\alpha(t)$ satisfy the conditions (1.5) and (1.6), we will show that the solution $x(t)$ of (2.1) exists for all time $t \geq 0$ and $x(t) \to x^\dagger$ as $t \to \infty$ provided that $(K_0 + K_1)||x_0 - x^\dagger||$ is suitably small. Throughout this section we will use the notations

$$e_0 := x_0 - x^\dagger, \quad e(t) := x(t) - x^\dagger \quad \tag{2.2}$$

and

$$T := F'(x^\dagger), \quad T(t) := F'(x(t)), \quad A := T^*T, \quad A(t) := T(t)^*T(t). \quad \tag{2.3}$$

We also introduce the functions

$$g(\alpha, \lambda) = \frac{1}{\alpha + \lambda} \quad \text{and} \quad r(\alpha, \lambda) = \frac{\alpha}{\alpha + \lambda}. \quad \tag{2.4}$$
Then
\[
\Theta(x, t) = -(x - x^t) + r(\alpha(t), F'(x)F'(x))(x_0 - x^t) \\
- g(\alpha(t), F'(x)F'(x))F'(x)(y - F'(x)(x - x^t)).
\]

We start with some consequences of condition (1.6). Clearly (1.6) implies for any \( x, z \in B_{\rho}(x^t) \) that
\[
\|F(x) - F(z) - F'(z)(x - z)\| \leq \frac{1}{2}(K_0 + K_1)\|x - z\|\|F'(z)(x - z)\| \tag{2.5}
\]
and
\[
\|F(x) - F(z) - F'(z)(x - z)\| \leq \frac{3}{2}(K_0 + K_1)\|x - z\|\|F'(x)(x - z)\|. \tag{2.6}
\]
The following lemma gives additional consequences of (1.6) which will be used frequently in this article.

**Lemma 2.1** Let \( F \) satisfy condition (1.6). Then for the functions \( g(\alpha, \lambda) \) and \( r(\alpha, \lambda) \) defined in (2.4) there hold
\[
\|r(\alpha, A_x) - r(\alpha, A_z)\| \leq \frac{1}{2}K_0\|x - z\| + \frac{1}{2\sqrt{\alpha}}K_1(\|F'(x)(x - z)\| + \|F'_z(x - z)\|), \tag{2.7}
\]
\[
\|r(\alpha, B_x) - r(\alpha, B_z)\| \leq \frac{1}{2}K_0\|x - z\| + \frac{1}{2\sqrt{\alpha}}K_1(\|F'(x)(x - z)\| + \|F'_z(x - z)\|), \tag{2.8}
\]
\[
\|F'_x[r(\alpha, A_x) - r(\alpha, A_z)]\| \leq \sqrt{\alpha}K_0\|x - z\| + \frac{1}{4}K_1\|F'(x)(x - z)\| + K_1\|F'_z(x - z)\| \tag{2.9}
\]
and
\[
\|F'_x[g(\alpha, A_x) - g(\alpha, A_z)]\| \leq \frac{1}{\sqrt{\alpha}}K_0\|x - z\| + \frac{1}{4\alpha}K_1\|F'(x)(x - z)\| + \frac{1}{\alpha}K_1\|F'_z(x - z)\| \tag{2.10}
\]
for all \( \alpha > 0 \) and \( x, z \in B_{\rho}(x^t) \), where \( F_x := F'(x), A_x = F_x^{21}F_x, \) and \( B_x = F'_xF_x^{21} \).

**Proof** The proof is straightforward. Let us prove (2.10) as an illustration. Since \( g(\alpha, \lambda) = (\alpha + \lambda)^{-1} \), we have
\[
F'_x[g(\alpha, A_x) - g(\alpha, A_z)] = F'_x(\alpha I + A_z)^{-1}(A_z - A_x)(\alpha I + A_z)^{-1} \tag{2.11}
\]
\[
= F'_x(\alpha I + A_x)^{-1}(F'_x - F'_z)(\alpha I + A_x)^{-1} \tag{2.12}
\]
\[
+ (\alpha I + B_x)^{-1}B_x(F'_z - F'_x)(\alpha I + A_z)^{-1}. \tag{2.13}
\]
Therefore
\[
\|F'_x[g(\alpha, A_x) - g(\alpha, A_z)]\| \leq \frac{1}{2\sqrt{\alpha}}\|F'_x - F'_z\|\|(\alpha I + A_x)^{-1}\|F'_x^{21}\| \tag{2.14}
\]
\[
+ \|F'_x - F'_z\|\|(\alpha I + A_x)^{-1}\|. \tag{2.15}
\]
From (1.6) it follows that
\[
\|F'_{\chi}[g(\alpha, A_{\chi}) - g(\alpha, A_{\chi})]\| \leq \frac{1}{2\sqrt{\alpha}} K_0 \|x - z\|^{2} \|B_{\chi}(\alpha I + B_{\chi})^{-1}\|
\]
\[
+ \frac{1}{2\sqrt{\alpha}} K_1 \|F_{\chi}(x - z)\|^{2} \|(\alpha I + A_{\chi})^{-1} F'_{\chi}\|
\]
\[
+ K_0 \|x - z\|^{2} \|F'_{\chi}(\alpha I + A_{\chi})^{-1}\|
\]
\[
+ K_1 \|F'_{\chi}(x - z)\|^{2} \|(\alpha I + A_{\chi})^{-1}\|
\]
\[
\leq \frac{1}{\sqrt{\alpha}} K_0 \|x - z\| + \frac{1}{\alpha} K_1 \|F'_{\chi}(x - z)\|
\]
\[
+ \frac{1}{4\alpha} K_1 \|F'_{\chi}(x - z)\|.
\]
We thus complete the proof of (2.10).

Now we are ready to show a long-time existence result for the evolution equation (2.1).

**Lemma 2.2** Let \( F \) satisfy (1.6) and (1.7) in \( B_\rho(x^0) \), and let \( \alpha(t) \) satisfy (1.5) with \( c_\alpha \leq 1/2 \). If \( 2\|e_0\| < \rho \) and if \( (K_0 + K_1)\|e_0\| \) is suitably small, then (2.1) has a unique solution \( x(t) \) defined for all \( t \geq 0 \). Moreover, there hold the estimates
\[
\|e(t)\| \leq 2\|e_0\| \quad \text{and} \quad \|Te(t)\| \leq 2\|e_0\|\sqrt{\alpha(t)} \tag{2.11}
\]
for all \( t \geq 0 \).

**Proof** We will use the bootstrap argument. We define \( t_0 > 0 \) to be the largest number such that (2.1) has a solution \( x(t) \) defined on the interval \([0, t_0)\) with the properties
\[
\|e(t)\| \leq 2\|e_0\| \quad \text{and} \quad \|Te(t)\| \leq 2\|e_0\|\sqrt{\alpha(t)} \quad \forall 0 \leq t < t_0. \tag{2.12}
\]
Using the local existence result, the continuity of \( e(t) \) and \( \alpha(t) \), and the assumption \( \|T\| \leq \sqrt{\alpha(0)} \), it is easy to see that such \( t_0 > 0 \) is well-defined. It suffices to show \( t_0 = \infty \).

Suppose on the contrary that \( t_0 \) is finite. We will make use of (2.12) together with the conditions on \( F \) and \( \alpha(t) \) to show that (2.12) in fact can be improved as follows:
\[
\|e(t)\| \leq (2 - \varepsilon)\|e_0\| \quad \text{and} \quad \|Te(t)\| \leq (2 - \varepsilon)\|e_0\|\sqrt{\alpha(t)} \quad \forall 0 \leq t < t_0 \tag{2.13}
\]
for a sufficiently small number \( \varepsilon > 0 \). Thus (2.12) serves as the bootstrap assumption during the argument.

Once (2.13) is done, we can proceed as follows to get a contradiction. We first note that for any \( 0 \leq t_1 < t_2 < t_0 \) there holds
\[
x(t_2) - x(t_1) = \int_{t_1}^{t_2} \frac{dx(t)}{dt}dt = \int_{t_1}^{t_2} \Theta(x(t), t)dt.
\]
From the definition of \( \Theta(x, t) \) it follows readily that
\[
\Theta(x(t), t) = -e(t) + r(\alpha(t), A(t))e_0 - g(\alpha(t), A(t))T(t)^*u(t),
\]
where

\[ u(t) := F(x(t)) - y - T(t)e(t). \]

By using (2.6) and the bootstrap assumption (2.12), we have for \( 0 \leq t < t_0 \) that

\[
\|\Theta(x(t), t)\| \leq 3\|e_0\| + \frac{3}{4\sqrt{a(t)}}(K_0 + K_1)\|e(t)\|\|Te(t)\| \\
\leq 3\|e_0\| + 3(K_0 + K_1)\|e_0\|^2 := C_0.
\]

Consequently \( \|x(t_2) - x(t_1)\| \leq C_0(t_2 - t_1) \) for all \( 0 \leq t_1 < t_2 < t_0 \), which implies that \( x(t_0) := \lim_{t \to t_0} x(t) \) exists with the properties

\[
\|e(t_0)\| \leq (2 - \varepsilon)\|e_0\| \quad \text{and} \quad \|Te(t_0)\| \leq (2 - \varepsilon)\|e_0\|\sqrt{a(t_0)}.
\]

(2.14)

We may consider the initial value problem

\[
\frac{d}{dt} \tilde{x}(t) = \Theta(\tilde{x}(t), t), \quad \tilde{x}(t_0) = x(t_0),
\]

which, by the local existence result, has a unique solution \( \tilde{x}(t) \) defined on an interval \([t_0 - \varepsilon_1, t_0 + \varepsilon_1]\) for some \( \varepsilon_1 > 0 \). Since both \( x(t) \) and \( \tilde{x}(t) \) satisfy the same evolution equation on \([t_0 - \varepsilon_1, t_0]\) with the same initial condition at \( t_0 \), the uniqueness theorem gives \( x(t) = \tilde{x}(t) \) on \([t_0 - \varepsilon_1, t_0]\). Therefore \( x(t) \) extends to the larger interval \([0, t_0 + \varepsilon_1]\) by defining \( x(t) = \tilde{x}(t) \) for \( t \in [t_0, t_0 + \varepsilon_1] \). Using (2.14) and the continuity of \( e(t) \) and \( a(t) \) we can conclude the existence of \( 0 < \varepsilon_2 \leq \varepsilon_1 \) such that

\[
\|e(t)\| \leq 2\|e_0\| \quad \text{and} \quad \|Te(t)\| \leq 2\|e_0\|\sqrt{a(t)}
\]

(2.15)

for all \( 0 \leq t \leq t_0 + \varepsilon_2 \). This contradicts the maximality of \( t_0 \). Therefore \( t_0 = \infty \).

It remains to show (2.13) for some small number \( \varepsilon > 0 \). From the definition of \( x(t) \) it follows that

\[
\frac{d}{dt} e(t) = -e(t) + r(a(t), A(t))e_0 - g(a(t), A(t))T(t)^*u(t).
\]

We can continue to write

\[
\frac{d}{dt} e(t) = -e(t) + r(a(t), A)e_0 + \eta_1(t) + \eta_2(t),
\]

(2.16)

where

\[
\eta_1(t) := [r(a(t), A(t)) - r(a(t), A)]e_0,
\]

\[
\eta_2(t) := -g(a(t), A(t))T(t)^*u(t).
\]

By using the identities

\[
\frac{d}{dt}(\|e(t)\|^2) = 2\langle e(t), \frac{d}{dt} e(t) \rangle \quad \text{and} \quad \frac{d}{dt}(\|Te(t)\|^2) = 2\langle Te(t), T \frac{d}{dt} e(t) \rangle,
\]
it then follows from (2.16) that
\[
\frac{d}{dt}(\|e(t)\|^2) = -2\|e(t)\|^2 + 2(e(t), r(\alpha(t), A)e_0) + 2(e(t), \eta_1(t) + \eta_2(t))
\] (2.17)

and
\[
\frac{d}{dt}(\|Te(t)\|^2) = -2\|Te(t)\|^2 + 2(Te(t), Tr(\alpha(t), A)e_0)
\]
\[+ 2(Te(t), T(\eta_1(t) + \eta_2(t))).
\] (2.18)

We need to give estimates on \(\eta_i(t)\) and \(T\eta_i(t)\) for \(i = 1, 2\). We first estimate \(\eta_1(t)\).

By using (2.6) and the bootstrap assumption (2.12) we can estimate
\[
\|\eta_2(t)\| \leq \frac{3}{2\sqrt{\alpha(t)}}(K_0 + K_1)\|e(t)\|\|Te(t)\| \leq 3(K_0 + K_1)\|e_0\|\|e(t)\|.
\] (2.19)

In order to estimate \(\eta_1(t)\), we first use (2.7) in Lemma 2.1 to get
\[
\|\eta_1(t)\| \leq \frac{1}{2} K_0\|e_0\|\|e(t)\| + \frac{1}{2\sqrt{\alpha(t)}} K_1\|e_0\|\|Te(t)\| + \|T(t)e(t)\|.
\]

Note that (1.6) and the bootstrap assumption (2.12) imply
\[
\|(T - T(t))e(t)\| \leq (K_0 + K_1)\|e(t)\|\|Te(t)\| \leq 2(K_0 + K_1)\|e_0\|\|Te(t)\|.
\]

Thus, if \(6(K_0 + K_1)\|e_0\| \leq 1\), then the above inequality implies
\[
\frac{2}{3}\|Te(t)\| \leq \|T(t)e(t)\| \leq \frac{4}{3}\|Te(t)\|.
\] (2.20)

Therefore
\[
\|\eta_1(t)\| \leq \frac{1}{2} K_0\|e_0\|\|e(t)\| + \frac{7}{6\sqrt{\alpha(t)}} K_1\|e_0\|\|Te(t)\|.
\] (2.21)

Next we turn to estimate \(T\eta_2(t)\). From (2.9) in Lemma 2.1 and (2.20) it follows for \(T\eta_2(t)\) that
\[
\|T\eta_2(t)\| \leq \sqrt{\alpha(t)}K_0\|e_0\|\|e(t)\| + K_1\|e_0\|\|T(t)e(t)\| + \frac{1}{4} K_1\|e_0\|\|Te(t)\|
\]
\[\leq \sqrt{\alpha(t)}K_0\|e_0\|\|e(t)\| + \frac{5}{3} K_1\|e_0\|\|Te(t)\|.
\] (2.22)

Using \(T = [T - T(t)] + T(t)\), (1.6), (2.12), (2.20), (2.6), the estimate on \(\eta_2\), and
\(6(K_0 + K_1)\|e_0\| \leq 1\), we can estimate \(T\eta_2\) as follows:
\[
\|T\eta_2(t)\| \leq (1 + K_0\|e(t)\|)\|T(t)\eta_2(t)\| + K_1\|T(t)e(t)\|\|\eta_2(t)\|
\]
\[\leq \frac{3}{2}(1 + 2K_0\|e_0\|)(K_0 + K_1)\|e(t)\|\|Te(t)\|
\]
\[+ 4K_1(K_0 + K_1)\|e_0\|\|e(t)\|\|Te(t)\|
\]
\[\leq 5(K_0 + K_1)\|e_0\|\|Te(t)\|.
\] (2.23)
With the help of the above estimates (2.19), (2.21), (2.22) and (2.23) on $\eta(t)$ and $T\eta(t)$, it follows from (2.17) and (2.18) that
\[ \frac{d}{dt}(\|e(t)\|^2) \leq -2\|e(t)\|^2 + 2\langle e(t), r(\alpha(t), A)e_0 \rangle + (7K_0 + 6K_1)\|e_0\|\|e(t)\|^2 \\
+ \frac{7}{3\sqrt{\alpha(t)}}K_1\|e_0\|\|e(t)\||Te(t)| \tag{2.24} \]
and
\[ \frac{d}{dt}(\|Te(t)\|^2) \leq -2\|Te(t)\|^2 + 2\langle Te(t), Tr(\alpha(t), A)e_0 \rangle \\
+ 2K_0\|e_0\|\|e(t)\||Te(t)||\sqrt{\alpha(t)} \\
+ \left(10K_0 + \frac{40}{3}K_1\right)\|e_0\|\|Te(t)\|^2. \tag{2.25} \]
By using (2.24) and the bootstrap assumption (2.12) it yields
\[ \frac{d}{dt}(\|e(t)\|^2) \leq -2\|e(t)\|^2 + 2\|e_0\|\|e(t)\|^2 + 2\left(7K_0 + \frac{25}{3}K_1\right)\|e_0\|^2\|e(t)\|. \]
Therefore
\[ \|e(t)\| \leq \|e_0\| + \left(7K_0 + \frac{25}{3}K_1\right)\|e_0\|^2 \quad \forall 0 \leq t < t_0. \tag{2.26} \]
On the other hand, by making use of the condition $|\alpha'(t)|/\alpha(t) \leq c_\alpha$ with $c_\alpha \leq 1/2$ we have from (2.25) that
\[ \frac{d}{dt}\left(\frac{\|Te(t)\|^2}{\alpha(t)}\right) = \frac{1}{\alpha(t)}\frac{d}{dt}(\|Te(t)\|^2) - \frac{\alpha'(t)}{\alpha(t)}\|Te(t)\|^2 \\
\leq -\frac{3}{2}\|Te(t)\|^2 + 2\frac{\langle Te(t), Tr(\alpha(t), A)e_0 \rangle}{\alpha(t)} \\
+ 2K_0\|e_0\|\|e(t)\|\|Te(t)||/\alpha(t) \\
+ \left(10K_0 + \frac{40}{3}K_1\right)\|e_0\|\|Te(t)\|^2/\alpha(t). \]
By using the bootstrap assumption (2.12) and $\|Tr(\alpha(t), A)e_0\| \leq \|e_0\||\sqrt{\alpha(t)}$, we have with $\sigma := (12K_0 + \frac{40}{3}K_1)||e_0||$
\[ \frac{d}{dt}\left(\frac{\|Te(t)\|^2}{\alpha(t)}\right) \leq -\frac{3}{2}\|Te(t)\|^2 + 2(1 + \sigma)\|e_0\||\|Te(t)\||/\sqrt{\alpha(t)}. \]
Since $\|Te_0\| \leq \sqrt{\alpha(0)}\|e_0\|$, we obtain
\[ \frac{\|Te(t)\|}{\sqrt{\alpha(t)}} \leq \frac{4}{3}(1 + \sigma)\|e_0\| \quad \forall 0 \leq t < t_0. \tag{2.27} \]
If $(K_0 + K_1)||e_0||$ is suitably small such that $2\sigma < 1$, then (2.13) follows immediately from (2.26) and (2.27) with a suitable small number $\epsilon > 0$. \qed
In the next result we will give the convergence property of \( x(t) \), i.e. we will show that \( \| x(t) - x^\dagger \| \to 0 \) as \( t \to \infty \).

**Proposition 2.3** Let all the conditions in Lemma 2.2 be fulfilled. If \( x_0 - x^\dagger \in \mathcal{N}(T)^\perp \) then there hold

\[
\lim_{t \to \infty} \| e(t) \| = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{\| Te(t) \|}{\sqrt{\alpha(t)}} = 0.
\]

**Proof** In the proof of Lemma 2.2 we have obtained the differential inequalities (2.24) and (2.25). Let

\[
\Phi(t) := \| e(t) \|^2 + \frac{\| Te(t) \|^2}{\alpha(t)}.
\]

Then it follows from (2.24) and (2.25) that

\[
\frac{d}{dt} \Phi(t) \leq -\frac{3}{2} \Phi(t) + \left( 10K_0 + \frac{40}{3}K_1 \right) \| e_0 \| \Phi(t) + 2 \langle e(t), r(\alpha(t), A)e_0 \rangle \\
+ 2 \frac{\langle Te(t), Tr(\alpha(t), A)e_0 \rangle}{\alpha(t)} + \frac{7}{3} (K_0 + K_1) \| e_0 \| \| e(t) \| \frac{\| Te(t) \|}{\sqrt{\alpha(t)}}.
\]

Using the inequalities \( 2 \| e(t) \| \| Te(t) \| /\sqrt{\alpha(t)} \leq \Phi(t) \) and

\[
2 \langle e(t), r(\alpha(t), A)e_0 \rangle + 2 \frac{\langle Te(t), Tr(\alpha(t), A)e_0 \rangle}{\alpha(t)} \leq \varepsilon \Phi(t) + \varepsilon^{-1} \Psi(t),
\]

where

\[
\Psi(t) := \| r(\alpha(t), A)e_0 \|^2 + \frac{\| Tr(\alpha(t), A)e_0 \|^2}{\alpha(t)}
\]

and \( \varepsilon > 0 \) is an arbitrarily small number, we then obtain

\[
\frac{d}{dt} \Phi(t) \leq - \left[ \frac{3}{2} - \varepsilon - \frac{1}{6} (67K_0 + 87K_1) \| e_0 \| \right] \Phi(t) + \varepsilon^{-1} \Psi(t). \tag{2.28}
\]

Now we assume that \( (67K_0 + 87K_1) \| e_0 \| < 9 \) and choose \( \varepsilon > 0 \) such that \( \theta := 3/2 - \varepsilon - \frac{1}{6} (67K_0 + 87K_1) \| e_0 \| > 0 \). Then

\[
\frac{d}{dt} \Phi(t) \leq -\theta \Phi(t) + \varepsilon^{-1} \Psi(t) \quad \forall t \geq 0.
\]

Since \( e_0 \in \mathcal{N}(T)^\perp \) and \( \alpha(t) \to 0 \), we have \( \Psi(t) \to 0 \) as \( t \to \infty \). This together with the above differential inequality implies \( \Phi(t) \to 0 \) as \( t \to \infty \). \( \blacksquare \)

### 3. Stability estimates

In this section, we will concentrate on the study of the solution \( x^\delta(t) \) defined by (1.3). Due to the appearance of noise, we cannot expect the global existence in general. However, we are able to show that \( x^\delta(t) \) indeed exists in an interval larger enough for...
our purpose. We then derive the stability estimates for $x^\delta(t)$ on such an interval. To this end, we define $\tilde{t}_0 \geq 0$ to be the first number such that

$$\alpha(\tilde{t}_0) \leq \left(\frac{(\tau - 1)\delta}{4\|e_0\|}\right)^2. \quad (3.1)$$

Since $\alpha(t) \to 0$ as $t \to \infty$, such $\tilde{t}_0$ exists. When $\tilde{t}_0 > 0$ we must have $\alpha(\tilde{t}_0) = ((\tau - 1)\delta/(4\|e_0\|))^2$. The following result shows that $x^\delta(t)$ exists at least on the time interval $[0, \tilde{t}_0]$.

**Lemma 3.1** Let $F$ satisfy (1.6) and (1.7) in $B_p(x^\delta)$, and let $\alpha(t)$ satisfy (1.5) with $c_\alpha \leq 1/2$. If $\frac{2\tau}{\tau - 1}\|e_0\| < \rho$ and if $(K_0 + K_1)\|e_0\|$ is suitably small, then the evolution equation (1.3) has a unique solution $x^\delta(t)$ defined at least on the interval $[0, \tilde{t}_0]$. Moreover, there hold the estimates

$$\|e^\delta(t)\| \leq \frac{2\tau}{\tau - 1}\|e_0\| \quad \text{and} \quad \|T_e^\delta(t)\| \leq \frac{2(\tau + 3)}{\tau - 1}\|e_0\|\sqrt{\alpha(t)} \quad (3.2)$$

for all $0 \leq t \leq \tilde{t}_0$, where $e^\delta(t) := x^\delta(t) - x^\dagger$.

**Proof** The proof follows by a bootstrap argument with slight modifications on the proof of Lemma 2.2. Since the changes are minor, we omit the details. ■

Next we will derive the stability estimates. We use the notations

$$T^\delta(t) := F'(x^\delta(t)) \quad \text{and} \quad A^\delta(t) := T^\delta(t)^* T^\delta(t) \quad \text{in addition to} \quad (2.2) \quad \text{and} \quad (2.3).$$

**Proposition 3.2** Let all the conditions in Lemma 3.1 be fulfilled. If $(K_0 + K_1)\|e_0\|$ is suitably small, then there hold

$$\|x^\delta(t) - x(t)\| \leq \frac{\delta}{\sqrt{\alpha(t)}} \quad (3.3)$$

and

$$\|F(x^\delta(t)) - F(x(t)) - y^\delta + y\| \leq (1 + C(K_0 + K_1)\|e_0\|)\delta \quad (3.4)$$

for all $0 \leq t \leq \tilde{t}_0$.

**Proof** We may assume $\tilde{t}_0 > 0$ since otherwise the result is trivial. We will apply the bootstrap argument. Let $0 < t_0 \leq \tilde{t}_0$ be the largest number such that

$$\|x^\delta(t) - x(t)\| \leq \frac{2\delta}{\sqrt{\alpha(t)}} \quad \text{and} \quad \|T(x^\delta(t) - x(t))\| \leq 3\delta \quad (3.5)$$

for all $0 \leq t \leq t_0$. We will establish (3.3) by showing that $t_0 = \tilde{t}_0$. We will assume $t_0 < \tilde{t}_0$ and show that (3.5) can be improved as

$$\|x^\delta(t) - x(t)\| \leq \frac{(2 - \varepsilon)\delta}{\sqrt{\alpha(t)}} \quad \text{and} \quad \|T(x^\delta(t) - x(t))\| \leq (3 - \varepsilon)\delta \quad (3.6)$$
for all $0 \leq t \leq t_0$ with a small number $\varepsilon > 0$. This together with the continuity of $x^\delta(t) - x(t)$ and $\alpha(t)$ then gives a contradiction to the maximality of $t_0$. In order to establish (3.6), we use the definitions of $x^\delta(t)$ and $x(t)$ to write

$$
\frac{d}{dt} (x^\delta(t) - x(t)) = -(x^\delta(t) - x(t)) + [r(\alpha(t), A^\delta(t)) - r(\alpha(t), A)]e_0
$$

$$
- g(\alpha(t), A^\delta(t)) T^\delta(t)^* (y^\delta - u^\delta(t))
$$

$$
+ g(\alpha(t), A(t)) T(t)^* u(t),
$$

where

$$
u(t) = F(x(t)) - y - T(t)e(t) \quad \text{and} \quad u^\delta(t) = F(x^\delta(t)) - y - T^\delta(t)e^\delta(t).
$$

By regrouping the terms on the right-hand side we can write

$$
\frac{d}{dt} (x^\delta(t) - x(t)) = -(x^\delta(t) - x(t)) + s(t),
$$

where $s(t) = s_1(t) + s_2(t) + s_3(t) + s_4(t)$ with

$$
s_1(t) := [r(\alpha(t), A^\delta(t)) - r(\alpha(t), A)]e_0,
$$

$$
s_2(t) := g(\alpha(t), A^\delta(t)) T^\delta(t)^* (y^\delta - y),
$$

$$
s_3(t) := [g(\alpha(t), A(t)) T(t)^* - g(\alpha(t), A^\delta(t)) T^\delta(t)^*] u(t),
$$

$$
s_4(t) := g(\alpha(t), A^\delta(t)) T^\delta(t)^* (u(t) - u^\delta(t)).
$$

Consequently

$$
\frac{d}{dt} (\|x^\delta(t) - x(t)\|^2) = -2 \|x^\delta(t) - x(t)\|^2 + 2 (x^\delta(t) - x(t), s(t))
$$

and

$$
\frac{d}{dt} (\|T(x^\delta(t) - x(t))\|^2) = -2 \|T(x^\delta(t) - x(t))\|^2 + 2 (T(x^\delta(t) - x(t)), Ts(t)).
$$

In order to proceed further, we will use the bootstrap assumption (3.5) to derive suitable estimates on $s(t)$ and $Ts(t)$ for all $0 \leq t \leq t_0$. From the condition (1.6) and the estimates in Lemma 2.2 it follows

$$
\|(T(t) - T)(x^\delta(t) - x(t))\|
$$

$$
\leq K_0 \|e(t)\| \|T(x^\delta(t) - x(t))\| + K_1 \|Te(t)\| \|x^\delta(t) - x(t)\|
$$

$$
\leq K_0 \|e_0\| \|T^\delta(t) - x(t)\| + \sqrt{\alpha(t)} K_1 \|e_0\| \|x^\delta(t) - x(t)\|.
$$

The bootstrap assumption (3.5) then gives

$$
\|T(t)(x^\delta(t) - x(t))\| \leq \|T(x^\delta(t) - x(t))\| + K_1 \|e_0\| \delta \leq \delta.
$$

Similarly, (1.6), Lemma 3.1 and the bootstrap assumption (3.5) imply

$$
\|T^\delta(t)(x^\delta(t) - x(t))\| \leq \|T(x^\delta(t) - x(t))\| + K_1 \|e_0\| \delta \leq \delta.
$$
By (2.7) and (2.9) in Lemma 2.1 we have
\[
\|s_1(t)\| \leq K_0 \|e_0\| \|x^\delta(t) - x(t)\| + \frac{1}{\sqrt{\alpha(t)}} K_1 \|e_0\| \left( \|T(t)(x^\delta(t) - x(t))\| + \|T^\delta(t)(x^\delta(t) - x(t))\| \right)
\]
and
\[
\|T^\delta(t)s_1(t)\| \leq K_0 \|e_0\| \|x^\delta(t) - x(t)\| \sqrt{\alpha(t)}
\]
\[
+ K_1 \|e_0\| \left( \|T(t)(x^\delta(t) - x(t))\| + \|T^\delta(t)(x^\delta(t) - x(t))\| \right).
\]
With the help of (3.10), (3.11) and the bootstrap assumption (3.5), we obtain
\[
\|s_1(t)\| \leq (K_0 + K_1)\|e_0\| \frac{\delta}{\sqrt{\alpha(t)}}, \quad \|T^\delta(t)s_1(t)\| \leq (K_0 + K_1)\|e_0\| \delta. \quad (3.12)
\]
It is easy to see that
\[
\|s_2(t)\| \leq \frac{\delta}{2\sqrt{\alpha(t)}}, \quad \|T^\delta(t)s_2(t) - y^\delta + y\| \leq \delta. \quad (3.13)
\]
In order to estimate \(s_3(t)\) and \(T^\delta(t)s_3(t)\), we note that
\[
s_3(t) = g(\alpha(t), A(t))[T(t)^* - T^\delta(t)^*]u(t)
\]
\[
+ \left[ g(\alpha(t), A(t)) - g(\alpha(t), A^\delta(t)) \right] T^\delta(t)^*u(t)
\]
and
\[
T^\delta(t)s_3(t) = [T(t) - T^\delta(t)] g(\alpha(t), A(t)) T(t)^*u(t)
\]
\[
+ \left[ r(\alpha(t), B(t)) - r(\alpha(t), B^\delta(t)) \right] u(t).
\]
From (1.6), Lemma 2.1, (2.10) and (2.8) it follows that
\[
\|s_3(t)\| = \sup_{w \in X, \|w\| = 1} |\langle s_3(t), w \rangle|
\]
\[
\leq \sup_{w \in X, \|w\| = 1} |\langle u(t), [T(t) - T^\delta(t)] g(\alpha(t), A(t))w \rangle|
\]
\[
+ \sup_{w \in X, \|w\| = 1} |\langle u(t), T^\delta(t) \left[ g(\alpha(t), A(t)) - g(\alpha(t), A^\delta(t)) \right]w \rangle|
\]
\[
\leq \frac{1}{\sqrt{\alpha(t)}} K_0 \|x^\delta(t) - x(t)\| \|u(t)\|
\]
\[
+ \frac{1}{\alpha(t)} K_1 \left( \|T(t)(x^\delta(t) - x(t))\| + \|T^\delta(t)(x^\delta(t) - x(t))\| \right) \|u(t)\|
\]
and
\[
\|T^\delta(t)s_3(t)\| \leq K_0 \|x^\delta(t) - x(t)\| \|u(t)\|
\]
\[
+ \frac{1}{\sqrt{\alpha(t)}} K_1 \left( \|T(t)(x^\delta(t) - x(t))\| + \|T^\delta(t)(x^\delta(t) - x(t))\| \right) \|u(t)\|.
\]
Note that (2.6) and Lemma 2.2 imply \( \| u(t) \| \leq (K_0 + K_1) \| e_0 \| \sqrt{\alpha(t)} \). Therefore, it follows from (3.10), (3.11) and the bootstrap assumption (3.5) that

\[
\| s_3(t) \| \leq (K_0 + K_1) \| e_0 \| \frac{\delta}{\sqrt{\alpha(t)}}, \quad \| T^\delta(t)s_3(t) \| \leq (K_0 + K_1) \| e_0 \| \delta. \tag{3.14}
\]

For \( s_4(t) \) and \( T^\delta(t)s_4(t) \) we first have

\[
\| s_4(t) \| \leq \frac{1}{\sqrt{\alpha(t)}} \| u(t) - u^\delta(t) \|, \quad \| T^\delta(t)s_4(t) \| \leq \| u(t) - u^\delta(t) \|. \tag{3.15}
\]

By using the condition (1.6), the estimates in Lemma 2.2 and Lemma 3.1, the bootstrap assumption (3.5), (3.10) and the fact \( \delta/\sqrt{\alpha(t)} \leq \| e_0 \| \) for \( 0 \leq t \leq \bar{t}_\delta \) we have

\[
\| u(t) - u^\delta(t) \| \leq \| F(x^\delta(t)) - F(x(t)) - T(t)(x^\delta(t) - x(t)) \| + \| (T^\delta(t) - T(t))e^\delta(t) \|
\leq (K_0 + K_1) \| x^\delta(t) - x(t) \| \| T(t)(x^\delta(t) - x(t)) \|
+ K_0 \| x^\delta(t) - x(t) \| \| T(t)e^\delta(t) \| + K_1 \| e^\delta(t) \| \| T(t)(x^\delta(t) - x(t)) \|
\leq (K_0 + K_1) \| e_0 \| \delta.
\]

Therefore

\[
\| s_4(t) \| \leq (K_0 + K_1) \| e_0 \| \frac{\delta}{\sqrt{\alpha(t)}}, \quad \| T^\delta(t)s_4(t) \| \leq (K_0 + K_1) \| e_0 \| \delta. \tag{3.15}
\]

Combining the estimates (3.12)–(3.15) yields

\[
\| s(t) \| \leq (1 + C(K_0 + K_1) \| e_0 \|) \frac{\delta}{2\sqrt{\alpha(t)}} \tag{3.16}
\]

and

\[
\| T^\delta(t)s(t) - y^\delta + y \| \leq (1 + C(K_0 + K_1) \| e_0 \|) \delta. \tag{3.17}
\]

Consequently, it follows from (1.6) and the estimates in Lemma 3.1 that

\[
\| (T^\delta(t) - T)s(t) \| \leq K_0 \| e^\delta(t) \| \| T^\delta(t)s(t) \| + K_1 \| T^\delta(t)e^\delta(t) \| \| s(t) \|
\leq (K_0 + K_1) \| e_0 \| \delta.
\]

Hence

\[
\| Ts(t) \| \leq \| T^\delta(t)s(t) \| + C(K_0 + K_1) \| e_0 \| \delta \leq (2 + C(K_0 + K_1) \| e_0 \|) \delta. \tag{3.18}
\]

By using the estimates (3.16) and (3.18) together with \( \alpha'(t) \leq 0 \), it follows from (3.8) and (3.9) that

\[
\frac{d}{dt} (\alpha(t) \| x^\delta(t) - x(t) \|^2) \leq -2\alpha(t) \| x^\delta(t) - x(t) \|^2
+ (1 + C(K_0 + K_1) \| e_0 \|) \delta \sqrt{\alpha(t)} \| x^\delta(t) - x(t) \|
\]
\[
\frac{d}{dt} \left( \| T(x^\delta(t) - x(t)) \|^2 \right) \leq -2\| T(x^\delta(t) - x(t)) \|^2 \\
+ 2(2 + C(K_0 + K_1)\|e_0\|)\delta \| T(x^\delta(t) - x(t)) \|.
\]

We therefore obtain
\[
\sqrt{\alpha(t)}\| x^\delta(t) - x(t) \| \leq \frac{1}{2} (1 + C(K_0 + K_1)\|e_0\|)\delta
\]
and
\[
\| T(x^\delta(t) - x(t)) \| \leq (2 + C(K_0 + K_1)\|e_0\|)\delta.
\]
Thus, if \((K_0 + K_1)\|e_0\|\) is suitably small, then the above two inequalities imply (3.6) for all \(0 \leq t \leq t_0\) with a small number \(\varepsilon > 0\).

Next we prove (3.4). In the above we have obtained (3.17) for all \(0 \leq t \leq \tilde{t}_\delta\). It then follows (3.7) and (3.17) that
\[
\frac{d}{dt} (\| T(x^\delta(t) - x(t)) - y^\delta + y \|^2 ) \\
= 2(T(x^\delta(t) - x(t)) - y^\delta + y, T \frac{d}{dt} (x^\delta(t) - x(t))) \\
\leq -2\| T(x^\delta(t) - x(t)) - y^\delta + y \|^2 \\
+ 2(1 + C(K_0 + K_1)\|e_0\|)\delta \| T(x^\delta(t) - x(t)) - y^\delta + y \|,
\]
which implies for all \(0 \leq t \leq \tilde{t}_\delta\) that
\[
\| T(x^\delta(t) - x(t)) - y^\delta + y \| \leq (1 + C(K_0 + K_1)\|e_0\|)\delta.
\]
Thus, from (1.6), Lemma 2.2, Lemma 3.1 and (3.3) it follows that
\[
\| T(t)(x^\delta(t) - x(t)) - y^\delta + y \| \\
\leq (1 + C(K_0 + K_1)\|e_0\|)\delta + \| (T(t) - T)(x^\delta(t) - x(t)) \| \\
\leq (1 + C(K_0 + K_1)\|e_0\|)\delta + K_0\|e(t)\|\| T(x^\delta(t) - x(t)) \| \\
+ K_1\| T(x^\delta(t) - x(t)) \|\| x^\delta(t) - x(t) \| \\
\leq (1 + C(K_0 + K_1)\|e_0\|)\delta.
\]
Consequently, noting that \(\delta / \sqrt{\alpha(t)} \leq \|e_0\|\) for \(0 \leq t \leq \tilde{t}_\delta\), we have
\[
\| F(x^\delta(t)) - F(x(t)) - y^\delta + y \| \\
\leq \| F(x^\delta(t)) - F(x(t)) - T(t)(x^\delta(t) - x(t)) \| + (1 + C(K_0 + K_1)\|e_0\|)\delta \\
\leq \frac{1}{2}(K_0 + K_1)\| x^\delta(t) - x(t) \|\| T(t)(x^\delta(t) - x(t)) \| + (1 + C(K_0 + K_1)\|e_0\|)\delta \\
\leq C(K_0 + K_1)\frac{\delta}{\sqrt{\alpha(t)}}\delta + (1 + C(K_0 + K_1)\|e_0\|)\delta \\
\leq (1 + C(K_0 + K_1)\|e_0\|)\delta.
\]
The proof of (3.4) is therefore complete.
We conclude this section by showing that the discrepancy principle (1.4) is well-defined for \( \tau > 1 \) and \( t_\delta \leq \tilde{t}_\delta \) if \( (K_0 + K_1)\|e_0\| \) is suitably small. It suffices to show
\[
\|F(\mathcal{x}^\delta) - y^\delta\| \leq \tau \delta.
\] (3.19)

To see this, we first have from (3.4) in Proposition 3.2 that
\[
\|F(\mathcal{x}^\delta) - y^\delta\| \leq \|F(\mathcal{x}(\tilde{t}_\delta)) - y\| + (1 + C(K_0 + K_1)\|e_0\|)\delta.
\]

From (2.5) and Lemma 2.2 it follows that
\[
\|F(\mathcal{x}(\tilde{t}_\delta)) - y - Te(\tilde{t}_\delta)\| \leq (K_0 + K_1)\|e_0\||Te(\tilde{t}_\delta)|.
\]

Thus, if \((K_0 + K_1)\|e_0\| \) is suitably small, then
\[
\|F(\mathcal{x}(\tilde{t}_\delta)) - y\| \leq \frac{3}{2} \|Te(\tilde{t}_\delta)\| \leq 3\|e_0\|\sqrt{\alpha(\tilde{t}_\delta)}.
\]

Using the definition of \( \tilde{t}_\delta \), we have \( \|F(\mathcal{x}(\tilde{t}_\delta)) - y\| \leq 3(\tau - 1)\delta/4 \). Therefore
\[
\|F(\mathcal{x}^\delta) - y^\delta\| \leq \left(1 + \frac{3(\tau - 1)}{4} + C(K_0 + K_1)\|e_0\|\right)\delta.
\]

We therefore obtain (3.19) if \((K_0 + K_1)\|e_0\| \) is suitably small.

4. Convergence and rate of convergence

Let \( t_\delta \) be the number determined by the discrepancy principle (1.4) with \( \tau > 1 \). In this section we will show that \( \mathcal{x}^\delta(t_\delta) \rightarrow \mathcal{x}_0^{\dagger} \) as \( \delta \rightarrow 0 \). Moreover, we will derive the rate of convergence when \( e_0 = x_0 - \mathcal{x}_0^{\dagger} \) satisfies suitable source-wise conditions.

We need further estimates on \( \|e(t)\| \) and \( \|Te(t)\| \). We first note that the function \( r(\alpha, \lambda) = \alpha(\alpha + \lambda)^{-1} \) verifies
\[
0 \leq \alpha \frac{\partial r}{\partial \alpha}(\alpha, \lambda) \leq r(\alpha, \lambda) \quad \forall \alpha > 0 \quad \text{and} \quad \lambda \geq 0,
\] (4.1)

which can be used to derive some useful differential inequalities on \( \|r(\alpha(t), \mathcal{A})e_0\|^2 \) and \( \|Tr(\alpha(t), \mathcal{A})e_0\|^2 \). To see this, let \( \{E_\lambda\} \) be the spectral family generated by the nonnegative self-adjoint operator \( \mathcal{A} \). Then
\[
\frac{d}{dt}(\|r(\alpha(t), \mathcal{A})e_0\|^2) = \frac{d}{dt}\left(\int_0^1 r(\alpha(t), \lambda)^2 d\|E_\lambda e_0\|^2\right)
= 2\alpha'(t) \int_0^1 r(\alpha(t), \lambda) \frac{\partial r}{\partial \alpha}(\alpha(t), \lambda)d\|E_\lambda e_0\|^2.
\]

It then follows from the condition \( 0 \leq -\alpha'(t)/\alpha(t) \leq c_\alpha \) and (4.1) that
\[
0 \leq -\frac{d}{dt}(\|r(\alpha(t), \mathcal{A})e_0\|^2) \leq 2c_\alpha\|r(\alpha(t), \mathcal{A})e_0\|^2
\] (4.2)

for all \( t \geq 0 \). Using the same argument we have
\[
0 \leq -\frac{d}{dt}(\|Tr(\alpha(t), \mathcal{A})e_0\|^2) \leq 2c_\alpha\|Tr(\alpha(t), \mathcal{A})e_0\|^2
\] (4.3)

for all \( t \geq 0 \).
LEMMA 4.1 Let all the conditions in Lemma 2.2 hold. If \((K_0 + K_1)\|e_0\|\) is suitably small, then there holds
\[
\|e(t)\| + \|Te(t)\| \leq \|r(\alpha(t), \mathcal{A})e_0\| + \|Tr(\alpha(t), \mathcal{A})e_0\| \sqrt{\alpha(t)}
\]
for all \(t \geq 0\).

Proof Let \(\Phi(t)\) and \(\Psi(t)\) be the same functions defined in the proof of Proposition 2.3. It suffices to show that \(\Phi(t) \leq \Psi(t)\) for all \(t \geq 0\). From (4.2), (4.3) and \(\alpha'(t) \leq 0\), it follows that \(\Psi(t)\) satisfies the differential inequality
\[
- \frac{d}{dt} \Psi(t) \leq 2c_\alpha \Psi(t) \quad \forall t \geq 0.
\]
With the help of the differential inequalities (2.28) and (4.4) we have for the function \(\beta(t) := \Phi(t)/\Psi(t)\) that
\[
\frac{d}{dt} \beta(t) \leq - \left( \frac{3}{2} - \varepsilon - 2c_\alpha - C(K_0 + K_1)\|e_0\| \right) \beta(t) + \varepsilon^{-1}.
\]
Now we choose \(\varepsilon = 1/4\), use \(c_\alpha \leq 1/2\) and let \((K_0 + K_1)\|e_0\|\) be small, then we obtain
\[
\frac{d}{dt} \beta(t) \leq - \beta(t)/8 + 4,
\]
which implies that \(\beta(t) \leq \max\{\beta(0), 32\}\). Since \(\|T\| \leq \sqrt{\alpha(0)}\) implies \(\beta(0) \leq 8\), the proof is thus complete.

LEMMA 4.2 Let all the conditions in Lemma 2.2 hold but with \(c_\alpha \leq 1/4\), and let \(x(t)\) be the solution of the evolution equation (2.1). If \(e_0 \in \mathcal{N}(T)^\perp\) and \((K_0 + K_1)\|e_0\|\) is suitably small, then
\[
\|Tr(\alpha(t), \mathcal{A})e_0\| \leq \|Te(t)\| \leq \|Tr(\alpha(t), \mathcal{A})e_0\|
\]
for all \(t \geq 0\).

Proof We will use (2.18). We need to derive a refined estimate on \(\|T\eta_1(t)\|\). Since \(r(\alpha, \lambda) = \alpha(\alpha + \lambda)^{-1}\). We can write \(T\eta_1(t) = u_1(t) + u_2(t)\), where
\[
u_1(t) := T(\alpha(t)I + \mathcal{A}(t))^{-1} T(t)^*(T - T(t))r(\alpha(t), \mathcal{A})e_0,\]
\[
u_2(t) := T(\alpha(t)I + \mathcal{A}(t))^{-1} (T^* - T(t)^*) Tr(\alpha(t), \mathcal{A})e_0.
\]
Using the condition (1.6), the inequality (2.20) and the estimates in Lemma 2.2, we obtain
\[
\|T(\alpha(t)I + \mathcal{A}(t))^{-1} T(t)^*\| \leq 1 + \|(T - T(t))(\alpha(t)I + \mathcal{A}(t))^{-1} T(t)^*\|
\leq 1 + K_0\|e(t)\| + \frac{1}{2\sqrt{\alpha(t)}} K_1\|T(t)e(t)\|
\leq 1 + C(K_0 + K_1)\|e_0\| \leq 1
\]
and

\[
\|T(\alpha(t)I + A(t))^{-1}\| \leq \frac{1}{2\sqrt{\alpha(t)}} + \|(T - T(t))(\alpha(t)I + A(t))^{-1}\|
\]

\[
\leq \frac{1}{2\sqrt{\alpha(t)}} (1 + K_0\|e(t)\|) + \frac{1}{\alpha(t)}K_1\|T(t)e(t)\|
\]

\[
\leq \frac{1}{\sqrt{\alpha(t)}} (1 + C(K_0 + K_1)\|e_0\|) \lesssim \frac{1}{\sqrt{\alpha(t)}}.
\]

By the above two estimates together with (1.6), (2.20) and Lemma 2.2 it yields for any \(v \in Y\) and \(w \in X\) satisfying \(\|v\| = \|w\| = 1\) that

\[
\langle T(\alpha(t)I + A(t))^{-1}(T^*-T(t)^*)v, w \rangle = \langle v, (T - T(t))(\alpha(t)I + A(t))^{-1}T^*w \rangle 
\]

\[
\leq K_0\|e(t)\|\|T(t)(\alpha(t)I + A(t))^{-1}T^*\|
\]

\[
+ K_1\|T(t)e(t)\|\|T(t)(\alpha(t)I + A(t))^{-1}T^*\|
\]

\[
\lesssim (K_0 + K_1)\|e_0\|.
\]

which implies \(\|T(\alpha(t)I + A(t))^{-1}(T^*-T^*)\| \lesssim (K_0 + K_1)\|e_0\|.\) Consequently

\[
\|u_1(t)\| \lesssim (\|T - T(t)\|\|r(\alpha(t), A)e_0\|.
\]

\[
\|u_2(t)\| \lesssim (K_0 + K_1)\|e_0\|\|Tr(\alpha(t), A)e_0\|.
\]

With the help of (1.6), \(u_1(t)\) can be further estimated as

\[
\|u_1(t)\| \lesssim K_0\|e(t)\|\|Tr(\alpha(t), A)e_0\| + K_1\|Te(t)\|\|r(\alpha(t), A)e_0\|
\]

\[
\lesssim (K_0 + K_1)\|e_0\|\|Tr(\alpha(t), A)e_0\| + K_1\|e_0\|\|Te(t)\|.
\]

Therefore for \(T_{\eta_1}(t)\) there holds the estimate

\[
\|T_{\eta_1}(t)\| \lesssim (K_0 + K_1)\|e_0\|\|Tr(\alpha(t), A)e_0\| + K_1\|e_0\|\|Te(t)\|.\] (4.5)

This together with the estimate (2.23) on \(T_{\eta_2}(t)\) and Equation (2.18) gives

\[
\frac{d}{dt}(\|Te(t)\|^2) \leq -2\|Te(t)\|^2 + 2\|Te(t)\|\|Tr(\alpha(t), A)e_0\|
\]

\[
+ C(K_0 + K_1)\|e_0\|(\|Te(t)\|^2 + \|Tr(\alpha(t), A)e_0\|^2)
\]

and

\[
\frac{d}{dt}(\|Te(t)\|^2) \geq -2\|Te(t)\|^2 + 2(\langle Te(t), Tr(\alpha(t), A)e_0 \rangle)
\]

\[
- C(K_0 + K_1)\|e_0\|(\|Te(t)\|^2 + \|Tr(\alpha(t), A)e_0\|^2).
\]

With the help of the differential inequality (4.3), it follows that the function

\[
\phi(t) := \frac{\|Te(t)\|^2}{\|Tr(\alpha(t), A)e_0\|^2}
\]
satisfies the differential inequalities
\[
\frac{d}{dt} \varphi(t) \leq -(2 - 2c_a - C(K_0 + K_1)\|e_0\|)\varphi(t) + 2\sqrt{\varphi(t)} + C(K_0 + K_1)\|e_0\| \tag{4.6}
\]
and
\[
\frac{d}{dt} \varphi(t) \geq -(2 + C(K_0 + K_1)\|e_0\|)\varphi(t) + 2\psi(t) - C(K_0 + K_1)\|e_0\|, \tag{4.7}
\]
where
\[
\psi(t) := \frac{(Te(t), Tr(\alpha(t), A)e_0)}{\|Tr(\alpha(t), A)e_0\|^2}.
\]

We first use (4.6) to derive an upper bound on \( \varphi(t) \). To this end, using \( c_a \leq 1/4 \) and \( 2\sqrt{\varphi(t)} \leq \varphi(t) + 1 \), we have for small \( (K_0 + K_1)\|e_0\| \) that
\[
\frac{d}{dt} \varphi(t) \leq -\varphi/3 + 4/3.
\]
Consequently \( \varphi(t) \leq \max\{\varphi(0), 4\} \). By using \( \|T\| \leq \sqrt{\alpha(0)} \) we have \( \varphi(0) \leq 4 \). Therefore \( \varphi(t) \leq 4 \) for all \( t \geq 0 \). We thus obtain
\[
\|Te(t)\| \leq 2\|Tr(\alpha(t), A)e_0\| \quad \forall t \geq 0. \tag{4.8}
\]

We next derive the lower bound on \( \varphi(t) \). We first prove that \( \psi(t) \geq 1/4 \) for all \( t \geq 0 \). Since
\[
\frac{d}{dt} \psi(t) = \frac{1}{\|Tr(\alpha(t), A)e_0\|^2} \left( T \frac{d}{dt} e(t), Tr(\alpha(t), A)e_0 \right)
+ \frac{1}{\|Tr(\alpha(t), A)e_0\|^2} \left( Te(t), \alpha'(t) T \frac{\partial r}{\partial \alpha}(\alpha(t), A)e_0 \right)
- \psi(t) \frac{d}{dt} (\log \|Tr(\alpha(t), A)e_0\|^2),
\]
we have from (2.16) and (4.1) that
\[
\frac{d}{dt} \psi(t) \geq -\psi(t) + 1 - \frac{\|Te(t)\|}{\|Tr(\alpha(t), A)e_0\|} - C_a \frac{\|Te(t)\|}{\|Tr(\alpha(t), A)e_0\|}
- \psi(t) \frac{d}{dt} (\log \|Tr(\alpha(t), A)e_0\|^2).
\]

Using the estimates (4.5), (2.23) on \( \|T\eta_1\| \) and \( \|T\eta_2\| \) together with (4.8) it yields
\[
\frac{d}{dt} \psi(t) \geq -\psi(t) + 1 - 2C_a - C(K_0 + K_1)\|e_0\| - \psi(t) \frac{d}{dt} (\log \|Tr(\alpha(t), A)e_0\|^2). \tag{4.9}
\]

Now we are able to conclude that \( \psi(t) > 0 \) for all \( t \geq 0 \). It is clear that \( \psi(0) \geq 1 \). If there was a finite number \( t_0 > 0 \) such that \( \psi(t_0) = 0 \) for the first time, then \( \frac{d}{dt} \psi(t_0) \leq 0 \). It then follows from (4.9) that
\[
0 \geq \frac{d}{dt} \psi(t_0) \geq 1 - 2C_a - C(K_0 + K_1)\|e_0\|.
\]
Since $c_\alpha \leq 1/4$, the right-hand side is positive if $(K_0 + K_1)\|e_0\|$ is small, which is a contradiction. Thus $\psi(t) > 0$ for all $t \geq 0$.

Note that $\frac{d}{dt}(\log \text{Tr}(\alpha(t), A)e_0)^2) \leq 0$, we have from $\psi(t) > 0$ and (4.9) that

$$\frac{d}{dt} \psi(t) \geq -\psi(t) + 1 - 2c_\alpha - C(K_0 + K_1)\|e_0\|.$$

Consequently, $\psi(t) \geq \min\{\psi(0), 1 - 2c_\alpha - C(K_0 + K_1)\|e_0\|\} \geq 1/4$ if $(K_0 + K_1)\|e_0\|$ is small.

Finally, we return to the derivation of a positive lower bound on $\varphi(t)$. It follows from (4.7) and $\psi(t) \geq 1/4$ that

$$\frac{d}{dt} \varphi(t) \geq -3\varphi(t) + 1/4$$

for small $(K_0 + K_1)\|e_0\|$ which implies that $\varphi(t) \geq \min\{\varphi(0), 1/12\} \geq 1/12$ because $\varphi(0) \geq 1$. The proof is therefore complete.

Now we are ready to give the main result of this article concerning the convergence and rate of convergence.

**Theorem 4.3** Let all the conditions in Lemma 3.1 hold, let $x^\delta(t)$ be the solution of the evolution equation (1.3), and let $t_\delta$ be the number determined by the discrepancy principle (1.4) with $\tau > 1$. If $(K_0 + K_1)\|e_0\|$ is suitably small and if $e_0 \in \mathcal{N}(T)^\perp$, then

$$\lim_{\delta \to 0} x^\delta(t_\delta) = x^\dagger.$$

If, in addition, $e_0 = A^*\omega$ for some $0 < \nu \leq 1/2$ and $\omega \in X$, then

$$\|x^\delta(t_\delta) - x^\dagger\| \leq C_\nu\|\omega\|^{1/(1+2\nu)}\delta^{\nu/(1+2\nu)},$$

where $C_\nu$ is a constant depending only on $\nu$.

**Proof** We may assume $x_0 \neq x^\dagger$. From the stability estimate (3.3) in Proposition 3.2 and the fact $t_\delta \leq t_\delta$ we have

$$\|x^\delta(t_\delta) - x^\dagger\| \leq \|e(t_\delta)\| + \frac{\delta}{\sqrt{\alpha(t_\delta)}}. \quad (4.10)$$

We only need to estimate the two terms on the right-hand side.

From the definition of $t_\delta$ and the estimate (3.4) in Proposition 3.2 it follows that

$$\|F(x(t_\delta)) - y\| \leq \|F(x^\delta(t_\delta)) - y^\delta\| + \|F(x^\delta(t_\delta)) - F(x(t_\delta)) - y^\delta + y\|$$

and

$$\tau\delta = \|F(x^\delta(t_\delta)) - y^\delta\|
\leq \|F(x(t_\delta)) - y\| + \|F(x^\delta(t_\delta)) - F(x(t_\delta)) - y^\delta + y\|
\leq \|F(x(t_\delta)) - y\| + (1 + C(K_0 + K_1)\|e_0\|)\delta.$$
Since \( \tau > 1 \), we have \( \delta \leq \| F(x(t_{\delta})) - y \| \) if \( (K_{0} + K_{1})\| e_{0} \| \) is suitably small. Therefore

\[
\delta \leq \| F(x(t_{\delta})) - y \| \leq \delta.
\]

By (2.5) and the estimates in Lemma 2.2 we have

\[
\| F(x(t_{\delta})) - y - Te(t_{\delta}) \| \leq (K_{0} + K_{1})\| e_{0} \| \| Te(t_{\delta}) \|,
\]

which implies for sufficiently small \( (K_{0} + K_{1})\| e_{0} \| \) that

\[
\frac{1}{2} \| Te(t) \| \leq \frac{3}{2} \| F(x(t_{\delta})) - y \| \leq \frac{3}{2} \| Te(t_{\delta}) \|.
\]

Consequently \( \delta \leq \| Te(t_{\delta}) \| \leq \delta \). Thus we have from Lemma 4.2 that

\[
\delta \leq \| Tr(a(t_{\delta}), A)e_{0} \| \leq \delta. \tag{4.11}
\]

We first prove \( x^{\delta}(t_{\delta}) \to x^{+} \) as \( \delta \to 0 \). From (4.11), \( e_{0} \in \mathcal{N}(T)^{\perp} \) and \( e_{0} \neq 0 \) it follows easily that \( t_{\delta} \to \infty \) as \( \delta \to 0 \). This together with Proposition 2.3 and (4.11) implies \( \| e(t_{\delta}) \| \to 0 \) and \( \delta/\sqrt{\alpha(t_{\delta})} \to 0 \) as \( \delta \to 0 \). The convergence result thus follows from (4.10).

Next we derive the rate of convergence under the source condition \( e_{0} = A^{\nu} \omega \) for some \( 0 < \nu \leq 1/2 \) and \( \omega \in X \). Since \( \| Tr(\alpha, A)e_{0} \| \leq \alpha^{\nu+1/2}\| \omega \| \), we have from (4.11) that \( \delta \leq \alpha(t_{\delta})^{\nu+1/2}\| \omega \| \) which implies

\[
\delta \sqrt{\alpha(t_{\delta})} \leq \| \omega \|^{1/(1+2\nu)} \delta^{2\nu/(1+2\nu)}. \tag{4.12}
\]

By the interpolation inequality and (4.11) we also have

\[
\| r(\alpha(t_{\delta}), A)e_{0} \| \leq \| r(\alpha(t_{\delta}), A)\omega \|^{1/(1+2\nu)} \| Tr(\alpha(t_{\delta}), A)e_{0} \|^{2\nu/(1+2\nu)} \\
\leq \| \omega \|^{1/(1+2\nu)} \delta^{2\nu/(1+2\nu)}.
\]

It follows from this estimate, Lemma 4.1, (4.11) and (4.12) that

\[
\| e(t_{\delta}) \| \leq \| \omega \|^{1/(1+2\nu)} \delta^{2\nu/(1+2\nu)}. \tag{4.13}
\]

Therefore, the desired rate of convergence is the direct consequence of (4.10), (4.12) and (4.13).

\[ \blacksquare \]

5. An example

In this section we will consider a parameter identification problem in partial differential equations and show that the condition (1.6) is satisfied.

Let \( \Omega \subset \mathbb{R}^{N} \), \( N = 2, 3 \), be a bounded domain with smooth boundary \( \partial \Omega \). Consider the identification of the diffusion parameter \( a \) in

\[
\begin{cases}
-\text{div}(a \nabla u) = f & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega
\end{cases} \tag{5.1}
\]

from the \( L^{2} \) measurement of \( u \), where \( f \in H^{-1}(\Omega) \) and \( g \in H^{1/2}(\partial \Omega) \) are given. It is well-known that for \( a \in L^{\infty}(\Omega) \) with \( a \geq a > 0 \) on \( \Omega \), (5.1) has a unique solution \( u = u(a) \in H^{1}(\Omega) \). In order to put the problem into the framework of Hilbert space,
we assume \( a^+ \in H^2(\Omega) \) is the sought solution with \( a^+ \geq 2a > 0 \) on \( \Omega \). We then define \( F \) as

\[
F : H^2(\Omega) \to L^2(\Omega), \quad F(a) := u(a)
\]

with

\[
D(F) := \{ a \in H^2(\Omega) : a \geq a > 0 \text{ on } \Omega \}.
\]

Since \( H^2(\Omega) \) embeds into \( L^\infty(\Omega) \), such \( F \) is well-defined.

This is the inverse groundwater filtration problem corresponding to the steady-state case studied in [8] in which it has been shown that \( F \) is Fréchet differentiable and there is a neighbourhood \( B_{\rho}(a^+) \) of \( a^+ \) such that

\[
\| F(\tilde{a}) - F(a) - F'(a)(\tilde{a} - a) \|_{L^2} \leq \| \tilde{a} - a \|_{H^2} \| F(\tilde{a}) - F(a) \|_{L^2}
\]

for all \( \tilde{a}, a \in B_{\rho}(a^+) \).

In the following we will verify the condition (1.6). For \( \tilde{a}, a \in B_{\rho}(a^+) \) and \( h \in H^2(\Omega) \) we set

\[
u = u(a), \quad \tilde{u} = u(\tilde{a}), \quad u' = F'(a)h, \quad \tilde{u}' = F'(\tilde{a})h.
\]

Recall that \( u' \) is the weak solution of the boundary value problem

\[
\begin{align*}
-\text{div}(a \nabla u') &= \text{div}(h \nabla u) & \text{in } \Omega, \\
u' &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

The same is true for \( \tilde{u}' \). Thus

\[
\begin{align*}
-\text{div}(\tilde{a} \nabla (\tilde{u}' - u')) &= \text{div}(h \nabla (\tilde{u} - u)) + \text{div}((\tilde{a} - a) \nabla u') & \text{in } \Omega, \\
\tilde{u}' - u' &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

Since the operator \( A(\tilde{a}) : V := H^1_0(\Omega) \cap H^2(\Omega) \to L^2(\Omega) \) defined by \( A(\tilde{a})w = -\text{div}(\tilde{a} \nabla w) \) can be extended as an isomorphism \( A(\tilde{a}) : L^2(\Omega) \to V' \) so that \( A(\tilde{a})^{-1} : V' \to L^2(\Omega) \) is uniformly bounded around \( a^+ \), where \( V' \) denotes the anti-dual of \( V \) with respect to the bilinear form \( \int_\Omega \varphi \psi \text{d}x \), we have from the above equation that

\[
\| \tilde{u}' - u' \|_{L^2} \leq \| \text{div}((\tilde{a} - a) \nabla u') \|_{V'} + \| \text{div}(h \nabla (\tilde{u} - u)) \|_{V'}.
\]

In order to proceed further, note that for \( h \in H^2(\Omega), \varphi \in H^1_0(\Omega) \) and \( \psi \in V \), we have

\[
\int_\Omega \text{div}(h \nabla \varphi) \psi \text{d}x = \int_\Omega \varphi \text{div}(h \nabla \psi) \text{d}x \\
\leq \| \varphi \|_{L^2(\Omega)} \| \text{div}(h \nabla \psi) \|_{L^2(\Omega)}.
\]

For a bounded domain \( \Omega \subset \mathbb{R}^N \) with \( N=2,3 \), there hold the embedding \( H^2(\Omega) \hookrightarrow L^\infty(\Omega) \) and \( H^1(\Omega) \hookrightarrow L^4(\Omega) \). Thus

\[
\| \text{div}(h \nabla \psi) \|_{L^2(\Omega)} \leq \| h \Delta \psi \|_{L^2(\Omega)} + \| \nabla h \cdot \nabla \psi \|_{L^2(\Omega)} \\
\leq \| h \|_{L^\infty(\Omega)} \| \Delta \psi \|_{L^2(\Omega)} + \| \nabla h \|_{L^4(\Omega)} \| \nabla \psi \|_{L^4(\Omega)} \\
\leq \| h \|_{H^2(\Omega)} \| \psi \|_{V'}.
\]
Therefore, for all $\psi \in V$,
\[
\int_{\Omega} \text{div}(h\nabla \psi) \, dx \leq \|h\|_{H^2} \|\varphi\|_{L^2} \|\psi\|_{V}.
\]
Hence
\[
\|\text{div}(h\nabla \psi)\|_{V} \leq \|h\|_{H^2} \|\varphi\|_{L^2}.
\]
Applying this inequality to estimate the two terms on the right-hand side of (5.3), we obtain
\[
\|F'(\tilde{a}) - F'(a)\|_{L^2} \leq \|\tilde{a} - a\|_{H^2} \|F'(a)h\|_{L^2} + \|h\|_{H^2} \|F(\tilde{a}) - F(a)\|_{L^2}  
\]
for all $h \in H^2(\Omega)$ and $\tilde{a}, a \in B_{\rho}(a^\dagger)$. From (5.2) it follows
\[
\|F(\tilde{a}) - F(a)\|_{L^2} \leq \|F'(a)(\tilde{a} - a)\|_{L^2}
\]
for $\tilde{a}, a \in B_{\rho}(a^\dagger)$ by shrinking the ball $B_{\rho}(a^\dagger)$ if necessary. This together with (5.4) implies that there is a ball $B_{\rho}(a^\dagger) \subset D(F)$ such that for $\tilde{a}, a \in B_{\rho}(a^\dagger)$
\[
\|F'(\tilde{a}) - F'(a)\|_{L^2} \leq \|\tilde{a} - a\|_{H^2} \|F'(a)h\|_{L^2} + \|F'(a)(\tilde{a} - a)\|_{L^2} \|h\|_{H^2}.
\]
Thus the condition (1.6) is verified.

References


