APPLICATIONS OF THE MODIFIED DISCREPANCY PRINCIPLE TO TIKHONOV REGULARIZATION OF NONLINEAR ILL-POSED PROBLEMS∗

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Abstract. In this paper, we consider the finite-dimensional approximations of Tikhonov regularization for nonlinear ill-posed problems with approximately given right-hand sides. We propose an a posteriori parameter choice strategy, which is a modified form of Morozov’s discrepancy principle, to choose the regularization parameter. Under certain assumptions on the nonlinear operator, we obtain the convergence and rates of convergence for Tikhonov regularized solutions. This paper extends the results, which were developed by Plato and Vainikko in 1990 for solving linear ill-posed equations, to nonlinear problems.

Key words. nonlinear ill-posed problems, Tikhonov regularization, the modified discrepancy principle, convergence and rates of convergence

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1. Introduction. In this paper we consider the nonlinear problems of the form

\( F(x) = y_0, \)

(1.1)

where \( F : D(F) \subset X \longrightarrow Y \) is a nonlinear operator between real Hilbert spaces \( X \) and \( Y \). Throughout this paper it is assumed that the domain \( D(F) \) is closed and convex, \( y_0 \in \mathcal{R}(F) \) is the range, and \( F \) is weakly closed, continuous, and Fréchet differentiable with compact Fréchet derivatives. Our interest is to find the \( x^* \)-minimum-norm solution (\( x^*-\text{MNS} \)) \( x_0 \) of problem (1.1), i.e., to find an element \( x_0 \in X \) such that

\[ F(x_0) = y_0 \]

(1.2)

and

\[ \|x_0 - x^*\| = \min_{x \in D(F)} \{\|x - x^*\| : F(x) = y_0\}. \]

(1.3)

The existence of an \( x^*-\text{MNS} x_0 \) of (1.1) for exact data \( y_0 \) is well known. Note that due to the nonlinearity of \( F \), this solution need not be unique. The element \( x^* \in D(F) \) is an a priori guess of \( x_0 \), and it plays the role of a selection criterion. We are mainly interested in problems of the form (1.1) for which the solution \( x_0 \) does not depend continuously on the right-hand side data \( y_0 \). Such problems are called ill-posed and need to be regularized to obtain reasonable approximations to their solutions. We refer to [5] for a number of important inverse problems in natural sciences which lead to such ill-posed problems.

In recent years, a few regularization methods have been proposed for solving nonlinear ill-posed problems [25, 6, 19, 26, 10, 14]. Among these methods, Tikhonov
regularization is the best-known one, and some studies of it have been done recently [25, 6, 18, 23, 13]. In Tikhonov regularization, the solution $x^\delta$ of the minimization problem

$$\min_{x \in D(F)} \{\|F(x) - y_\delta\|^2 + \alpha\|x - x^*\|^2\}$$

is used to approximate the $x^*$-MNS of problem (1.1), where $\alpha > 0$ is called the regularization parameter, $x^\delta$ is the regularized solution, and $y_\delta$ is an approximation of $y_0$ satisfying

$$\|y_\delta - y_0\| \leq \delta.$$ (1.4)

As is well known, the choice of the regularization parameter plays an important role in regularization methods, and how to choose the regularization parameter attracts many mathematicians. For linear ill-posed equations, this problem has been well studied [27, 9, 2], and many parameter choice strategies, such as Morozov’s discrepancy principle [16], the general Arcangeli’s method [12], the modified Arcangeli’s method [4], and Gfrerer’s method [7], have been proposed. For nonlinear ill-posed problems, some modified forms of Morozov’s discrepancy principle have been investigated in [6, 15, 22], and the extensions of Gfrerer’s principle and Arcangeli’s method have been considered in [23] and [13], respectively; however, many questions regarding choosing the regularization parameter remain to be resolved.

Until now, all the studies concerning the choice of the regularization parameter for Tikhonov regularization of nonlinear ill-posed problems were confined to the infinite-dimensional setting. For numerical implementation, one should consider the finite-dimensional approximation of Tikhonov regularization. Some works have been presented in [18, 20]. Since there the regularization parameter has been chosen a priori, the results are not really applicable in practice because the information on the smoothness of the exact solution is often unknown a priori. As we know, a parameter choice based on a wrong guess of the smoothness on the exact solution will lead to a bad convergence, so an a posteriori parameter choice strategy for choosing the regularization parameter is necessary. In this paper we will contribute to such research and propose a parameter choice strategy based on the discrepancy principle.

This paper is organized as follows. In section 2 a rule is proposed for Tikhonov regularization of nonlinear ill-posed problems in finite-dimensional settings to choose the regularization parameter, and the justification is given under appropriate conditions. Section 3 contributes to the analyses of the convergence and rates of convergence. The uniqueness of the Tikhonov regularized solution, which is crucial for our analyses, is considered in section 4. Finally, two concrete examples from the parameter estimations of partial differential equations are given in section 5 to illustrate the applicability of our results.

2. The modified discrepancy principle. Let us approximate $X$ by a sequence of finite-dimensional subspaces $\{X_n\}$ with the property

$$\lim_{n \to \infty} \inf_{x_n \in X_n} \|x - x_n\| = 0 \quad \text{for all} \ x \in X$$ (2.1)

and

$$V_n := D(F) \cap X_n \neq \emptyset \quad \text{for all} \ n.$$ (2.2)
Then we use the solution \( x_{\alpha}^{\delta,n} \) of the finite-dimensional minimization problem

\[
\min_{x \in V_n} \{ \| F(x) - y_\delta \|^2 + \alpha \| x - x^* \|^2 \}
\]

to approximate the \( x^* \)-MNS \( x_0 \) of (1.1). As illustrated in [18], the quantity

\[
\gamma_n := \| F'(x_0)(I - P_n) \|
\]

plays an important role in the study of finite-dimensional approximation of Tikhonov regularization, and \( \gamma_n \to 0 \) as \( n \to \infty \) due to the compactness of \( F'(x_0) \) [9], where \( P_n \) denotes the ortho-projector from \( X \) onto \( X_n \). Clearly \( \gamma_n = \| (I - P_n)F'(x_0)^* \| \). Also as illustrated in [18, 20], the choice of the regularization parameter \( \alpha \) is crucial, and how to choose the regularization parameter directly affects the convergence rate of \( x_{\alpha}^{\delta,n} \). In this paper we propose the following a posteriori parameter choice strategy based on the discrepancy principle.

**Rule 2.1.** Let \( c_1, c_2 > 1, d_2 \geq d_1 > 1 \), and suppose \( \| F(P_n x^*) - y_\delta \| > d_1 \delta \). Then

(i) choose \( \alpha \geq \alpha_0 := (c_1 \delta + c_2 \gamma_n)^2 \) such that

\[
\| F(x_{\alpha}^{\delta,n}) - y_\delta \| \geq d_1 \delta,
\]

(ii) if there is no \( \alpha \geq \alpha_0 \) such that (2.6) holds, then choose \( \alpha := \alpha_0 \).

Our rule can be viewed as an extension of the method in [21], which is proposed for solving a linear ill-posed equation, to the nonlinear case with a slight modification. For the linear case, Plato and Vainikko [21] have obtained the convergence and rates of convergence with their parameter choice strategy. In the following we will do this for nonlinear ill-posed problems. Before doing so, let us explain why Rule 2.1 is reasonable. The following assumption is needed.

**Assumption 2.2.** For each \( \alpha \geq \alpha_0 \), the minimization problem (2.3) has a unique solution \( x_{\alpha}^{\delta,n} \).

Some sufficient conditions will be given in section 4 to guarantee the validity of Assumption 2.2. Here let us prove a claim concerning the continuity of the mapping \( \alpha \mapsto x_{\alpha}^{\delta,n} \) for \( \alpha \in [\alpha_0, \infty) \) which is more general than Lemma 2.8 in [15]. The proof can be carried out by a method similar to that in [6] which was presented to prove the stability of Tikhonov regularized solutions. For completeness, we include a derivation here.

Let \( \alpha \in [\alpha_0, \infty) \) be fixed and \( \{ \alpha_k \} \subset [\alpha_0, \infty) \) be a sequence satisfying \( \lim_{k \to \infty} \alpha_k = \alpha \). By the definition of \( x_{\alpha_k}^{\delta,n} \) we have

\[
\| F(x_{\alpha_k}^{\delta,n}) - y_\delta \|^2 + \alpha_k \| x_{\alpha_k}^{\delta,n} - x^* \|^2 \leq \| F(x_{\alpha}^{\delta,n}) - y_\delta \|^2 + \alpha_k \| x_{\alpha}^{\delta,n} - x^* \|^2.
\]

This implies the boundedness of \( \{ F(x_{\alpha_k}^{\delta,n}) \} \) and \( \{ x_{\alpha_k}^{\delta,n} \} \); hence there are elements \( \bar{x}_{\alpha}^{\delta,n} \in X \) and \( \bar{y}_\delta \in Y \) and a subsequence of \( \{ x_{\alpha_k}^{\delta,n} \} \) (for simplicity, still denote it by \( \{ x_{\alpha_k}^{\delta,n} \} \)) such that

\[
F(x_{\alpha_k}^{\delta,n}) \rightharpoonup \bar{y}_\delta, \quad x_{\alpha_k}^{\delta,n} \rightharpoonup \bar{x}_{\alpha}^{\delta,n},
\]

where here and below “\( \rightharpoonup \)” denotes the weakly convergence. From the weakly closedness of \( F \) it follows that \( \bar{x}_{\alpha}^{\delta,n} \in D(F) \) and \( F(\bar{x}_{\alpha}^{\delta,n}) = \bar{y}_\delta \). By the weak lower semicontinuity of the Hilbert space norm we have

\[
\| F(x_{\alpha_k}^{\delta,n}) - y_\delta \| \leq \liminf_{k \to \infty} \| F(x_{\alpha_k}^{\delta,n}) - y_\delta \|, \quad \| \bar{x}_{\alpha}^{\delta,n} - x^* \| \leq \liminf_{k \to \infty} \| x_{\alpha_k}^{\delta,n} - x^* \|.
\]
Combining (2.7) and (2.8) gives
\[
\|F(\bar{x}_\alpha^{\delta,n}) - y_k\|^2 + \alpha \|\bar{x}_\alpha^{\delta,n} - x^*\|^2 \leq \liminf_{k \to \infty} \{\|F(x_{\alpha_k}^{\delta,n}) - y_k\|^2 + \alpha_k \|x_{\alpha_k}^{\delta,n} - x^*\|^2\}
\]
\[
\leq \|F(x_{\alpha_k}^{\delta,n}) - y_k\|^2 + \alpha \|x_{\alpha_k}^{\delta,n} - x^*\|^2.
\]
Now the definition of $x_{\alpha_k}^{\delta,n}$ and Assumption 2.2 imply $x_{\alpha_k}^{\delta,n} = x_{\alpha_k}^{\delta,n}$. Hence $x_{\alpha_k}^{\delta,n} \to x_{\alpha_k}^{\delta,n}$. To prove $x_{\alpha_k}^{\delta,n} \to x_{\alpha_k}^{\delta,n}$, it remains to prove $\|x_{\alpha_k}^{\delta,n} - x^*\| = \lim_{k \to \infty} \|x_{\alpha_k}^{\delta,n} - x^*\|$. Because of (2.8), we need only to prove
\[
a := \|x_{\alpha_k}^{\delta,n} - x^*\|^2 = \liminf_{k \to \infty} \|x_{\alpha_k}^{\delta,n} - x^*\|^2 := b.
\]
Suppose to the contrary that $a < b$. It is easy to show that there is a subsequence of $\{x_{\alpha_k}^{\delta,n}\}$—say, $\{x_{\alpha_{k_l}}^{\delta,n}\}$—such that
\[
\|x_{\alpha_{k_l}}^{\delta,n} - x^*\|^2 = \limsup_{k \to \infty} \|x_{\alpha_{k_l}}^{\delta,n} - x^*\|^2 - \frac{b - a}{4},
\]
\[
\|F(x_{\alpha_{k_l}}^{\delta,n}) - y_{k_l}\|^2 \geq \liminf_{k \to \infty} \|F(x_{\alpha_k}^{\delta,n}) - y_k\|^2 - \frac{\alpha(b - a)}{4}.
\]
Therefore
\[
\|F(x_{\alpha_k}^{\delta,n}) - y_k\|^2 + \alpha_k \|x_{\alpha_k}^{\delta,n} - x^*\|^2 \leq \liminf_{k \to \infty} \|F(x_{\alpha_k}^{\delta,n}) - y_k\|^2 + \alpha_k \|x_{\alpha_k}^{\delta,n} - x^*\|^2 - \alpha_k(b - a)
\]
\[
\leq \|F(x_{\alpha_{k_l}}^{\delta,n}) - y_{k_l}\|^2 + \alpha_k \|x_{\alpha_{k_l}}^{\delta,n} - x^*\|^2 - \frac{3\alpha_k(b - a)}{4}
\]
\[
< \|F(x_{\alpha_{k_l}}^{\delta,n}) - y_{k_l}\|^2 + \alpha_k \|x_{\alpha_{k_l}}^{\delta,n} - x^*\|^2
\]
for sufficiently large $l$. This contradicts the definition of $x_{\alpha_{k_l}}^{\delta,n}$.

Combining the above we obtain the continuity of the mapping $\alpha \mapsto x_{\alpha_k}^{\delta,n}$ on $[\alpha_0, \infty)$. By the continuity of $F$ we know at once that the function $\rho(\alpha) := \|F(x_{\alpha_k}^{\delta,n}) - y_k\|$ is continuous with respect to $\alpha$ on $[\alpha_0, \infty)$. Therefore our rule is well defined. Note that $x_{\alpha_k}^{\delta,n}$ is also the minimizer of the minimization problem (2.3) with $x^*$ replaced by $P_n x^*$. We can show [27, 1] that $\rho(\alpha)$ is monotonically increasing on $[\alpha_0, \infty)$ and $\lim_{\alpha \to \infty} \rho(\alpha) = \|F(P_n x^*) - y^\delta\|$ for each $\delta > 0$ and $n$. Therefore there always holds $\|F(x_{\alpha_k}^{\delta,n}) - y^\delta\| \geq d_1 \delta$ with the $\alpha$ chosen by Rule 2.1 if $\|F(P_n x^*) - y^\delta\| > d_1 \delta$.

3. Convergence and rates of convergence. The purpose of this section is to establish the results on the convergence and rates of convergence of $x_{\alpha_k}^{\delta,n}$ with $\alpha$ chosen according to Rule 2.1. Some restrictions on the nonlinearity of $F$ are needed.

Assumption 3.1. There is a constant $\tau \geq 0$ such that for all $x \in D(F)$ there holds
\[
\|F(x) - F(x_0) - F'(x_0)(x - x_0)\| \leq \tau \|x - x_0\| \|F'(x_0)(x - x_0)\|.
\]
One can refer to [11] for some illustrations of this assumption by concrete problems.

Assumption 3.2. $P_n x_0 \in D(F)$ for $n$ large enough.

This assumption can be guaranteed when $x_0$ is an interior point of $D(F)$.

Now we can state the convergence result of $x_{\alpha_k}^{\delta,n}$ with the $\alpha$ chosen by Rule 2.1.

Theorem 3.3. Let Assumptions 2.2, 3.1, and 3.2 hold, and let $\alpha := \alpha(\delta, n)$ be chosen by Rule 2.1. Let $\{\delta_k\}$ and $\{n_k\}$ be any sequences satisfying $\delta_k \to 0$, $n_k \to \infty$. 

as \( k \to \infty \), let \( \alpha_k := \alpha(\delta_k, n_k) \), and let \( x_{\alpha_k}^{\delta_k, n_k} \) be the solution of the minimization problem (2.3) with \( y_k, v_\alpha \), and \( \alpha \) replaced by \( y_{\delta_k}, v_{\alpha_k} \), and \( \alpha_k \), respectively, where \( y_{\delta_k} \) denotes the perturbed data of \( y_0 \) satisfying \( \|y_{\delta_k} - y_0\| \leq \delta_k \). Then the sequence \( \{x_{\alpha_k}^{\delta_k, n_k}\} \) has a convergent subsequence. The limit of every convergent subsequence is an \( x^* \)-MNS of (1.1). If in addition the \( x^* \)-MNS \( x_0 \) of (1.1) is unique, then

\[
\lim_{k \to 0, n_k \to \infty} x_{\alpha_k}^{\delta_k, n_k} = x_0.
\]

**Proof.** From Assumptions 3.1 and 3.2 it is easy to show

\[
\|F(P_n x_0) - F(x_0)\| \leq \gamma_n (\|I - P_n\| x_0\| (\tau \|x_0\| + 1).
\]

Therefore the definition of \( x_{\alpha_k}^{\delta_k, n_k} \) gives

\[
\begin{align*}
&\|F(x_{\alpha_k}^{\delta_k, n_k}) - y_{\delta_k}\|^2 + \alpha_k \|x_{\alpha_k}^{\delta_k, n_k} - x^*\|^2 \leq \|F(P_n x_0) - y_{\delta_k}\|^2 + \alpha_k \|P_n x_0 - x^*\|^2 \\
&\leq (\delta_k + \gamma_n \|I - P_n\| x_0\| (\tau \|x_0\| + 1))^2 + \alpha_k \|P_n x_0 - x^*\|^2 \\
&\leq (1 + \mu)\delta_k^2 + C_1^2 \gamma_n^2 \|I - P_n\| x_0\|^2 + \alpha_k \|P_n x_0 - x^*\|^2,
\end{align*}
\]

where we have used the inequality \( 2ab \leq \mu a^2 + \frac{\delta_k^2}{\mu}, \mu > 0 \) is chosen such that \( 1 + \mu < d_1 \), and \( C_1^2 := (1 + \frac{1}{\mu})(\tau \|x_0\| + 1)^2 \). Since \( \|F(x_{\alpha_k}^{\delta_k, n_k}) - y_{\delta_k}\| \geq d_1 \delta_k \) we have

\[
\|x_{\alpha_k}^{\delta_k, n_k} - x^*\|^2 \leq \frac{C_1^2 \gamma_n^2 \|I - P_n\| x_0\|^2}{\alpha_k} + \|P_n x_0 - x^*\|^2.
\]

By the definition of \( \alpha_k \) we have \( \frac{\gamma_n^2}{\alpha_k} \leq 1 \), and hence

\[
\limsup_{k \to \infty} \|x_{\alpha_k}^{\delta_k, n_k} - x^*\| \leq \|x_0 - x^*\|.
\]

This implies that \( \{x_{\alpha_k}^{\delta_k, n_k}\} \) is bounded in \( X \). Therefore there exists \( \bar{x} \in X \) and a subsequence (for simplicity, we still denote it by \( \{x_{\alpha_k}^{\delta_k, n_k}\} \) such that \( x_{\alpha_k}^{\delta_k, n_k} \rightharpoonup \bar{x} \).

On the other hand, if \( \alpha_k = (c_1 \delta_k + c_2 \gamma_n) \), it follows from (3.2) that

\[
\|F(x_{\alpha_k}^{\delta_k, n_k}) - y_{\delta_k}\|^2 \leq (1 + \mu)\delta_k^2 + C_1^2 \gamma_n^2 \|I - P_n\| x_0\|^2 + (c_1 \delta_k + c_2 \gamma_n) \|P_n x_0 - x^*\|^2.
\]

If \( \alpha_k > (c_1 \delta_k + c_2 \gamma_n)^2 \), then the definition of \( \alpha_k \) implies

\[
\|F(x_{\alpha_k}^{\delta_k, n_k}) - y_{\delta_k}\| \leq d_2 \delta_k.
\]

Hence, for the \( \alpha_k \) chosen by Rule 2.1, we always have

\[
\lim_{k \to \infty} \|F(x_{\alpha_k}^{\delta_k, n_k}) - y_0\| = 0.
\]

By the weakly closedness of \( F \) we have \( \bar{x} \in D(F) \) and \( F(\bar{x}) = y_0 \). From the weak lower semicontinuity of the Hilbert space norm there holds

\[
\|\bar{x} - x^*\| \leq \liminf_{k \to \infty} \|x_{\alpha_k}^{\delta_k, n_k} - x^*\| \leq \limsup_{k \to \infty} \|x_{\alpha_k}^{\delta_k, n_k} - x^*\| \leq \|x_0 - x^*\|.
\]

Since \( x_0 \) is an \( x^* \)-MNS, we derive that \( \|\bar{x} - x^*\| = \|x_0 - x^*\| \) and hence \( \bar{x} \) is an \( x^* \)-MNS, and

\[
\lim_{k \to \infty} \|x_{\alpha_k}^{\delta_k, n_k} - x^*\| = \|x_0 - x^*\|.
\]
It follows from
\[ \|x^{\delta,k,n}_o - \bar{x}\|^2 = \|x^{\delta,k,n}_o - x^*\|^2 - 2\langle x^{\delta,k,n}_o - x^*, \bar{x} - x^* \rangle + \|\bar{x} - x^*\|^2 \]
that \( \lim_{k \to \infty} \|x^{\delta,k,n}_o - \bar{x}\| = 0 \). Now the assertion can be proved easily. \( \square \)

We now turn to the consideration of the rates of convergence of \( x^{\delta,n}_o \) with the \( \alpha \) chosen by Rule 2.1. Since the convergence speed of regularization methods may be arbitrarily slow [24], additional assumptions should be imposed on the \( x^* \)-MNS of (1.1) to yield reasonable rates; these conditions are the so-called “source conditions” and are contained in the following theorem.

**Theorem 3.4.** Let Assumptions 2.2, 3.1, and 3.2 hold. Let \( x^* \) be chosen such that \( \|(I - P_n)x^*\| \leq O(\gamma_n) \), let \( x_0 \) be the \( x^* \)-MNS of (1.1), and let \( \alpha \) be chosen by Rule 2.1. Suppose \( \|x_0 - x^*\| \) is sufficiently small such that \( 2\tau\|x_0 - x^*\| < 1 \) holds. If there is an \( 0 < \eta \leq 1 \) such that \( x_0 - x^* \in R((F'(x_0)*)^n)^{\eta/2}) \), then there is a positive constant \( M \) independent of \( \delta \) and \( n \) such that for \( \delta > 0 \) sufficiently small and \( n \) large enough, there holds
\[ \|x^{\delta,n}_o - x_0\| \leq M(\delta^{\eta/2} + \gamma_n^\eta). \]

**Remark 3.5.** 1. One can use the Morozov discrepancy principle, i.e., use the \( \alpha > 0 \) satisfying \( e_1\delta + f_1\gamma_n \leq \|F(x^{\delta,n}_o) - y_6\| \leq C_2\gamma_n \) with \( 1 < e_1 \leq e_2, 1 < f_1 \leq f_2 \) as the regularization parameter. With this \( \alpha \), we can obtain the error estimate \( \|x^{\delta,n}_o - x_0\| \leq O((\delta + \gamma_n)^{\eta/(1 + \eta)}) \), which is worse than the assertion in Theorem 3.4. This phenomenon has been observed in [21] for linear ill-posed problems.

2. If we can prove the uniqueness of \( x^* \)-MNS of (1.1) for concrete problems directly, the closeness condition \( 2\tau\|x_0 - x^*\| < 1 \) can be removed.

To prove Theorem 3.4, we need the following two lemmas.

**Lemma 3.6.** Let Assumption 3.1 hold, and suppose there is an \( x^* \)-MNS \( x_0 \) of (1.1) satisfying \( 2\tau\|x_0 - x^*\| < 1 \). Then \( x_0 \) must be the unique \( x^* \)-MNS of (1.1).

**Proof.** Let \( \bar{x} \) be another \( x^* \)-MNS of (1.1); then by the definition of \( x^* \)-MNS we also have \( 2\tau\|\bar{x} - x^*\| < 1 \) and hence \( \tau\|x_0 - \bar{x}\| < 1 \). From Assumption 3.1 we know that for all \( x \in D(F) \) there holds
\[ (1 - \tau\|x - x_0\|)\|F'(x_0)(x - x_0)\| \leq \|F(x) - F(x_0)\| \leq (1 + \tau\|x - x_0\|)\|F'(x_0)(x - x_0)\|. \]
This implies that both \( \bar{x} \) and \( x_0 \) are the solutions of the linear operator equation \( F'(x_0)\bar{x} = F'(x_0)x_0 \) which are nearest to \( x^* \). According to [8], it follows that \( \bar{x} = x_0 \), and the proof is complete. \( \square \)

**Lemma 3.7.** Let \( 0 < p \leq 1 \), \( A : X \mapsto Y \) be a bounded linear operator, \( A^* \) be the adjoint of \( A \), and \( P \) be an orthogonal projection on \( X \). Then
\[ \|(I - P)(A^*A)^{\frac{1}{2}}\| \leq \|A(I - P)\|^p. \]

**Proof.** See [21]. \( \square \)

**Proof of Theorem 3.4.** The proof can be carried out by estimating \( \|x^{\delta,n}_o - P_nx_0\| \) and \( \|(I - P_n)x_0\| \) separately. Similar to the derivation of (3.2) we obtain
\[
\begin{align*}
\|F(x^{\delta,n}_o) - y_6\|^2 + \alpha\|x^{\delta,n}_o - P_nx_0\|^2 &\leq (1 + \mu)\delta^2 + C_2\gamma_n^2\|x_0 - P_nx_0\|^2 \\
&\quad + \alpha\|P_nx_0 - x^*\|^2 - \|x^{\delta,n}_o - x^*\|^2 + \|x^{\delta,n}_o - P_nx_0\|^2 \\
&= (1 + \mu)\delta^2 + C_2\gamma_n^2\|x_0 - P_nx_0\|^2 + 2\alpha(x_0 - x^*, P_nx_0 - x^{\delta,n}_o) \\
&= (1 + \mu)\delta^2 + C_2\gamma_n^2\|x_0 - P_nx_0\|^2 + 2\alpha(x_0 - x^*, P_nx_0 - x^{\delta,n}_o).
\end{align*}
\]
Using the assumption on $x_0 - x^*$ it follows that there is an element $w \in X$ such that $x_0 - x^* = (F'(x_0)^* F'(x_0))^{\eta/2}w$. Therefore
\[
\|(x_0 - x^*, P_\alpha x_0 - x^{\delta,n}_\alpha)\| = \|(w, (F'(x_0)^* F'(x_0))^{\eta/2}(P_\alpha x_0 - x^{\delta,n}_\alpha))\|
\leq \|w\|\|(F'(x_0)^* F'(x_0))^{\eta/2}(P_\alpha x_0 - x^{\delta,n}_\alpha)\|.
\]
(3.4)

Since moment inequality implies for $0 < \eta \leq 1$ that
\[
\|(F'(x_0)^* F'(x_0))^{\eta/2}(P_\alpha x_0 - x^{\delta,n}_\alpha)\| \leq \|F'(x_0)(P_\alpha x_0 - x^{\delta,n}_\alpha)\|^\eta \|P_\alpha x_0 - x^{\delta,n}_\alpha\|^{1-\eta},
\]
we can collaborate (3.5) with (3.3) and (3.4) to obtain
\[
\|F(x^{\delta,n}_\alpha) - y_\delta\|^2 + \alpha \|x^{\delta,n}_\alpha - P_\alpha x_0\|^2 \leq (1 + \mu)\delta^2 + C_1^2 \gamma_n^2 \|(I - P_\alpha) x_0\|^2
\]
+ $2\alpha\|w\|\|F'(x_0)(P_\alpha x_0 - x^{\delta,n}_\alpha)\|^\eta \|P_\alpha x_0 - x^{\delta,n}_\alpha\|^{1-\eta}.
\]
(3.6)

Noting that $\|F(x^{\delta,n}_\alpha) - y_\delta\| \geq d_1 \delta$, we have
\[
\|x^{\delta,n}_\alpha - P_\alpha x_0\|^2 \leq A(\alpha, \delta, n) + B(\alpha, \delta, n),
\]
where
\[
A(\alpha, \delta, n) := \frac{C_1^2 \gamma_n^2 \|(I - P_\alpha) x_0\|^2}{\alpha},
\]
\[
B(\alpha, \delta, n) := 2\|w\|\|F'(x_0)(P_\alpha x_0 - x^{\delta,n}_\alpha)\|^\eta \|P_\alpha x_0 - x^{\delta,n}_\alpha\|^{1-\eta}.
\]

We will estimate the term $\|x^{\delta,n}_\alpha - P_\alpha x_0\|$ by dividing it into the following two cases:

(i) $A(\alpha, \delta, n) \geq B(\alpha, \delta, n)$,

(ii) $A(\alpha, \delta, n) < B(\alpha, \delta, n)$.

For case (i), by noting that $\frac{\gamma_n}{\sqrt{\alpha}} \leq 1$, from (3.7) we immediately get
\[
\|x^{\delta,n}_\alpha - P_\alpha x_0\| \leq \sqrt{2}C_1 \|(I - P_\alpha) x_0\|.
\]
(3.8)

In the following, let us consider case (ii). Clearly, (3.7) implies
\[
\|x^{\delta,n}_\alpha - P_\alpha x_0\|^{1+\eta} \leq 4\|w\|\|F'(x_0)(P_\alpha x_0 - x^{\delta,n}_\alpha)\|^\eta.
\]
(3.9)

Now we estimate $\|F'(x_0)(P_\alpha x_0 - x^{\delta,n}_\alpha)\|$. It is obvious that
\[
\|F'(x_0)(P_\alpha x_0 - x^{\delta,n}_\alpha)\| \leq \|F'(x_0)(I - P_\alpha) x_0\| + \|F'(x_0)(x_0 - x^{\delta,n}_\alpha)\|,
\]
(3.10)

\[
\|F'(x_0)(I - P_\alpha) x_0\| \leq \gamma_n \|(I - P_\alpha) x_0\|.
\]
(3.11)

By Assumption 3.1 we have
\[
\|F'(x_0)(x_0 - x^{\delta,n}_\alpha)\| \leq \|F'(x^{\delta,n}_\alpha) - y_\delta\| + \tau \|x^{\delta,n}_\alpha - x_0\| \|F'(x^{\delta,n}_\alpha) - x_0\|.
\]
(3.12)
Inserting (3.10)–(3.12) into (3.9) yields

\begin{equation}
\|x_{\alpha,n}^\delta - P_n x_0\|^{1+\eta} \leq 4\|w\| \|\gamma_n\|(I - P_n)x_0\| + 2(\delta + \|F(x_{\alpha,n}^\delta) - y_0\|)\eta.
\end{equation}

If \( \alpha \geq (c_1\delta^{\frac{1}{1+\eta}} + c_2\gamma_n)^2 \), the definition of \( \alpha \) implies \( \|F(x_{\alpha,n}^\delta) - y_0\| \leq d_2\delta \). Hence from (3.13) and the inequality \((a + b)^p \leq a^p + b^p\) for \( a, b \geq 0, 0 < p \leq 1 \), we have with \( C_2 := (4\|w\|)\frac{\eta}{1+\eta}(2 + 2d_2)^{\frac{\eta}{1+\eta}} \)

\begin{equation}
\|x_{\alpha,n}^\delta - P_n x_0\| \leq C_2 \left( (\gamma_n\|(I - P_n)x_0\|)\frac{\eta}{1+\eta} + \delta^{\frac{\eta}{1+\eta}} \right).
\end{equation}

Now we assume \( \alpha \leq (c_1\delta^{\frac{1}{1+\eta}} + c_2\gamma_n)^2 \). Substituting (3.9), (3.11), and (3.12) into (3.6) we obtain

\[
\begin{align*}
\|F(x_{\alpha,n}^\delta) - y_6\|^2 &\leq (1 + \mu)\delta^2 + 4\alpha\|w\|F'(x_0)(P_n x_0 - x_{\alpha,n}^\delta))\|P_n x_0 - x_{\alpha,n}^\delta\|^{1-\eta} \\
&\leq (1 + \mu)\delta^2 + \alpha(4\|w\|)^{\frac{n}{1+\eta}}\|F'(x_0)(P_n x_0 - x_{\alpha,n}^\delta))\|^{\frac{n}{1+\eta}} \\
&\leq (1 + \mu)\delta^2 + \alpha(4\|w\|)^{\frac{n}{1+\eta}}(2\delta + 2\|F(x_{\alpha,n}^\delta) - y_6\| + \gamma_n\|(I - P_n)x_0\|)^{\frac{2n}{1+\eta}} \\
&\leq (1 + \mu)\delta^2 + \alpha(4\|w\|)^{\frac{n}{1+\eta}}(I - P_n)x_0\|^{\frac{2n}{1+\eta}} \\
&\leq \alpha(4\|w\|)^{\frac{n}{1+\eta}}(\delta^{\frac{2n}{1+\eta}} + \|F(x_{\alpha,n}^\delta) - y_6\|^{\frac{2n}{1+\eta}}).
\end{align*}
\]

Using the implication

\[ a, b, c \geq 0, \quad 0 < p \leq 1, \quad a^2 \leq a^p b + c^2 \implies a \leq \sqrt{2}c + (2b)^{\frac{1}{2p}}, \]

we have

\begin{equation}
\|F(x_{\alpha,n}^\delta) - y_6\| \leq C_3(\delta + \sqrt{\alpha}\delta^{\frac{n}{1+\eta}} + \sqrt{\alpha}\gamma_n\|(I - P_n)x_0\|^{\frac{\eta}{1+\eta}}) + C_4\alpha^{\frac{1}{1+\eta}},
\end{equation}

where \( C_3 := \sqrt{2}\max\{\sqrt{1+\mu}, (2^{2+\eta}\|w\|)^{\frac{n}{1+\eta}}\} \), \( C_4 := 2^{\frac{1}{1+\eta}}\|w\| \).

Combining (3.15) and (3.13) it follows that

\begin{equation}
\|x_{\alpha,n}^\delta - P_n x_0\| \leq C_5 \left( (\gamma_n\|(I - P_n)x_0\|)^{\frac{\eta}{1+\eta}} + \delta^{\frac{\eta}{1+\eta}} + (\sqrt{\alpha}\delta^{\frac{n}{1+\eta}})^{\frac{\eta}{1+\eta}} + \alpha^{\frac{1}{2}} \right)
\end{equation}

with \( C_5 := (4\|w\|)^{\frac{n}{1+\eta}} \max\{(2 + 2C_3)^{\frac{n}{1+\eta}}, (2C_4)^{\frac{n}{1+\eta}}\} \).

Since \( \alpha \leq (c_1\delta^{\frac{1}{1+\eta}} + c_2\gamma_n)^2 \), we have

\begin{equation}
\alpha^{\frac{1}{2}} \leq c_1\delta^{\frac{n}{1+\eta}} + c_2\gamma_n^{\frac{n}{1+\eta}},
\end{equation}

\begin{equation}
\sqrt{\alpha}\delta^{\frac{n}{1+\eta}} \leq c_1\delta + c_2\gamma_n\delta^{\frac{n}{1+\eta}} \leq \left( c_1 + \frac{c_2\gamma_n}{1+\eta} \right)\delta + \frac{c_2}{1+\eta}\gamma_n^{1+\eta},
\end{equation}

and

\begin{equation}
\sqrt{\alpha}\gamma_n\|(I - P_n)x_0\|^{\frac{n}{1+\eta}} \leq \frac{1}{1+\eta}\alpha^{\frac{1+\eta}{2}} + \frac{\eta}{1+\eta}\gamma_n\|(I - P_n)x_0\|,
\end{equation}

respectively.
where we have used Young’s inequality: \( a, b \geq 0, \ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \) then \( ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \)

Inserting (3.17), (3.18), and (3.19) into (3.16) and setting \( C_6 := C_5 \max \{ 1 + (\frac{1}{1+r} \frac{2}{p+q}) c_0^2 + (1 + (\frac{1}{1+r} \frac{2}{p+q}) c_0^2 + (\frac{c_2}{1+r} \frac{2}{p+q}) \} \) give

\[
\| x^{k,n}_\alpha - P_n x_0 \| \leq C_6 \left( (\gamma_n \| (I - P_n) x_0 \|) \frac{\eta}{\gamma_n} + \delta \frac{\eta}{\gamma_n} + \gamma_n^\alpha \right).
\]

From (3.8), (3.14), and (3.20) it follows that no matter whether (i) or (ii) is valid, we always have the estimate

\[
\| x^{k,n}_\alpha - P_n x_0 \| \leq C \max \left\{ \| (I - P_n) x_0 \|, (\gamma_n \| (I - P_n) x_0 \|) \frac{\eta}{\gamma_n} + \delta \frac{\eta}{\gamma_n} + \gamma_n^\alpha \right\},
\]

where \( C := \max \{ \sqrt{2} C_1, C_2, C_6 \}. \)

To complete the proof, now we need only to estimate \( \| (I - P_n) x_0 \|. \) From the assumptions on \( x^* \) and \( x_0 - x^* \) we have

\[
\| (I - P_n) x_0 \| \leq \| (I - P_n) (F'(x_0)^* F'(x_0))^{\eta/2} w \| + O(\gamma_n).
\]

Since Lemma 3.7 implies

\[
\| (I - P_n) (F'(x_0)^* F'(x_0))^{\eta/2} w \| \leq \gamma_n^{\eta} \| w \|,
\]

there exists a positive constant \( D \) independent of \( n \) such that

\[
\| (I - P_n) x_0 \| \leq D \gamma_n^{\eta}.
\]

Combining (3.21) and (3.22), the proof follows from the triangle inequality

\[
\| x^{k,n}_\alpha - x_0 \| \leq \| x^{k,n}_\alpha - P_n x_0 \| + \| (I - P_n) x_0 \|. \quad \Box
\]

Remark 3.8. When \( F \) is a linear operator, Assumptions 2.2 and 3.1 hold automatically. Therefore Rule 2.1 provides an a posteriori parameter choice strategy for constrained Tikhonov regularization of ill-posed linear operator equations [17].

4. Sufficient conditions for the uniqueness of the Tikhonov regularized solution. We have seen in section 2 that Assumption 2.2 is crucial in showing the well-definedness of Rule 2.1. So it is necessary to give some sufficient conditions to guarantee its validity. We consider the weakly nonlinear inverse problems first (see [3]).

Definition 4.1. A nonlinear mapping \( F : D(F) \subset x \mapsto Y \) is called weakly nonlinear over \( D(F) \) if there exists \( \beta_M > 0 \) and \( R > 0 \) such that for every pair \( x_0, x_1 \in D(F) \), the path \( \tilde{P} \) defined by

\[
\tilde{P} : t \in [0, 1] \mapsto F((1 - t)x_0 + tx_1)
\]

belongs to \( W^{2,\infty}([0, 1]; Y) \), and for the velocity \( V(t) = \tilde{P}'(t) \) and the acceleration \( A(t) = \tilde{P}''(t) \) there holds

\[
\| V(t) \| \leq \beta_M \| x_0 - x_1 \|, \quad \| A(t) \| \leq \frac{1}{R} \| V(t) \|^2
\]

for all \( t \in [0, 1] \).

For weakly nonlinear problems we have the following lemma.
Lemma 4.2. If $F$ is weakly nonlinear over $D(F)$ and

$$
\theta := \frac{\beta_M}{R} \text{diam}(D(F)) < \pi,
$$

then the $x^*$-MNS of (1.1) is unique.

Proof. This result follows immediately from the convexity of the solution set

$$
\{x \in D(F) \mid F(x) = y_0\} := F^{-1}(y_0) \text{ of (1.1)},
$$

which is guaranteed by [3, Theorem 2.7(iii)]. □

As in [3], we define some quantity as follows:

$$
\text{Rad} := \sup_{x \in D(F)} \|x - x^*\|,
\quad R_G = \begin{cases} 
R & \text{if } 0 \leq \theta \leq \frac{\pi}{2}, \\
R \sin \theta & \text{if } \frac{\pi}{2} < \theta < \pi.
\end{cases}
$$

It is easy to show there is a constant $d > 0$ such that $R^2_G(1 + \frac{\alpha}{\beta_M})^2 - \alpha \text{Rad}^2 \geq d > 0$

for all $\alpha \in (0, \alpha)$, where $\alpha := \frac{\alpha}{2}$ and $\alpha$ is defined by

$$
\frac{\alpha}{\beta_M} \geq 1 + \frac{2}{\beta_M \text{Rad}} \quad \text{if } \beta_M \text{Rad} < 2R_G
$$

or

$$
1 + \frac{\alpha}{\beta_M} = \frac{2}{1 + \sqrt{1 - \left(\frac{2R_G}{\beta_M \text{Rad}}\right)^2}} \quad \text{if } \beta_M \text{Rad} \geq 2R_G.
$$

Theorem 4.3. If $F$ is weakly nonlinear over $D(F)$, $\theta < \pi$, and $P_n x_0 \in D(F)$

for sufficiently large $n$, then the minimization problem (2.3) has a unique solution

provided that $\delta \in (0, d)$ and $\alpha \in (0, \alpha)$ and $n$ is large enough.

Proof. Since $V_n \subset D(F)$ for all $n$, the conclusion can be proved by [3, Theorem 2.8] and its proof if $\text{dist}(F(V_n), y_6) < d$. Since $P_n x_0 \in V_n$ for $n$ large enough, we have

$$
\text{dist}(F(V_n), y_6) \leq \delta + \|F(x_0) - F(P_n x_0)\|.
$$

By using the continuity of $F$ it gives $\|F(P_n x_0) - F(x_0)\| \to 0$ as $n \to \infty$. Hence the proof is complete. □

In view of Theorem 4.3, Assumption 2.2 is valid if $\alpha = +\infty$, which depends on

the quantity $\theta$ (cf. [3]).

Now we present another sufficient condition, which can be viewed as a replenishment

when $\alpha < +\infty$ occurs. Here some assumptions, which are different from those

in Definition 4.1, are imposed on the $x^*$-MNS $x_0$ and $F$. The first one is as follows.

Assumption 4.4. There is a number $r > 4\|x_0 - x^*\|$ such that $B_r(x_0) := \{x \in X \mid \|x - x_0\| \leq r\} \subset D(F)$.

This assumption indicates that $x_0$ is an interior point of $D(F)$ and imposes some

restrictions on $x^*$: $x^*$ should be a good initial guess of the $x^*$-MNS $x_0$.

Assumption 4.5. There is a constant $K_1$ such that for every $(x, z, y) \in B_r(x_0) \times B_r(x_0) \times Y$, there exists $l(x, z, y) \in Y$ such that

$$
(F'(x)^* - F'(z)^*)y = F'(z)^*l(x, z, y),
$$

where

$$
\|l(x, z, y)\| \leq K_1 \|y\| \|x - z\|.
$$
One can refer to [23, 10] for some illustrations of this assumption. Now we show that Assumption 4.5 implies Assumption 3.1.

**Lemma 4.6.** Let Assumptions 4.4 and 4.5 hold; then for all \( x, z \in B_r(x_0) \) there holds

\[
\|F(z) - F(x) - F'(x)(z - x)\| \leq \frac{K_1}{2} \|z - x\| \|F'(x)(z - x)\|.
\]

**Proof.** Clearly we can write

\[
F(z) - F(x) - F'(x)(z - x) = \int_0^1 (F'(x + t(z - x)) - F'(x))(z - x) dt.
\]

Let \( w \in Y \) be arbitrary. By Assumption 4.5 we obtain

\[
\|((F'(x + t(z - x)) - F'(x))(z - x), w)\| = \|(z - x, (F'(x + t(z - x)) - F'(x))^* w)\|
\]

\[
= \|(z - x, F'(x)^* l(x + t(z - x), x, w)\|
\]

\[
= \|(F'(x)(z - x), l(x + t(z - x), x, w)\|
\]

\[
\leq K_1 t \|z - x\| \|F'(x)(z - x)\| \|w\|.
\]

Hence, the definition of the norm of an element in Hilbert space implies

\[
\|(F'(x + t(z - x)) - F'(x))(z - x)\| \leq K_1 t \|z - x\| \|F'(x)(z - x)\|.
\]

Substituting this into (4.5) yields the result. \( \square \)

**Remark 4.7.** In [22], Scherzer has used Assumption 4.5 together with Assumption 3.1 (i.e., (3.1), (3.2) in [22]) to prove the saturation property of Morozov’s discrepancy principle for Tikhonov regularization of nonlinear ill-posed problems in infinite-dimensional space. Now Lemma 3.2 reveals the relations between Assumptions 4.5 and 3.1. So the assumptions (3.1), (3.2) in [22] need not be stated explicitly.

**Lemma 4.8.** Let Assumptions 4.4 and 4.5 hold; then for all \( x \in B_r(x_0) \),

\[
\|((I - P_n)F'(x))^*\| \leq \gamma_n(1 + K_1 \|x_0 - x\|).
\]

**Proof.** For every \( w \in Y \),

\[
\|((I - P_n)F'(x))^* w\| \leq \|(I - P_n)F'(x_0)^* w\| + \|(I - P_n)(F'(x)^* - F'(x_0)^*) w\|
\]

\[
\leq \gamma_n \|w\| + \|(I - P_n)F'(x_0)^* l(x, x_0, w)\|
\]

\[
\leq \gamma_n(1 + K_1 \|x_0 - x\|) \|w\|. \quad \square
\]

Before stating the result on the uniqueness of Tikhonov regularized solution, we give an additional assumption [23].

**Assumption 4.9.** There is a constant \( K_2 \) such that for every \((x, z, v) \in B_r(x_0) \times B_r(x_0) \times X\) there is an element \( k(x, z, v) \in X\) such that

\[
(F'(x) - F'(z)) v = F'(z) k(x, z, v),
\]

where

\[
\|k(x, z, v)\| \leq K_2 \|v\| \|x - z\|.
\]
THEOREM 4.10. Let Assumptions 4.4, 4.5, and 4.9 hold, let $c_1, c_2$ be chosen such that $c_1 r \geq 2$, and let $\tau_0 := \frac{K_2}{c_1} + \frac{K_4}{c_1 c_2} + \frac{K_5}{c_2} < 1$. If $\|x_0 - x^*\|$ is sufficiently small such that

$$K_2 \left(1 + \frac{1}{c_2} + \frac{3K_1}{c_1 c_2} + \frac{2K_1}{c_2} \|x_0 - x^*\| \right) \|x_0 - x^*\| < 1 - \tau_0,$$

then for $\delta > 0$ sufficiently small and $n$ large enough, the minimization problem (2.3) has a unique solution $x_{\delta,n}^\alpha$ for $\alpha \geq \alpha_0$.

**Proof.** Let $\alpha \geq \alpha_0$ be fixed, and let $\tilde{x}_{\alpha}^{\delta,n}$ be another solution of the minimization problem (2.3). Since by Assumption 4.4, $P_n x_0 \in B_r(x_0)$ for $n$ large enough, we can use Lemma 4.6 and obtain, with $M_0 := \frac{K_5}{c_1} \|x_0\| + 1$,

$$\|F(P_n x_0) - y_0\| \leq M_0 \gamma_n \|(I - P_n)x_0\|. \tag{4.8}$$

By using the definition of $\tilde{x}_{\alpha}^{\delta,n}$ it follows that

$$\|F(\tilde{x}_{\alpha}^{\delta,n}) - y_0\|^2 + \alpha \|\tilde{x}_{\alpha}^{\delta,n} - x^*\|^2 \leq \|F(P_n x_0) - y_0\|^2 + \alpha \|P_n x_0 - x^*\|^2,$$

which together with (4.8) gives

$$\|y_0 - F(\tilde{x}_{\alpha}^{\delta,n})\| \leq \delta + M_0 \gamma_n \|(I - P_n)x_0\| + \sqrt{\alpha} \|P_n x_0 - x^*\|. \tag{4.9}$$

$$\|\tilde{x}_{\alpha}^{\delta,n} - x^*\| \leq \frac{\delta + M_0 \gamma_n \|(I - P_n)x_0\|}{\sqrt{\alpha}} + \|P_n x_0 - x^*\|. \tag{4.10}$$

From (4.10) one can derive for $\alpha \geq \alpha_0 := (c_1 \delta + c_2 \gamma_n)^2$ that

$$\|\tilde{x}_{\alpha}^{\delta,n} - x^*\| \leq \frac{1}{c_1} + \|x_0 - x^*\| + O(\|(I - P_n)x_0\|) \text{ as } n \to \infty. \tag{4.11}$$

Since $c_1 r \geq 2$, we have $\frac{1}{c_1} + 2\|x_0 - x^*\| < r$, which implies that $\tilde{x}_{\alpha}^{\delta,n}$ is an interior point of $D(F)$ for sufficiently large $n$. Therefore we have the following first order necessary optimality condition for $\tilde{x}_{\alpha}^{\delta,n}$:

$$P_n F'(\tilde{x}_{\alpha}^{\delta,n})(F(\tilde{x}_{\alpha}^{\delta,n}) - y_0) + \alpha (\tilde{x}_{\alpha}^{\delta,n} - P_n x^*) = 0. \tag{4.12}$$

Now by using the definition of $x_{\alpha}^{\delta,n}$ and (4.12) we can obtain

$$\|F(x_{\alpha}^{\delta,n}) - F(\tilde{x}_{\alpha}^{\delta,n})\|^2 + \alpha \|x_{\alpha}^{\delta,n} - \tilde{x}_{\alpha}^{\delta,n}\|^2 = 2(F(x_{\alpha}^{\delta,n}) - F(\tilde{x}_{\alpha}^{\delta,n}), y_0 - F(\tilde{x}_{\alpha}^{\delta,n})) + 2\alpha (x_{\alpha}^{\delta,n} - \tilde{x}_{\alpha}^{\delta,n}, x^* - \tilde{x}_{\alpha}^{\delta,n})$$

$$= 2(F(x_{\alpha}^{\delta,n}) - F(\tilde{x}_{\alpha}^{\delta,n}) - F'(\tilde{x}_{\alpha}^{\delta,n})(x_{\alpha}^{\delta,n} - \tilde{x}_{\alpha}^{\delta,n}), y_0 - F(\tilde{x}_{\alpha}^{\delta,n})).$$

With the application of Assumption 4.9 we have

$$F(x_{\alpha}^{\delta,n}) - F(\tilde{x}_{\alpha}^{\delta,n}) - F'(\tilde{x}_{\alpha}^{\delta,n})(x_{\alpha}^{\delta,n} - \tilde{x}_{\alpha}^{\delta,n}) = F'(\tilde{x}_{\alpha}^{\delta,n}) \int_0^1 \tilde{k}_t dt.$$
Hence, to prove \( \tilde{x}_{\alpha}^{h,n} = x_{\alpha}^{h,n} \), we obtain \( \| F(x_{\alpha}^{h,n}) - F(\tilde{x}_{\alpha}^{h,n}) \|^2 + \alpha \| x_{\alpha}^{h,n} - \tilde{x}_{\alpha}^{h,n} \|^2 \)

Therefore by using (4.12) again we obtain

\[
\| F(x_{\alpha}^{h,n}) - F(\tilde{x}_{\alpha}^{h,n}) \|^2 + \alpha \| x_{\alpha}^{h,n} - \tilde{x}_{\alpha}^{h,n} \|^2 \\
\leq 2 \left( \int_0^1 \tilde{k}_t \, dt, F' (\tilde{x}_{\alpha}^{h,n})^* (y_\delta - F(\tilde{x}_{\alpha}^{h,n})) \right) \\
= 2 \alpha \left( \int_0^1 \tilde{k}_t \, dt, x_{\alpha}^{h,n} - x^* \right) + 2 \left( \int_0^1 \tilde{k}_t \, dt, (I - P_n) F'(\tilde{x}_{\alpha}^{h,n})^* (y_\delta - F(\tilde{x}_{\alpha}^{h,n})) \right) \\
\leq (\alpha K_2 \| x_{\alpha}^{h,n} - x^* \| + K_2 \| (I - P_n) F'(\tilde{x}_{\alpha}^{h,n})^* (y_\delta - F(\tilde{x}_{\alpha}^{h,n})) \|) \| x_{\alpha}^{h,n} - \tilde{x}_{\alpha}^{h,n} \|^2.
\]

Hence, to prove \( \tilde{x}_{\alpha}^{h,n} = x_{\alpha}^{h,n} \), now we need only to show

\[
Q_{\alpha}^{h,n} := 1 - K_2 \| x_{\alpha}^{h,n} - x^* \| - \frac{K_2 \| (I - P_n) F'(\tilde{x}_{\alpha}^{h,n})^* (y_\delta - F(\tilde{x}_{\alpha}^{h,n})) \|}{\alpha} > 0.
\]

By virtue of (4.9) we have for \( \alpha \geq \alpha_0 \) that

\[
\| y_\delta - F(\tilde{x}_{\alpha}^{h,n}) \| \leq \frac{1}{c_1} + \| x_0 - x^* \| + O(\| (I - P_n)x_0 \|),
\]

which together with (4.10) and Lemma 4.8 gives

\[
\frac{K_2 \| (I - P_n) F'(\tilde{x}_{\alpha}^{h,n})^* (y_\delta - F(\tilde{x}_{\alpha}^{h,n})) \|}{\alpha} \leq K_2 (1 + K_1 \| x_{\alpha}^{h,n} - x_0 \|) \| y_\delta - F(\tilde{x}_{\alpha}^{h,n}) \| \\
\leq \frac{K_2}{c_2} \left( 1 + \frac{K_1}{c_1} + 2 K_1 \| x_0 - x^* \| \right) \left( \frac{1}{c_1} + \| x_0 - x^* \| \right) + O(\| (I - P_n)x_0 \|).
\]

Therefore, from (4.11) and (4.13) we obtain

\[
Q_{\alpha}^{h,n} \geq 1 - \tau_0 - K_2 \left( 1 + \frac{3 K_1}{c_1 c_2} + \frac{2 K_1}{c_2} \| x_0 - x^* \| \right) \| x_0 - x^* \| - O(\| (I - P_n)x_0 \|) \\
> 0
\]

for \( n \) large enough; hence the proof is complete. \( \square \)

5. Applications. Our results can be applied to many concrete problems. Here we present two examples, which involve the parameter estimations of partial differential equations, to illustrate the assumptions and the choice of \( X_\alpha \).

Example 5.1. Consider the problem of determining the source term \( g \) in the equation

\[
- \Delta u + q \cdot \nabla u + K(u) = g \quad \text{in} \quad \Omega,
\]

\[
u|_{\partial \Omega} = 0
\]

from the noisy data \( u_\delta \) of the state \( u_0 \) which corresponds to the sought source term \( g_0 \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n \leq 3 \) with the cone property, and the nonlinear function \( K \) has the properties that \( K \in W^{2,\infty}(\mathbb{R}) \), \( K(0) = 0 \), and \( K'(x) \geq 0 \) for all \( x \in \mathbb{R} \) and the vector valued function \( q \) satisfies \( q \in W^{2,\infty}(\Omega, \mathbb{R}^n) \) and \( \nabla \cdot q = 0 \) on \( \Omega \). This problem is a special case of an example in [3]. It has been proved that problem
(5.1), (5.2) has a unique variational solution \( u := u(g) \in H^1_0(\Omega) \) for every \( g \in L^2(\Omega) \), and there holds the following a priori estimate with some constant \( M_1 \):

\[
\|u(g_1) - u(g_2)\|_{H^1} \leq M_1 \|g_1 - g_2\|_{L^2} \quad \text{for all } g_1, g_2 \in L^2(\Omega).
\]

If we choose \( X = Y := L^2(\Omega) \) and define the nonlinear operator \( F \) by

\[
F : D(F) := L^2(\Omega) \to L^2(\Omega),
\]

\[
g \mapsto F(g) := u(g),
\]

then this inverse problem can be transformed to solve an equation like (1.1).

Clearly Assumption 4.4 holds automatically, and the weak nonlinearity of \( F \) over \( L^2(\Omega) \) has been proved in [3]. Now we consider the validity of Assumption 3.1. From Lemma 4.6, we need only to verify Assumption 4.5. It is easy to show that the operator \( F \) is weakly closed, continuous, and Fréchet differentiable with compact Fréchet derivative. The Fréchet derivative \( F'(g) \) of \( F \) at \( g \) and its adjoint are given by

\[
F'(g) h = A(g)^{-1} h \quad \text{for all } h \in L^2(\Omega),
\]

\[
F'(g)^* v = B(g)^{-1} v \quad \text{for all } v \in L^2(\Omega),
\]

where \( A(g) : H^1_0 \cap H^2(\Omega) \to L^2(\Omega) \) and \( B(g) : H^1_0 \cap H^2(\Omega) \to L^2(\Omega) \) are defined by

\[
A(g)w := -\Delta w + q \cdot \nabla w + K'(u(g))w \quad \text{and} \quad B(g)^{-1} w := -\Delta w - \text{div}(qw) + K'(u(g))w
\]

for all \( w \in H^1_0 \cap H^2(\Omega) \). From the theory of elliptic equation it follows that \( B(g)^{-1} v \in H^1_0 \cap H^2(\Omega) \) for every \( v \in L^2(\Omega) \), and there is a constant \( M_2 > 0 \) independent of \( g \) such that

\[
M_2^{-1} \|v\|_{L^2} \leq \|B(g)^{-1} v\|_{H^2} \leq M_2 \|v\|_{L^2} \quad \text{for all } v \in L^2(\Omega).
\]

Since for arbitrary \( g_1, g_2, v \in L^2(\Omega) \) there holds \( B(g_1)F'(g_1)^* v = B(g_2)F'(g_2)^* v \), we have

\[
(F'(g_2)^* - F'(g_1)^*) v = B(g_1)^{-1}(B(g_1) - B(g_2))F'(g_2)^* v
\]

\[
= B(g_1)^{-1}(K'(u(g_1)) - K'(u(g_2)))B(g_2)^{-1} v.
\]

Therefore by letting \( l(g_2, g_1, v) := (K'(u(g_1)) - K'(u(g_2)))B(g_2)^{-1} v \), we obtain

\[
(F'(g_2)^* - F'(g_1)^*) v = F'(g_1)^* l(g_2, g_1, v).
\]

Noting that

\[
\|K'(u(g_1)) - K'(u(g_2))\|_{L^2} \leq \|K''\|_{L^\infty} \|u(g_1) - u(g_2)\|_{H^1},
\]

from (5.3) and (5.6) it follows that

\[
\|l(g_2, g_1, v)\|_{L^2} \leq \|K'(u(g_1)) - K'(u(g_2))\|_{L^2} \|B(g_2)^{-1} v\|_{L^2}
\]

\[
\leq \|K''\|_{L^\infty} \|u(g_1) - u(g_2)\|_{H^1} \|B(g_2)^{-1} v\|_{H^2}
\]

\[
\leq M_1 M_2 \|K''\|_{L^\infty} \|g_1 - g_2\|_{L^2} \|v\|_{L^2}.
\]

Hence we verify Assumption 4.5 for this inverse problem. By the same procedure, we can verify Assumption 4.9.

Finally we consider the choice of the sequence of finite-dimensional subspaces. We confine ourselves to the case \( \Omega \subset \mathbb{R}^2 \) and, for simplicity, we assume that \( \Omega \) is a
convex polygon. Let $T_h$ be a regular triangulation of $\Omega$ by triangles and $K$ be the typical element of $T_h$. Here $h = \max \{ h_K \mid K \in T_h \}$ and $h_K$ denotes the diameter of $K$. We choose the vertexes of all the triangles $K \in T_h$ as the nodal points and let $X_h$ be the finite-element space corresponding to $T_h$, which consists of the continuous piecewise polynomials of degree $\leq 1$, namely, $v_h \in X_h$ if and only if $v_h \in C^0(\overline{\Omega})$ and $v_h|_K \in \mathcal{P}_1(K)$, where $\mathcal{P}_1(K)$ is the space of polynomials of degree $\leq 1$ defined on $K$. Clearly $X_h$ is a finite-dimensional subspace of $L^2(\Omega)$. We denote by $P_h$ the ortho-projector of $L^2(\Omega)$ onto $X_h$. Now $X_h$ and $P_h$ play the roles of $X_n$ and $P_n$ in section 2, respectively. From the theory of finite element it follows that there is a constant $M_3$ independent of $h$ such that

$$
\left\| (I - P_h)u \right\|_{L^2} \leq M_3 h^2 \| u \|_{H^2}, \quad u \in H^2(\Omega).
$$

(5.7)

Now, instead of $\gamma_n$, we use $\gamma_h := \| (I - P_h)F'(g_0) \|$, and from (5.5), (5.6) we can obtain the estimate

$$
\gamma_h = \sup_{\| v \|_{L^2} \leq 1} \| (I - P_h)A(g_0)^{-1}v \|_{L^2} \leq M_2 h^2 \sup_{\| v \|_{L^2} \leq 1} \| A(g_0)^{-1}v \|_{H^2} \leq M_2 M_3 h^2.
$$

**Example 5.2.** We treat the problem of estimating the diffusion coefficient $a$ in two point boundary value problem

$$
-f(x) = f \quad \text{in} \quad (0,1),
$$

(5.8)

$$
u_0(0) = 0 = u(1)
$$

(5.9)

with $f \in L^2[0,1]$ from the noisy data $u_\delta$ of the state $u_0$, $\| u_\delta - u_0 \|_{L^2} \leq \delta$. Let $a_0$ be the sought solution and $u_0 = u(a_0)$. To put this problem into our framework, we choose $X = H^1[0,1]$, $Y = L^2[0,1]$ and define the nonlinear operator $F$ by

$$
F : D(F) := \{ a \in H^1[0,1] \mid a(x) \geq \mu \text{ a.e.} \} \subset H^1[0,1], \rightarrow L^2[0,1],
$$

$$
a \rightarrow F(a) := u(a),
$$

where $u(a)$ is the unique solution of problem (5.8), (5.9) and $\mu > 0$ is a given constant. It is well known that $F$ is weakly closed, continuous, compact, and Fréchet differentiable. The Fréchet derivative and its adjoint are given by

$$
F'(a)v = A(a)^{-1}(u_x(a)v_x(a)), \quad a, v \in H^1[0,1],
$$

$$
F'(a)^*w = -B^{-1}(u_x(a)(A(a)^{-1}w_x)), \quad a \in H^1[0,1], w \in L^2[0,1],
$$

where $B : D(B) := \{ \phi \in H^2[0,1] \mid \phi'(0) = \phi'(1) = 0 \} \rightarrow L^2[0,1]$ is defined by $B\phi = -\phi_{xx} + \phi$, and $A(a) : H^2_0 \cap H^2[0,1] \rightarrow L^2[0,1]$ is defined by $A(a)u = -(au_x)_x$. It has been shown [11] that there exists an $r > 0$ and a constant $\tau$ such that

$$
\| F(a) - F(a_0) - F'(a_0)(a - a_0) \|_{L^2} \leq \tau \| a - a_0 \|_{H^1}\| F'(a_0)(a - a_0) \|_{L^2}
$$

for all $a \in D(F) \cap B_r(a_0)$.

To give the finite-dimensional approximation, let $X_n$ be the space of the linear splines on a uniform grid of $n + 1$ points in $[0,1]$ and let $P_n$ be the ortho-projector of $H^1[0,1]$ onto $X_n$. It has been shown in [20] that

$$
\gamma_n := \| (I - P_n)F'(a_0)^* \| \leq O(n^{-1}).
$$

To illustrate the applicability of our results, now we only need to verify Assumptions 2.2 and 3.2. Assumption 3.2 can be guaranteed if $a_0$ is an interior point of $D(F)$, and Assumption 2.2 is also valid since it has been pointed out in [15] that the minimization problem (2.3) for this inverse problem has a unique solution for each $\alpha > 0$. 
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