ELLIPSE SYSTEMS

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1 Introduction

Elliptic equations model the behaviour of scalar quantities \( u \), such as temperature or gravitational potential, which are in an equilibrium situation subject to certain imposed boundary conditions. In his first four lectures, John Urbas discussed linear\(^1\) elliptic equations. In his lectures on the minimal surface equation, Graham Williams discussed the minimal surface equation, a quasilinear\(^2\) elliptic equation in divergence form. Neil Trudinger and Tim Cranny will discuss fully nonlinear\(^3\) elliptic equations.

Elliptic systems model vector-valued quantities in an equilibrium situation subject to certain imposed boundary conditions. Examples are a vector-field describing the molecular orientation of a liquid crystal, and the displacement of an elastic body under an external force.

Solutions of elliptic equations are typically as smooth as the data allows (e.g. are \( C^\infty \) if the given data is \( C^1 \)). Solutions of elliptic systems typically have singularities.


2 A Model, Harmonic Map, Problem

Suppose \( \Omega \subset \mathbb{R}^n \) is an elastic membrane, “stretched” via the function \( w \) over a part of the \( n \)-dimensional sphere \( S^n \subset \mathbb{R}^{n+1} \), where \( w \) is specified on the boundary \( \partial \Omega \). As a simple approximation to the physical situation, we can regard \( w \) as a minimiser of the Dirichlet energy

\[
\frac{1}{2} \int_{\Omega} |Dw|^2, \quad (1)
\]

amongst all maps \( w: \Omega \to \mathbb{R}^{n+1} \) such that

\[ |w| = 1, \quad w|_{\partial \Omega} \text{ specified.} \]

\(^1\)The unknown function \( u \) and its first and second derivatives occur linearly. The coefficients of \( u \) and its derivatives may be nonlinear, but usually smooth, functions of the domain variables \( x_1, \ldots, x_n \).

\(^2\)Linear in the second derivatives of \( u \), but not necessarily linear in \( u \) or its first derivatives.

\(^3\)Not even linear in the second derivatives of \( u \).

\(^4\)Where \( |Dw|^2 = \sum_{i,n} |D_i w^n|^2 \). The \( \frac{1}{2} \) is merely a convenient normalisation constant.
A simpler related problem, without the constraint \(|w| = 1\), is obtained as follows. Let \(\psi: S^n \to \mathbb{R}^n\) be stereographic projection from the north pole. If \(w[\Omega]\) avoids a neighbourhood of the south pole then \(u = \psi \circ w\) solves the problem:  

Minimise 

\[
E(u) = \frac{1}{2} \int_\Omega a(u)|Du|^2, 
\]

amongst all maps \(u: \Omega \to \mathbb{R}^n\) such that  

\(u|_{\partial \Omega}\) specified.

Here \(a(u)\) is a smooth positive function (which is determined\(^5\) by \(\psi\)).

We will consider this simpler problem

**Euler Lagrange System** We now derive the Euler Lagrange system for minimisers of \(E\). Arguing formally, if \(u\) is a minimiser of \(E\) (subject to fixing the boundary values of \(u\)), then for all \(\phi \in C^1_c(\Omega; \mathbb{R}^n)\)\(^6\)

\[
0 = \left. \frac{d}{dt} \right|_{t=0} \left( \frac{1}{2} \int_\Omega a(u + t\phi)|D(u + t\phi)|^2 \right) 
= \int_\Omega a(u) D_i u^\alpha D_i \phi^\alpha + \frac{1}{2} D_\alpha a(u) \phi^\alpha |Du|^2 
= \int_\Omega a(u) Du D\phi + B(u) |Du|^2 \phi. 
\]

\(^5\)\(a(u) = |\nabla \psi|^{-2}\), where \(\nabla \psi\) is the tangential gradient, defined in a natural manner.

\(^6\)\(C^1_c(\Omega; \mathbb{R}^n)\) consists of all compactly supported \(C^1\) functions \(\phi: \Omega \to \mathbb{R}^n\).
We sum over repeated indices in the second line, and in the last line we repress the indices.

If \( u \) satisfies the above integral equation for all \( \phi \in C^1_c(\Omega; \mathbb{R}^n) \), we say that \( u \) is a \textit{weak solution} of the system

\[
D_i(a(u)D_iu^\alpha) = \frac{1}{2} D_{\alpha} a(u)|Du|^2
\]

for \( \alpha = 1, \ldots, n \).\(^7\) We abbreviate this to

\[
D(a(u)Du) = B(u)|Du|^2.
\]

If \( u \) is \( C^1 \) then being weak solution is the same as satisfying (3) in the usual sense.

Important features to note are the positivity of \( a(u) \), which makes the system \textit{elliptic},\(^8\) and the quadratic nature of \( |Du|^2 \) on the right.\(^9\)

**Solutions with Singularities** In the theory of elliptic P.D.E’s, you considered the class of \( W^{1,2}(\Omega) \) functions largely for technical reasons.\(^{10}\) It was “simple” to show the existence of weak solutions in this class, and then one considered the question of regularity of solutions. In the vector-valued setting, solutions need not be smooth, and it becomes even more natural to work in the \( W^{1,2} \) setting.

Thus we define

\[
W^{1,2}(\Omega; \mathbb{R}^N)
\]

to be the class of functions \( u: \Omega \rightarrow \mathbb{R}^N \) such that each component function belongs to \( W^{1,2}(\Omega) \).

Note that the energy \( E(u) \) is well defined for arbitrary functions \( u \in W^{1,2}(\Omega; \mathbb{R}^n) \). In particular, the function \( x/|x| \) has partial derivatives which “behave like” \( 1/|x| \), and so \( x/|x| \in W^{1,2}(B_1(0); \mathbb{R}^n) \) if \( n \geq 3 \). But note that \( x/|x| \) has a singularity at the origin.

Let \( \Omega = B_1(0) \). The function

\[
w(x) = (x/|x|, 0)
\]

maps \( B_1(0) \) “radially” onto the equator of \( S^n \subset \mathbb{R}^{n+1} \). The function \( x/|x| \), and hence \( w \), is a \( W^{1,2} \) function if \( n \geq 3 \). One can show that if \( n \geq 7 \) then \( w \)

\(^7\)The fact that the number of “dependent” variables \( u_1, \ldots, u_n \) and the number of “independent” variables \( x_1, \ldots, x_n \) are the same is just a consequence of this particular problem. It is not the case in general.

\(^8\)More generally, if instead of \( a(u)D_iu^\alpha D_\alpha \phi^\alpha \) we had \( \sum_{\alpha=1}^{n} A_{ij}^\alpha D_\alpha u^\alpha D_j \phi^\beta \), then we say the system is \textit{elliptic} if \( A_{ij}^\alpha \xi_i^\alpha \xi_j^\beta \geq \lambda |\xi|^2 \) for some constant \( \lambda > 0 \) and all \( \xi \in \mathbb{R}^{n+N} \). In many physical problems it is important to have a weaker form of ellipticity, namely \( A_{ij}^\alpha \xi_i^\alpha \eta_j^\beta \geq \lambda |\xi|^2 \) for some constant \( \lambda > 0 \) and all \( \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^N \).

\(^9\)An exponent less than two is “easier” to handle; an exponent greater than two is more difficult. But two is the “natural” exponent for many problems, as is the case here.

\(^{10}\)See also my lectures on measure theory.
has least energy amongst all functions mapping $B_1(0)$ onto the unit sphere and having the same boundary values as $w$. Similarly, if $n \geq 7$, $u = \psi \circ w$ minimises $E(u)$ in (2) amongst all maps having the same boundary values. In particular, $u$ satisfies the system of equations (2), i.e. (3). If $3 \leq n < 7$ then $u$ is no longer a minimiser, but it still satisfies the system (3). If $n = 2$ it turns out that solutions of (3), and in particular minimisers of $E(u)$, are smooth.

We have just noted that a solution of (3) may have a singularity. If $u(x)$ is a solution, then clearly so is $u(x-a)$ for any $a \in \mathbb{R}^n$. Since a sum of solutions is also solution, we obtain solutions with any finite number of singularities.

In general, a solution of (3) is said to be stationary, or an equilibrium solution, for the energy $E$. Thus minimisers are solutions of the Euler Lagrange system, but not necessarily conversely. Since the energy is the Dirichlet Energy (for $w$, and also for $u$ if we choose the appropriate metric), stationary functions for this particular problem are called harmonic.

### 3 A Simpler Model Problem

Our intention is to provide a reasonably complete analysis for solutions of systems of the form (3), but with zero right side. Thus we consider systems of the form

$$D(a(u)Du) = 0,$$

which may or may not be an Euler Lagrange system.

Systems of the type (4) were the first type of nonlinear elliptic system to be analysed. (In the next Section we briefly remark on linear elliptic systems.) If the right side is nonzero, as in (3), then the problem is considerably more complicated. In particular, minimisers will have “nicer” properties than merely stationary solutions. See [G] for more details.

We remark (4) may also have singular solutions. For example, $x/|x|$ is a weak solution of (4) if

$$A^{ij}_\alpha(u) = \delta_{ij}\delta_{\alpha\beta} + \left(\delta_{ij} + \frac{4}{n-2}\frac{u^j u^\beta}{1 + |u|^2}\right) \left(\delta_{\alpha i} + \frac{4}{n-2}\frac{u^i u^\beta}{1 + |u|^2}\right),$$

where $n = N \geq 3$. See [G, p. 57]. Note that the $A$ are $C^\infty$, in fact analytic. The system of equations in this case is an Euler Lagrange system for a certain energy functional. Moreover, for sufficiently large $n$, $x/|x|$ is the (unique) minimiser of this particular energy functional.

\footnote{The analogy is that a function $E$ defined on $\mathbb{R}^k$ can have equilibrium points which are not minimisers.}
4 Linear Elliptic Systems

For completeness, we briefly discuss linear elliptic systems. Suppose $\Omega \subset \mathbb{R}^n$ and

$$u : \Omega \to \mathbb{R}^N.$$  

We say $u$ satisfies a linear elliptic system in integral form if

$$\int_{\Omega} \sum_{i=1}^{n} A_{ij}^{\alpha\beta}(x) D_i u^\alpha D_j \phi^\beta = 0$$

for all $\phi \in C_0^1(\Omega; \mathbb{R}^N)$. The $A_{ij}^{\alpha\beta}(x)$ are required to satisfy the ellipticity condition

$$A_{ij}^{\alpha\beta}(x) \xi^\alpha \xi^\beta \geq \lambda |\xi|^2$$

for some constant $\lambda > 0$ and all $\xi \in \mathbb{R}^{nN}$. Note that the coefficients $A_{ij}^{\alpha\beta}(x)$ depend only on $x$ and not on $u$. The summation sign is usually dropped, and we even suppress all indices and write

$$\int_{\Omega} A(x) D u \phi = 0.$$  

The ellipticity condition is then written

$$A_{ij} \xi^i \xi^j \geq \lambda |\xi|^2.$$  

Assuming the $A_{ij}^{\alpha\beta}(x)$ are bounded, it is straightforward to show by an approximation argument that we may take $\phi \in W^{1,2}_0(\Omega; \mathbb{R}^N)$ in (5). Recall that $W^{1,2}_0(\Omega; \mathbb{R}^N)$ consists of those $W^{1,2}(\Omega; \mathbb{R}^N)$ functions which are zero on $\partial \Omega$ in a natural way.

Motivated by integration by parts, we usually write the system as

$$D_j \left( A_{ij}^{\alpha\beta}(x) D_i u^\alpha \right) = 0$$

for $\beta = 1, \ldots, N$. This abbreviates to

$$D(A(x) Du) = 0.$$  

If $u \in W^{1,2}(\Omega; \mathbb{R}^N)$ satisfies (5) (i.e. (6)) we say $u$ is a weak solution of the system (7) (i.e. (8)). If $A(x)$ and $u$ are $C^1$, then it follows from integration by parts that a weak solution is a solution in the classical pointwise sense.

The theory of linear elliptic systems is similar to the theory of linear equations. In particular, one obtains an analogous Schauder theory (for $C^{k,\alpha}$ solutions) and Sobolev theory (for $W^{k,2}$ solutions). The main difference is that if the functions $A_{ij}^{\alpha\beta}(x)$ are merely bounded, then there exist solutions with singularities. This is not the case for a single equation. See [G, p. 54].

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12Although the details can be considerably more complicated, at least when one considers other than second-order elliptic systems.
5 Regularity Results, Summary

We now consider the question of partial regularity (i.e. smoothness) of solutions of (4).

More precisely, suppose \( u \in W^{1,2}(\Omega; \mathbb{R}^N) \) and
\[
D(A(u)Du) = 0, \tag{9}
\]
where
1. \(|A(z)| \leq M \ldots \forall z \in \mathbb{R}^N\),
2. \(A\xi \xi \geq \lambda|\xi|^2 \ldots \forall \xi \in \mathbb{R}^{nN}, \) where \( \lambda > 0 \),
3. \( A \in C^0(\mathbb{R}^N) \) is uniformly continuous.

More precisely, we are using an abbreviated notation as in the previous section. By \( u \) satisfying the system (9) we mean that the corresponding integral equations (as in (5) or (6) but with \( A_{ij}^{\alpha\beta}(u) \) instead of \( A_{ij}^{\alpha\beta}(x) \), are satisfied for all test functions \( \phi \in W^{1,2}_0(\Omega; \mathbb{R}^N) \).

We will see that \( u \in C^0(\Omega_0) \) for some open \( \Omega_0 \subset \Omega \), where \( \Omega \setminus \Omega_0 \) is a set of dimension \( \leq n - 2 \) (in a sense to be explained later). If \( A \) is smoother than \( C^0 \), then \( u \) is correspondingly smoother in \( \Omega_0 \). In particular, if \( A \) is \( C^\infty \) then \( u \in C^\infty(\Omega_0) \).

More can be proved. It is only necessary that \( A \) be continuous, not uniformly continuous. Moreover, \( \Omega \setminus \Omega_0 \) is in fact a set of dimension \( p \) for some \( p < n - 2 \), and is empty if \( n = 2 \).

The idea of the proof is that if the graph of a solution \( u \) is sufficiently “flat” in the \( L^2 \) sense near \( x_0 \in \Omega \), then in fact \( u \) is smooth in a neighbourhood of \( x_0 \). We will see that the “flatness” condition holds at all except a “small” set of points.

The key technical point in the proof is to consider the quantity
\[
U(x_0, R) = \bar{\int}_{B_R(x_0)} |u - (u)_{x_0,R}|^2,
\]
for \( B_R(x_0) \subset \Omega \). This measures the \( L^2 \) mean oscillation of \( u \) in \( B_R(x_0) \). Here \( \bar{\int} \) denotes the average, and is obtained by dividing by the volume \( \omega_n R^n \) of \( B_R(x_0) \). The quantity \((u)_{x_0,R}\) is the average of \( u \) in \( B_R(x_0) \) and is given by
\[
(u)_{x_0,R} = \bar{\int}_{B_R(x_0)} u.
\]

We will see that if \( U(x_0, R) \) is sufficiently small then in fact \( U(x_0, r) \) approaches zero like a power of \( r \). From this, one deduces the Hölder continuity of \( u \) in a neighbourhood of \( x_0 \). One also shows that except for a set of \( x_0 \) of dimension \( n - 2 \), \( U(x_0, R) \) is indeed small for some \( R = R(x_0) \).
6 Some Important Preliminaries

We discuss a number of fundamental results that are used in the proof of partial regularity.

6.1 Integral Characterisation of Hölder Continuity

**Theorem**  If \( \Omega \) has Lipschitz boundary, then

\[
\begin{align*}
\forall \Omega, \quad u & \in C^{0, \alpha}(\overline{\Omega})^{13} \iff \int_{B_R(x_0)} |u - (u)_{x_0, R}|^2 \leq cR^{n+2\alpha} \\
\end{align*}
\]

for all \( B_R(x_0) \subset \Omega \), and some constant \( c \).

**Remark**  More precisely, if the integral condition holds, then the precise representative \( u^* \) of \( u \), defined by

\[
\begin{align*}
u^*(x_0) &= \lim_{R \to 0} \int_{B_R(x_0)} u,
\end{align*}
\]

satisfies \( u^* \in C^{0, \alpha}(\overline{\Omega}) \). Since \( u^* = u \) a.e., and changing \( u \) on a set of measure zero does not change the integral, this is the best one can expect.

**Proof:** If \( u \) is Hölder continuous, the integral inequality is straightforward. For the other direction, one works from the definition of \( u^* \), see [G; Ch. III,1].

6.2 Energy (or Caccioppoli) Inequality

**Theorem**  If \( u \) is a solution of (9) and \( B_R(x_0) \subset \Omega \), then

\[
\begin{align*}
\int_{B_{R/2}(x_0)} |Du|^2 \leq \frac{c}{R^2} \int_{B_R(x_0)} |u|^2.
\end{align*}
\]

**Philosophy**  The important point here is that we are bounding the \( L^2 \) norm of the derivative of \( u \) in some ball in terms of the \( L^2 \) norm of \( u \) in a larger ball. Such an estimate is not true for arbitrary functions \( u \), but it is typical of solutions of elliptic equations or systems that we can often bound integrals of higher derivatives in terms of integrals of lower derivatives, usually over a slightly larger set.

\[^{13}\] Suppose \( 0 < \alpha \leq 1 \). Then

\[
\begin{align*}
\forall \Omega, \quad u & \in C^{0, \alpha}(\overline{\Omega}) \iff |u(x) - u(y)| \leq M|x - y|^\alpha
\end{align*}
\]

for some \( M > 0 \) and all \( x, y \in \Omega \). Note that if \( \alpha > 1 \) then the derivative of \( u \) would be everywhere zero, and so \( u \) is constant!
Conversely, bounding integrals of lower derivatives in terms of integrals of higher derivatives is something we can do for arbitrary functions, by means of Sobolev or Poincaré inequalities. In particular, note the Poincaré inequality

\[
\int_{B_R(x_0)} |u - (u)_{x_0,R}|^2 \leq cR^2 \int_{B_R(x_0)} |Du|^2.
\]

**Proof:** Since the proof is one of the simplest examples of a test function argument, we sketch it here.

As is usual in P.D.E.’s, in the following, \( c \) denotes a constant which may change from line to line. But it will only depend on the dimension and constants such as \( M \) and \( \lambda \) which appear at the beginning of Section 5.

Let \( \phi = \eta^2 u \), where \( \eta \) is smooth, \( \eta \geq 0 \), \( \eta = 1 \) on \( B_{R/2}(x_0) \), \( \eta = 0 \) outside \( B_R(x_0) \), and \( |D\eta| \leq 3/R \). Substituting this in the integral form of (9),

\[
0 = \int A Du D\phi = \int A Du (\eta^2 Du + 2\eta Du D\eta)
\]

Hence

\[
\int \eta^2 A Du Du = -\int 2A \eta D\eta u Du.
\]

Hence

\[
\lambda \int \eta^2 |Du|^2 \leq c \int \eta |D\eta| |u| |Du|
\]

\[
\leq \epsilon \int \eta^2 |Du|^2 + c(\epsilon) \int |D\eta|^2 |u|^2,
\]

by Young’s inequality. Taking \( \epsilon = \lambda/2 \),

\[
\int_{B_{R/2}(x_0)} |Du|^2 \leq \frac{c}{R^2} \int_{B_R(x_0)} |u|^2,
\]

as required.

6.3 A Decay estimate for Solutions of Constant Coefficient Systems

**Theorem** Suppose \( u \) satisfies (9) where the \( A \) are constant and \( \Omega = B_1(0) \) for simplicity of notation. Then for \( 0 < r \leq 1 \),

\[
U(0, r) \leq cr^2 U(0, 1)
\]

for some constant \( c \).

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14See the last Section of my measure theory notes.
Proof: We may assume \( r \leq 1/4 \), since if \( r > 1/4 \) we can take \( c \geq 4^{n+2} \).

Then
\[
\begin{align*}
r^{-2}U(0, r) &= \omega_n^{-1}r^{-2-n} \int_{B_r(0)} |u - (u)_r|^2 \\
&\leq cr^{-n} \int_{B_r(0)} |Du|^2 \quad \text{Poincaré's inequality} \\
&\leq c \sup_{B_r(0)} |Du|^2 \\
&\leq c \int_{B_{\frac{1}{2}}(0)} |Du|^2 \quad \text{a standard elliptic estimate} \\
&\leq c \int_{B_1(0)} |u - (u)_1|^2 \quad \text{by Caccioppoli's inequality}
\end{align*}
\]

The “standard elliptic estimate” above is that one can typically bound higher norms (here \( L^\infty \)) of solutions and their derivatives in terms of lower norms (here \( L^2 \)) over a larger domain. “Caccioppoli’s inequality” is applied to the solution \( u - (u)_1 \).

This gives the result. \( \blacksquare \)

7 Outline of Proof of Partial Regularity

Lemma Suppose \( u \) is a solution of (9). Then there exist constants \( \epsilon > 0 \) and \( \tau \in (0, 1) \) such that
\[
U(x_0, r) < \epsilon
\]
implies
\[
U(x_0, \tau r) < \frac{1}{2}U(x_0, r).
\]

Proof: Suppose \( \tau \in (0, 1) \) and the conclusion of the lemma is false for each \( \epsilon > 0 \) (the intention is to obtain a contradiction if \( \tau \) is sufficiently small).

Then there exist balls \( B_{r_k}(x_k) \subset \Omega \) such that
\[
U(x_k, r_k) = \lambda_k^2 \rightarrow 0 \quad \text{(10)}
\]
but
\[
U(x_k, \tau r_k) \geq \frac{1}{2} \lambda_k^2. \quad \text{(11)}
\]

Rescale to the unit ball by setting
\[
v_k(z) = \frac{u(x_k + r_kz) - a_k}{\lambda_k}
\]
for \( z \in B_1(0) \), where \( a_k = (u)_{x_k, r_k} \).
Then, using the integral form of (9),

$$\int A(\lambda_k v_k + a_k) Dv_k \, D\phi = 0$$

for all $\phi \in W^{1,2}_0(B_1(0); \mathbb{R}^N)$.

Moreover, from (10) and (11),

$$\begin{align*}
(v_k)_1 &= 0 \\
\int_{B_1} |v_k|^2 &= 1 \\
\int_{B_r} |v_k - (v_k)_r|^2 &\geq 1/2.
\end{align*}$$

From Caccioppoli’s inequality, $\int_{B_1} |Dv_k|^2$ is bounded independently of $k$. This allows one to pass to a subsequence of the $v_k$ which converges weakly in $W^{1,2}$, strongly in $L^2$, and pointwise a.e., to some function $v$. Moreover, $a_k \to a$, say. From this it is not difficult to show that $v$ will satisfy the “limit” equation

$$\int A(a) \, Dv \, D\phi = 0$$

for all $\phi \in W^{1,2}_0(B_1(0); \mathbb{R}^N)$.

From the decay estimate for constant coefficient equations,

$$\int_{B_r} |v - (v)_r|^2 \leq c r^2 \int_{B_1} |v - (v)_1|^2 = c r^2.$$  

On the other hand,

$$\int_{B_r} |v - (v)_r|^2 \geq \frac{1}{2},$$

using continuity of the $L^2$ norm under $L^2$ convergence in both lines.

This is a contradiction for sufficiently small $\tau$.

The Lemma is now used as follows. The inequality $U(x_0, r) < \epsilon$ must hold in an open subset of $\Omega$. Moreover, it can be iterated to show

$$U(x_0, \tau^j r) < \left( \frac{1}{2} \right)^j U(x_0, r).$$

This, together with the integral characterisation of Hölder continuity and elementary arguments, shows $u \in C^{0,\alpha}(\Omega_0)$ for some $\alpha$. Higher smoothness follows by fairly standard iteration techniques (although $C^{0,\alpha}$ to $C^{1,\alpha}$ is not quite so standard).

The estimate on the dimension of $\Omega \setminus \Omega_0$ follows from noting

$$U(x_0, r) \leq cr^{2-n} \int_{B_r(x_0)} |Du|^2$$

by Poincaré’s inequality, and the fact (using a Vitali covering argument) that the right side approaches zero except on a set $E$ with $\mathcal{H}^{n-2}(E) = 0$. 