$L^2$ Estimates for Approximations to Minimal Surfaces

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Abstract In a previous paper the authors developed a new algorithm for finding discrete approximations to (possibly unstable) disc-like minimal surfaces. Optimal convergence rates in the $H^1$ norm were obtained. Here we recall the key ideas and prove optimal $L^2$ convergence rates.

1 Introduction

Suppose $\Gamma$ is a smooth curve in $\mathbb{R}^n$. We are interested in the problem of obtaining discrete approximations to (possibly unstable) disc-like minimal surfaces spanning $\Gamma$.

Let $D$ be the unit disc in $\mathbb{R}^2$ and let

$$\mathcal{C} = \{ u : D \to \mathbb{R}^n \mid \Delta u = 0, \ u|_{\partial D} \text{ is a monotone parametrisation of } \Gamma \}.$$ 

Denote the Dirichlet energy by

$$\mathcal{D}(u) = \frac{1}{2} \int_D |Du|^2.$$ 

It is well-known that if $u$ is a critical point for $\mathcal{D}$ restricted to $\mathcal{C}$ then $u[D]$ is a minimal surface spanning $\Gamma$. Moreover, $u$ is then conformal. Conversely, any minimal surface spanning $\Gamma$ can be obtained in this manner. We will make this the basis of the numerical algorithm.

Harmonic maps are uniquely determined by their boundary values. Thus if $\Gamma = \gamma[S^1]$ is given by the parametrisation

$$\gamma : S^1 \to \Gamma,$$

\footnotetext[1]{Lecture delivered by Hutchinson.}
then instead of $C$ one can equivalently consider the class

$$\overline{\mathcal{M}} = \{ s : \partial D \to S^1 \mid s \text{ is monotone} \}.$$ 

For $s \in \overline{\mathcal{M}}$ the corresponding harmonic map spanning $\Gamma$ is

$$u = \Phi(\gamma \circ s)$$

where $\Phi$ denotes harmonic extension.

The energy functional on $\mathcal{M}$ is defined by

$$E(s) = \mathcal{D}(\Phi(\gamma \circ s)) = \frac{1}{2} \int_{\partial D} |D(\Phi(\gamma \circ s))|^2$$

$$= \frac{1}{16\pi} \int_{\partial D} \int_{\partial D} \frac{|(\gamma \circ s)(\phi) - (\gamma \circ s)(\phi')|^2}{\sin^2 \left( \frac{\phi - \phi'}{2} \right)} d\phi d\phi'.$$

The last integral is known as the Douglas Integral, c.f. [N2; §§310–311].

There is a three parameter family of conformal maps from the unit disc $D$ parametrising a given simply connected smooth surface. The usual normalisation is to specify the image of three points on $\partial D$. Here it is theoretically more convenient, and numerically more stable, to consider maps $s$ such that

$$\int_0^{2\pi} (s(\theta) - \theta) \, d\theta = 0,$$

$$\int_0^{2\pi} (s(\theta) - \theta) \cos \theta \, d\theta = 0,$$

$$\int_0^{2\pi} (s(\theta) - \theta) \sin \theta \, d\theta = 0.$$

Thus we define

$$\mathcal{M} = \{ s \in C^0(\partial D, S^1) : s \text{ is monotone, } s \text{ satisfies (2), } E(s) < \infty \}.$$ 

See [St; Section II.2].

If $s$ is critical for $E$ restricted to $\mathcal{M}$ we say $s$ is stationary and the corresponding harmonic map $u = \Phi(\gamma \circ s)$ is called the minimal surface corresponding to $s$.

Given a fixed grid $\phi_j$, $j = 1, \ldots, N$, on $\partial D$, with typical grid-size $h$ (i.e. the distance between successive points is controlled above and below by multiples of $h$), the discrete analogue of (3) is

$$M_h = \{ s_h = (s_1, \ldots, s_N) : s_h \text{ is a monotone sequence of points on } S^1 \}.$$ 

It is convenient to identify both $\partial D$ and $S^1$ with the interval $[0, 2\pi)$, and we will often do this. It is also convenient to identify $s_h \in M_h$ with the corresponding piecewise linear map $s_h : \partial D \to S^1$ for which $s_h(\phi_j) = s_j$ ("piecewise linear" with respect to arc length, i.e. angle variable).
The discrete energy functional is simply the restriction of the energy functional $E$ to $\mathcal{M}_h$ and is defined by

$$E_h(s_h) = \frac{1}{2} \int_D |D(\Phi(\gamma \circ s_h))|^2.$$ 

If $s_h$ is critical for $E_h$ restricted to $\mathcal{M}_h$ we say $s_h$ is stationary and the corresponding harmonic map $u_h = \Phi(\gamma \circ s_h)$ is called the semi-discrete minimal surface corresponding to $s_h$.

Numerically, one approximates the Douglas functional in order to compute $E_h$, and one computes discrete harmonic approximations to $u_h$ with boundary data $u_h(\phi_j) = \gamma(s_j)$. The numerical algorithm for finding critical points for $E_h$ restricted to $\mathcal{M}_h$ is:

**Algorithm** Given a grid $\phi_j$, $j = 1, \ldots, n$, on $\partial D$, initial values $s_h = (s_1, \ldots, s_n)$ and parametrisation $\gamma$:

1. Compute the derivative of the approximate energy $E_h'(s_h)$.
2. If $|E_h'(s_h)|/|s_h| \leq \epsilon$ then stop.
3. Compute the second derivative of the approximate energy $E_h''(s_h)$.
4. Solve the linear system $E_h''(s_h)d = -E_h'(s_h)$, update the solution $s_h := s_h + d$ and go to step 1.

Here $|s_h|$ is the $l^2$-norm of $s_h$ and $\epsilon$ is a given tolerance.

Suppose $s_0$ is stationary and $u_0$ is the corresponding minimal surface. In [DH] we showed, as $h \to 0$, the existence of a sequence of discrete stationary $s_h$ and corresponding semi-discrete minimal surfaces $u_h$, such that

$$\|s_h - s_0\|_{H^{1/2}(\partial D)} \leq ch^{3/2},$$

$$\|u_h - u_0\|_{H^1(D)} \leq ch^{3/2}.$$  

(4)  

(5)

In this paper we show that

$$\|s_h - s_0\|_{H^{-1/2}(\partial D)} \leq ch^{5/2},$$

$$\|u_h - u_0\|_{L^2(D)} \leq ch^{5/2}.$$  

(6)  

(7)

The proof of (6) and (7) will use a variant of the Aubin-Nitsche technique.

The computational significance of our results is as follows. Suppose $s_h \to s_0$ in $C^0 \cap H^{1/2}(\partial D)$ as $h \to 0$, where the $s_h$ are discrete stationary points. Then it is straightforward to prove $s_0$ is stationary, see [DH; Theorem 6.4]. Moreover, if $s_0$ is monotone and non-degenerate and $|\ln h|^{3/2}\|s_h - s_0\|_{C^0 \cap H^{1/2}(\partial D)} \to 0$, then the convergence rates of (4)–(7) will apply. By non-degeneracy we mean that there are no non-zero Jacobi fields for $s_0$. If $s_0$ has no branch points (and this can be determined by observation of the approximating sequence) then non-degeneracy is generically true, see [BT].

The theoretically predicted rates of convergence typically appear after a small number of iterations, and provide strong evidence that the sequence of discrete stationary points (or
corresponding sequence of semi-discrete minimal surfaces) is indeed converging towards a non-degenerate stationary point (or corresponding non-degenerate minimal surface).

For related results and further references, see [DH]. This research has been partially supported by the Australian Research Council.

2 Background Material

We recall the main ideas and results from [DH]. We follow the approach (in the non-discrete setting) of [St1, St2].

Assume $\gamma$ is $C^r$ where $r \geq 5$.

It is necessary to enlarge $\mathcal{M}$ as it is not linear, or even affine. We do this by first selecting a fixed member of $\mathcal{M}$, which for convenience we take to be the identity map

$$\text{id}: \partial D \to S^1, \quad \text{id}(\phi) = \phi.$$ 

We will consider maps

$$s = \text{id} + \sigma$$

such that $\sigma \in H^{1/2}(\partial D; \mathbb{R})$ and

$$\int_0^{2\pi} \sigma(\phi) \, d\phi = 0, \quad \int_0^{2\pi} \sigma(\phi) \cos \phi \, d\phi = 0, \quad \int_0^{2\pi} \sigma(\phi) \sin \phi \, d\phi = 0,$$

(8)

c.f. (2). Thus we define

$$H = H^{1/2}(\partial D; \mathbb{R}) \cap \{ \xi : (8) \text{ is satisfied with } \sigma \text{ replaced by } \xi \},$$

$$\mathcal{H} = \text{id} + H.$$ 

The $H^{1/2}$ semi inner product is defined by

$$(\xi, \eta)_{H^{1/2}} = \int_{\partial D} \int_{\partial D} \frac{(\xi(\phi) - \xi(\phi')) \cdot (\eta(\phi) - \eta(\phi'))}{|\phi - \phi'|^2} \, d\phi \, d\phi'$$

(9)

The corresponding seminorm $| \cdot |_{H^{1/2}}$ is in fact a norm on $H$, by the first equality in (8) and the Poincaré inequality.

The definition of $E$ is extended to $\mathcal{H}$ by (1).

Unfortunately $E$ is not $C^1$ on $\mathcal{H}$, and so for this reason we define

$$T = H \cap C^0(\partial D; \mathbb{R})$$

$$\mathcal{T} = \text{id} + T$$

$$\| \xi \| = | \xi |_{H^{1/2}(\partial D)} + \| \xi \|_{C^0}.$$ 

In particular,

$$\mathcal{M} \subset \mathcal{T} \subset \mathcal{H}, \quad \mathcal{T} \subset H.$$
If $s \in \mathcal{H}$ the corresponding harmonic map spanning $\Gamma$ is denoted by

$$\tau = \Phi(\gamma \circ s).$$

If $s \in \mathcal{M}$ is fixed and $\xi \in H$, then the corresponding vector field along $\gamma \circ s$ will be denoted by

$$\bar{\xi} = (\gamma'\circ s) \xi = \gamma'(s) \xi.$$

The harmonic extension of $\bar{\xi}$, which is an harmonic vector field over $\gamma[D]$, will also be denoted by $\bar{\xi}$.

Then one has

**Proposition 2.1** The energy functional $E : \mathcal{T} \to \mathbb{R}$ is $C^{r-1}$. Let $s = \text{id} + \sigma$. Then

$$dE(s)(\xi) = \int_D D\tau \cdot D\bar{\xi},$$

$$d^2E(s)(\xi_1, \xi_2) = \int_D D\bar{\xi}_1 \cdot D\bar{\xi}_2 + \int_{\partial D} \frac{\partial \tau}{\partial \nu} \cdot \gamma''(s) \xi_1 \xi_2.$$

Also

$$E(s) \leq c(||\gamma||_{C^1}) \left(1 + |\sigma|_{H^{1/2}}^2\right),$$

$$|dE(s)(\xi_1, \ldots, \xi_j)| \leq c(||\gamma||_{C^{j+1}}) (1 + |\sigma|_{H^{1/2}}) ||\xi_1||_T \cdots ||\xi_j||_T \quad 1 \leq j \leq r - 1.$$

If $\sigma \in C^0(\partial D; S^1)$ then

$$|\sigma|_{H^{1/2}}^2 \leq c(E(s) + 1),$$

where $c$ depends on $||\gamma^{-1}||_{C^1}$ and the modulus of continuity of $\sigma$.

The expressions for $dE$ and $d^2E$ are straightforward computations. For the remainder, see [St, Section II] and [DH, Proposition 4.3].

The following will be applied in case $s$ is stationary, see Proposition 2.5.

**Proposition 2.2** If $s$ is $C^2$ then $dE(s)$ and $d^2E(s)$ extend to bounded linear and bilinear operators respectively on $H$, and

$$|dE(s)(\xi)| \leq c(||\gamma||_{C^2}, ||s||_{C^1}) ||\xi||_{H^{1/2}},$$

$$|d^2E(s)(\xi_1, \xi_2)| \leq c(||\gamma||_{C^2}, ||s||_{C^1}) ||\xi_1||_{H^{1/2}} ||\xi_2||_{H^{1/2}}.$$

See [St, Section II] and [DH, Proposition 4.4].

**Definition 2.3** The function $s \in \mathcal{M}$ is a stationary point for $E$ if

$$\left.\frac{d}{dt}\right|_{t=0} E(s + t \xi) \geq 0$$

whenever $s + \xi \in \mathcal{M}$. If $s$ is stationary for $E$ then we say that the harmonic map $u = \Phi(\gamma \circ s)$ is a minimal surface or that $u$ is a solution of the Plateau Problem.
One has:

**Proposition 2.4** The function \( s \in \mathcal{M} \) is stationary for \( E \) with respect to monotone variations iff \( s \) is stationary in the sense of \( T \), i.e. iff

\[
dE(s)(\xi) = 0 \quad \forall \xi \in T. \tag{10}
\]

The regularity results of [Hil], [Ja], [N1], [He] imply the regularity of stationary \( s \).

**Proposition 2.5** If \( \gamma \) is \( C^{r,\alpha} \) where \( r \geq 1 \) and \( 0 < \alpha < 1 \), and \( s \in \mathcal{M} \) is stationary for \( E \), then \( s \) is \( C^{r,\alpha} \) and \( ||s||_{C^{r,\alpha}} \) is controlled by \( ||\gamma||_{C^{r,\alpha}} \).

**Definition 2.6** Suppose \( s \) is a stationary point for \( E \). The self-adjoint bounded linear map

\[
\nabla^2 E(s) : H \to H
\]

is defined by

\[
\left( \nabla^2 E(s)(\xi), \eta \right)_{H^{1/2}(\partial D)} = d^2 E(s)(\xi, \eta) \quad \forall \xi, \eta \in H.
\]

**Definition 2.7** A stationary \( s \in \mathcal{M} \) is non-degenerate if \( d^2 E(s) : H \times H \to IR \) is a non-degenerate bilinear operator. Equivalently, \( \nabla^2 E(s) \) is invertible with bounded inverse.

We next define discrete approximations to \( H \) as follows. For each \( h > 0 \) let \( G_h \) be any grid on \( \partial D \cong [0, 2\pi) \) such that

\[
c^{-1}h < |I| < ch \quad \forall I \text{ an interval of } G_h,
\]

where \( |I| \) is the length of \( I \) and \( c \) is independent of \( h \). If \( I \) is an interval, denote by

\[
P_1(I)
\]

the space of first order polynomials (in the arc length variable) defined over \( I \).

**Definition 2.8** Let

\[
H_h = \{ \xi_h \in C^0(\partial D, IR) : \xi_h \in P_1(I) \forall I \in G_h, \ (8) \text{ is satisfied} \}.
\]

Then

\[
H_h \subset T \subset H.
\]

Let

\[
\mathcal{H}_h = \{ \text{id} \} + H_h \subset T \subset \mathcal{H}
\]

be the corresponding finite dimensional affine space of continuous piecewise affine maps (with respect to arc length) from \( \partial D \) to \( S^1 \).
Definition 2.9 A function \( s_h \in \mathcal{H}_h \) is called a \textit{semi-discrete stationary point} for \( E \) if
\[
dE_h(s_h)(\xi_h) = 0 \quad \forall \xi_h \in H_h.
\]
The associated function \( u_h = \Phi(\gamma \circ s_h) \) is called a \textit{semi-discrete minimal surface}.

The main result from [DH] is:

Theorem 2.10 (Energy Estimate) Assume \( r \geq 5 \). Let \( s_0 \in \mathcal{M} \) be a non-degenerate stationary point for \( E \) with associated minimal surface \( u_0 = \Phi(\gamma \circ s_0) \).

Then there exist constants \( h_0, \epsilon_0 \) and \( c \), depending on \( s_0 \), such that if \( 0 < h \leq h_0 \) then there is a unique semi-discrete stationary point \( s_h \in \mathcal{H}_h \) such that
\[
|s_0 - s_h|_{H^{1/2}(\partial \Omega)} \leq \epsilon_0 |\ln h|^{-3/2}.
\]
Moreover,
\[
|s_h - s_0|_{H^{1/2}(\partial \Omega)} \leq ch^{3/2} \quad \text{and} \quad ||s_h - s_0||_{C^0(\partial \Omega)} \leq ch^{3/2} |\ln h|^{1/2}.
\]
Finally, if \( u_h = \Phi(\gamma \circ s_h) \) is the corresponding semi-discrete minimal surface, then
\[
||u_h - u_0||_{H^{1/2}(\partial \Omega)} \leq ch^{3/2} \quad \text{and} \quad ||u_h - u_0||_{C^0(\partial \Omega)} \leq ch^{3/2} |\ln h|^{1/2}.
\]

See [DH, Theorem 6.3]. The computational significance of this is a consequence of the following result:

Theorem 2.11 Suppose \( s_h \) is a sequence of semi-discrete stationary points and \( ||s_h - s_0||_T \to 0 \) as \( h \to 0 \). Then \( s_0 \) is a stationary point for the Plateau Problem.

If \( s_0 \) is monotone and non-degenerate and moreover \( |\ln h|^{3/2} ||s_h - s_0||_T \to 0 \), then the convergence rates of Theorem 2.10 will apply.

See [DH Theorem 6.4].

3 Some Technical Results

In this Section we recall or prove some technical results needed for the \( L^2(\partial D) \) estimate.

First note the following basic properties of the \( H^{1/2}(D) \) and \( T \) norms.

Proposition 3.1
(i) For any \( \xi, \eta \in T \)
\[
||\xi \cdot \eta||_T \leq c||\xi||_T ||\eta||_T.
\]
(ii) For any \( \xi \in C^1(\partial D; \mathbb{R}) \) and \( \eta \in H \)
\[
|\xi \cdot \eta|_{H^{1/2}(\partial D)} \leq c||\xi||_{C^1} ||\eta||_{H^{1/2}(\partial D)}.
\]
Approximations to elements of $T$ are given by the following result.

**Proposition 3.2** There is a bounded linear map

$$I_h : T \to H_h$$

such that

$$
\begin{align*}
\|\xi - I_h \xi\|_{H^{1/2}(\partial D)} &\leq ch^{3/2}\|\xi\|_{H^2(\partial D)}, \\
\|\xi - I_h \xi\|_{L^2(\partial D)} &\leq ch^{3/2}\|\xi\|_{H^{3/2}(\partial D)}, \\
\|\xi - I_h \xi\|_{H^{1/2}(\partial D)} &\leq ch\|\xi\|_{H^{3/2}(\partial D)} , \\
\|\xi - I_h \xi\|_{C^0(\partial D)} &\leq ch^2\|\xi\|_{C^2(\partial D)}.
\end{align*}
$$

**Proof:** The main point is to preserve the normalisation conditions. The first and last inequalities are from [DH, Proposition 5.2]. The other estimates follow in the same manner from the standard estimates

$$
\begin{align*}
\|\xi - \mathcal{T}_h \xi\|_{L^2(\partial D)} &\leq ch^{3/2}\|\xi\|_{H^{3/2}(\partial D)}, \\
\|\xi - \mathcal{T}_h \xi\|_{H^{1/2}(\partial D)} &\leq ch\|\xi\|_{H^{3/2}(\partial D)},
\end{align*}
$$

where $\mathcal{T}_h$ is the usual interpolation operator uniquely defined by requiring

$$\mathcal{T}_h \xi(\phi_i) = \xi(\phi_i)$$

for all vertices $\phi_i$ of $G_h$ and by requiring that $\mathcal{T}_h \xi$ be linear between vertices. 

The following estimate shows that the $\|\cdot\|_{H^{1/2}(\partial D)}$ and $\|\cdot\|_T$ norms are equivalent on $H_h$ up to a factor $|\ln h|^{1/2}$. In particular, this factor is dominated by any positive power of $h$.

**Proposition 3.3** If $\xi_h \in H_h$ then

$$
\|\xi_h\|_{H^{1/2}(\partial D)} \leq \|\xi_h\|_T \leq c \ln h|^{1/2} \|\xi_h\|_{H^{1/2}(\partial D)},
$$

for $h \leq 1/2$, say.

See [DH, Proposition 5.3].

Next define $H^{-1/2}(\partial D)$ to be the dual space of $H^{1/2}(\partial D)$ with the usual operator norm. There is a natural imbedding

$$H^{1/2}(D) \hookrightarrow H^{-1/2}(\partial D)$$

given by

$$\langle \zeta, \eta \rangle = \int_{\partial D} \zeta \eta, \quad \forall \eta \in H^{1/2}(D),$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing of $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$. Thus

$$\|\zeta\|_{H^{-1/2}(\partial D)} = \sup_{\|\eta\|_{H^{1/2}(\partial D)} = 1} \int_{\partial D} \zeta \eta.$$
Define
\[ H^{3/2}(\partial D) = \{ \xi \in H^{1/2}(\partial D) : \xi' \in H^{1/2}(\partial D) \}, \]
where \( \xi' \) is the distributional derivative of \( \xi \). Define the seminorm
\[ |\xi|_{H^{3/2}(\partial D)} = |\xi'|_{H^{1/2}(\partial D)} \]
and norm
\[ ||\xi||_{H^{3/2}(\partial D)} = |\xi|_{H^{3/2}(\partial D)} + ||\xi||_{L^2(\partial D)}. \]

We will need the simple interpolation result
\[ |||\cdot|||_{L^2(\partial D)} \leq c|\eta|^{1/2}_{H^{-1/2}(\partial D)} |||\eta|||_{L^1/2(\partial D)}, \tag{11} \]
which follows easily from the relevant definitions.
If
\[ \eta = \sum_{-\infty}^{\infty} a_n e^{in\theta} \]
then it is standard that \( |||\eta|||_{H^s(\partial D)} \) is comparable to
\[ \sum_{-\infty}^{\infty} (1 + n^2)^{s} |a_n|^2. \tag{12} \]

4 The \( L^2 \) estimate

The following Theorem will be established by a variant of the Aubin-Nitsche Lemma, c.f. [C; pp 140–143].

**Theorem 4.1** With the same hypotheses and notation as in Theorem 2.10, we have in addition that
\[ ||s_h - s_0||_{H^{-1/2}(\partial D)} \leq ch^{5/2}, \quad ||s_h - s_0||_{L^2(\partial D)} \leq ch^2, \quad ||u_h - u_0||_{L^2(\partial D)} \leq ch^{5/2}. \]

**Proof:** In the following, constants \( c \) may depend on \( s_0 \). \( I_h : T \to H_h \) is the interpolation-type operator from Proposition 3.2.

Consider an arbitrary \( \xi \in H \). From the following Lemma there is a unique \( \phi_\xi \in H \) solving the “adjoint problem”
\[ d^2 E(s_0)(\phi_\xi, \eta) = \int_{\partial D} \xi \eta \quad \forall \eta \in H. \]
Moreover \( \phi_\xi \in H^{3/2}(\partial D) \) and
\[ |\phi_\xi|_{H^{3/2}(\partial D)} \leq c|\xi|_{H^{1/2}(\partial D)}. \tag{13} \]
Hence
\[
\int_{\partial D} \xi(s_h - s_0) = d^2 E(s_0)(\phi_\xi, s_h - s_0) \\
= d^2 E(s_0)(\phi_\xi - I_h \phi_\xi, s_h - s_0) + \\
(\mathcal{D} E(s_0)(I_h \phi_\xi) + d^2 E(s_0)(I_h \phi_\xi, s_h - s_0) - dE(s_h)(I_h \phi_\xi)) \\
= A + B.
\]

But
\[
|A| \leq c ||s_h - s_0||_{H^{1/2}(\partial D)} ||\phi_\xi - I_h \phi_\xi||_{H^{1/2}(\partial D)} \quad \text{by Proposition 2.2}
\]
\[
\leq c h^{3/2} ||\phi_\xi||_{H^{1/2}(\partial D)} \quad \text{by Theorem 2.10 and Proposition 3.2}
\]
\[
\leq c h^{5/2} ||\xi||_{H^{1/2}(\partial D)} \quad \text{by (13)}.
\]

Also
\[
|B| \leq c ||s_h - s_0||_T^2 ||I_h \phi_\xi||_T \quad \text{by Proposition 2.1}
\]
\[
\leq c h^3 |\ln h|^{3/2} ||I_h \phi_\xi||_{H^{1/2}(\partial D)} \quad \text{by Theorem 2.10 and Proposition 3.3}
\]
\[
\leq c h^3 |\ln h|^{3/2} ||\xi||_{H^{1/2}(\partial D)} \quad \text{by Proposition 3.2 and (13)}.
\]

Thus
\[
\int_{\partial D} \xi(s_h - s_0) \leq c h^{5/2} ||\xi||_{H^{1/2}(\partial D)}
\]
for arbitrary \( \xi \in H^{1/2}(\partial D) \) (if \( \xi \in H^{1/2}(\partial D) \) is \( L^2 \)-orthogonal to \( H \) then the above integral is 0) and so
\[
||s_h - s_0||_{H^{-1/2}(\partial D)} \leq c h^{5/2}.
\]

The second inequality of the Theorem follows from (11) and Theorem 2.10.

To prove the third inequality of the Theorem we estimate
\[
||\gamma \circ s_h - \gamma \circ s_0||_{H^{-1/2}(\partial D; \mathbb{R}^n)} \leq \left\| \gamma' \circ s_0 \cdot (s_h - s_0) \right\|_{H^{-1/2}(\partial D; \mathbb{R}^n)} + \\
\left\| \left( \int_0^1 \gamma'' \circ (s_0 + t(s_h - s_0)) dt \right) (s_h - s_0)^2 \right\|_{H^{-1/2}(\partial D; \mathbb{R}^n)}
\]
\[
= C + D.
\]

But
\[
C \leq c ||s_h - s_0||_{H^{-1/2}(\partial D)} \leq c h^{5/2}
\]
from the operator definition of \( || \cdot ||_{H^{-1/2}(\partial D)} \), Proposition 3.1(ii), and the previous estimate for \( ||s_h - s_0||_{H^{-1/2}(\partial D)} \).

Also
\[
D \leq c \left\| \left( \int_0^1 \gamma'' \circ (s_0 + t(s_h - s_0)) dt \right) (s_h - s_0)^2 \right\|_{H^{-1/2}(\partial D; \mathbb{R}^n)}
\]
\[
\leq c ||s_h - s_0||_{C^0(\partial D)} ||s_h - s_0||_{H^{-1/2}(\partial D)}
\]
\[
\leq c h^4 |\ln h|^{1/2},
\]
from Proposition 3.1 and Theorem 2.10.

Hence
\[ \|\gamma \circ s_h - \gamma \circ s_0\|_{H^{-1/2}(\partial D; R^m)} \leq ch^{5/2} \]
and so
\[ \|u_h - u_0\|_{L^2(D; R^m)} \leq ch^{5/2}, \]
as required.

The following Lemma completes the proof of the previous Theorem.

**Lemma 4.2** Assume \( r \geq 5 \) and \( s_0 \) is a non-degenerate stationary point for \( E \). Suppose \( \xi \in H \). Then the “adjoint” problem
\[
d^2 E(s_0)(\phi_\xi, \eta) = \int_{\partial D} \xi \eta \quad \forall \eta \in H
\]
has a unique solution \( \phi_\xi \in H \). Moreover, \( \phi_\xi \in H^{3/2}(\partial D) \) and
\[
|\phi_\xi|_{H^{3/2}(\partial D)} \leq c|\xi|_{H^{1/2}(\partial D)}.
\]
The constant \( c \) depends on \( s_0 \).

**Proof:** In the following, constants \( c \) may depend on \( s_0 \).

Define \( \tilde{\xi} \in H \) by
\[
(\tilde{\xi}, \eta)_{H^{3/2}(\partial D)} = \int_{\partial D} \xi \eta \quad \forall \eta \in H.
\]
Then \( \tilde{\xi} \) exists and is unique by the Riesz representation theorem and
\[
|\tilde{\xi}|_{H^{1/2}(\partial D)} = ||\xi||_{H^{-1/2}(\partial D)} \leq c|\xi|_{H^{1/2}(\partial D)},
\]
using (12).

Define
\[
\phi_\xi = (\nabla^2 E(s_0))^{-1}(\tilde{\xi}),
\]
see Definition 2.6. It follows \( \phi_\xi \) satisfies (14). Moreover, \( \phi_\xi \) is the unique solution since \( d^2 E(s_0) \) is non-degenerate. Also
\[
|\phi_\xi|_{H^{1/2}(\partial D)} \leq c|\tilde{\xi}|_{H^{1/2}(\partial D)} \leq c|\xi|_{H^{1/2}(\partial D)}.
\]

We next estimate \( |\phi_\xi|_{H^{3/2}(\partial D)} \). See [St2, Section II.5] for similar arguments.

If \( \eta: \partial D \to \mathbb{R}^k \) and \( h > 0 \) is fixed we use the notation
\[
\eta_\pm(\theta) = \eta(\theta \pm h),
\]
\[
\partial_h \eta = \frac{1}{h}(\eta_+ - \eta).
\]
If \( \eta \in H \) then so are \( \eta_\pm \) and \( \partial_h \eta \), since the normalisation conditions (8) are preserved. Note that
\[
\partial_h(\psi \eta) = \partial_h \psi \eta_+ + \psi \partial_h \eta,
\]
\[
\partial_{-h} \partial_h(\psi \eta) = \partial_{-h} \partial_h \psi \eta + \partial_h \psi(\partial_{-h} \eta)_+ + \partial_{-h} \psi(\partial_h \eta)_+ + \psi \partial_{-h} \partial_h \eta.
\]
The operator \( \partial_h \) extends to a difference operator in the angle variable for functions defined over \( D \). Moreover, \( \partial_h \) commutes with the harmonic extension operator by the uniqueness of harmonic extension. Also

\[
\int_D D\Phi(f) \cdot D\Phi(\partial_h g) = - \int_D D\Phi(\partial_h f) \cdot D\Phi(g),
\]
as follows from integrating in polar coordinates and using parts.

Inequality (17) now follows from (11).

We next substitute \( \eta = \partial_h \partial_h \phi_\xi \) in (14). From Proposition 2.1 and the previous formulae,

\[
d^2 E(s_0)(\phi_\xi, \partial_h \partial_h \phi_\xi)
= \int_D D\left(\Phi(\gamma'(s_0)\phi_\xi)\right) \cdot D\left(\Phi(\gamma'(s_0)\partial_h \partial_h \phi_\xi)\right) + \int_{\partial D} \frac{\partial \sigma_0}{\partial \nu} \cdot \gamma''(s_0) \phi_\xi \partial_h \partial_h \phi_\xi
\]

\[
= - \int_D \left| D\left(\Phi(\partial_h(\gamma'(s_0)\phi_\xi))\right)\right|^2
\]

\[
- \int_D D\left(\Phi(\gamma'(s_0)\phi_\xi)\right) \cdot \left( D\left(\Phi(\partial_h \partial_h \gamma'(s_0) \phi_\xi)\right) + D\left(\Phi(\partial_h \gamma'(s_0) (\partial_h \phi_\xi)_+)\right) + D\left(\Phi(\partial_h \gamma'(s_0) (\partial_h \phi_\xi)_-)\right)\right)
\]

\[
+ \int_{\partial D} \frac{\partial \sigma_0}{\partial \nu} \cdot \gamma''(s_0) \phi_\xi \partial_h \partial_h \phi_\xi
\]

\[
= - A + B + C.
\]

Hence

\[
A = - \int_{\partial D} \xi \partial_h \partial_h \phi_\xi + B + C. \tag{16}
\]

To estimate \( A \) from below, we first claim for any \( \eta \in H \) that

\[
|\eta|^2_{H^{1/2}(\partial D)} \leq c |\gamma'(s_0)|^2_{H^{1/2}(\partial D, \mathbb{R}^n)} + c |\eta|^2_{H^{-1/2}(\partial D)}. \tag{17}
\]

To see this compute

\[
\left|\gamma'(s_0(\theta))\eta(\theta) - \gamma'(s_0(\theta'))\eta(\theta')\right|^2
\]

\[
\leq \left|\gamma'(s_0(\theta))(\eta(\theta) - \eta(\theta')) + \eta(\theta')(\gamma'(s_0(\theta)) - \gamma'(s_0(\theta')))\right|^2
\]

\[
\geq c_1 |\eta(\theta) - \eta(\theta')|^2 - |\eta(\theta')|^2 |\gamma'(s_0(\theta)) - \gamma'(s_0(\theta'))|^2
\]

where \( c_1 = \inf |\gamma'(s_0(\theta))|^2 > 0 \). Hence from (9),

\[
|\eta|^2_{H^{1/2}(\partial D)} \leq c |\gamma'(s_0)|^2_{H^{1/2}(\partial D, \mathbb{R}^n)} + c |\eta|^2_{L^2(\partial D)}.
\]

Inequality (17) now follows from (11).

Since

\[
\partial_h (\gamma'(s_0)\phi_\xi) = \partial_h \gamma'(s_0)(\phi_\xi)_+ + \gamma'(s_0) \partial_h \phi_\xi,
\]

it follows from (17) with \( \eta = \partial_h \phi_\xi \), the definition of \( A \) and Proposition 3.1 that

\[
|\partial_h \phi_\xi|^2_{H^{1/2}(\partial D)} \leq c \left( |\partial_h (\gamma'(s_0)\phi_\xi)|^2_{H^{1/2}(\partial D, \mathbb{R}^n)} + |\partial_h \gamma'(s_0)(\phi_\xi)_+|^2_{H^{1/2}(\partial D, \mathbb{R}^n)}
\]

\[
+ |\partial_h \phi_\xi|^2_{H^{-1/2}(\partial D)} \right)
\]

\[
\leq c \left( A + |\phi_\xi|^2_{H^{1/2}(\partial D)} + |\partial_h \phi_\xi|^2_{H^{-1/2}(\partial D)} \right).
\]
It follows from (12) that for any $\eta \in H$,

$$
\|\partial_\eta \eta\|_{H^{-1/2}(\partial D)} \leq c|\eta|_{H^{1/2}(\partial D)}
$$

(18)

Hence, from (15),

$$
|\partial_\eta \phi_\xi|^2_{H^{1/2}(\partial D)} \leq c \left(A + |\xi|^2_{H^{1/2}(\partial D)}\right).
$$

(19)

We next compute

$$
-B = \left(\gamma'(s_0)\phi_\xi, \partial_- h \partial_\gamma'(s_0)\phi_\xi + \partial_\gamma'(s_0)(\partial_- \phi_\xi)_+ + \partial_- \gamma'(s_0)(\partial_\phi_\xi)_-\right)_{H^{1/2}(\partial D; \mathbb{R}^n)}.
$$

Hence using Proposition 3.1 and then (15),

$$
|B| \leq c|\partial_\eta \phi_\xi|^2_{H^{1/2}(\partial D)} + c(\epsilon)|\phi_\xi|^2_{H^{1/2}(\partial D)}
\leq c|\partial_\eta \phi_\xi|^2_{H^{1/2}(\partial D)} + c(\epsilon)|\xi|^2_{H^{1/2}(\partial D)}.
$$

(20)

Also

$$
C \leq c \int_{\partial D} |\phi_\xi||\partial_- \partial_\eta \phi_\xi|
\leq c\|\partial_- \partial_\eta \phi_\xi\|_{H^{-1/2}(\partial D)} |\phi_\xi|_{H^{1/2}(\partial D)}
\leq c|\partial_\eta \phi_\xi|^2_{H^{1/2}(\partial D)} |\phi_\xi|_{H^{1/2}(\partial D)} \text{ from (18)}
\leq c|\partial_\eta \phi_\xi|^2_{H^{1/2}(\partial D)} + c(\epsilon)|\xi|^2_{H^{1/2}(\partial D)} \text{ from (15)}.
$$

(21)

Finally,

$$
\left|\int_{\partial D} \xi \partial_- \partial_\eta \phi_\xi\right| \leq \|\partial_- \partial_\eta \phi_\xi\|_{H^{-1/2}(\partial D)} |\xi|_{H^{1/2}(\partial D)}
\leq c|\partial_\eta \phi_\xi|^2_{H^{1/2}(\partial D)} |\xi|_{H^{1/2}(\partial D)}
\leq c|\partial_\eta \phi_\xi|^2_{H^{1/2}(\partial D)} + c(\epsilon)|\xi|^2_{H^{1/2}(\partial D)}.
$$

(22)

From (19), (16), (22), (20) and (21),

$$
|\partial_\eta \phi_\xi|^2_{H^{1/2}(\partial D)} \leq c|\xi|^2_{H^{1/2}(\partial D)}.
$$

Since this is true for any $h > 0$, it follows for example from (12) that

$$
|\phi_\xi|^2_{H^{1/2}(\partial D)} \leq c|\xi|^2_{H^{1/2}(\partial D)}.
$$

and so

$$
|\phi_\xi|^2_{H^{3/2}(\partial D)} \leq c|\xi|^2_{H^{3/2}(\partial D)}.
$$

This completes the proof of the Lemma and hence of the Theorem.
5 Numerical Results

For the sake of completeness we recall some of the numerical results from [DH].

The possibly unstable Enneper surface is given by the harmonic extension of

\[
\begin{align*}
\gamma_1(\phi) &= r_0 \cos \phi - r_0^3/3 \cos 3\phi, \\
\gamma_2(\phi) &= r_0 \sin \phi + r_0^3/3 \sin 3\phi, \\
\gamma_3(\phi) &= r_0^2 \cos 2\phi,
\end{align*}
\]

for \( \phi \in [0, 2\pi) \). It is well known that for \( 0 < r_0 < 1 \) there is exactly one solution of Plateau’s problem for \( \Gamma = \gamma([0,2\pi]) \) and for \( 1 < r_0 < \sqrt{3} \) there are two minima and one unstable minimal surface bounded by \( \Gamma \).

We computed the discrete analogue for \( r_0 = 0.5, r_0 = 1.0 \) and \( r_0 = 1.5 \) using the fixed parametrisation

\[
\gamma^* = \gamma \circ \tau, \quad \tau(s) = s + 0.1 \cos 2s.
\]

The error

\[
e_h = \|s_0 - s_h\|
\]

between the piecewise linear discrete solution \( s_h \) and the continuous solution \( s = \tau^{-1} \circ \text{id} \) was computed for various norms and for uniform grid sizes \( h = 2\pi/n \). The experimental order of convergence \( eoc \) between two grid sizes \( h_1 \) and \( h_2 \) is given by

\[
eoc = \ln \frac{e_{h_1}}{e_{h_2}} / \ln \frac{h_1}{h_2}.
\]

As the following tables show, the numerical results confirm the asymptotic convergence in the \( H^{1/2}(\partial D) \) and \( L^2(\partial D) \) norms predicted by our results. The experimental error in the \( H^{-1/2}(\partial D) \) norm behaves like \( O(h^2) \) only, due to the fact that we used an integration formula for \( E \) (and its derivatives) which restricts the order of convergence to 2. The use of a higher order quadrature would lead to a much more complicated scheme and would not change the order of convergence in the other norms.

<table>
<thead>
<tr>
<th>Stable Enneper Surface (r=0.5)</th>
</tr>
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<tr>
<td>( n )</td>
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</tr>
<tr>
<td>40</td>
</tr>
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<td>80</td>
</tr>
<tr>
<td>160</td>
</tr>
<tr>
<td>320</td>
</tr>
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</table>
### Enneper Surface (r = 1.0)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$H^{-1/2}$-error</th>
<th>$eoc$</th>
<th>$L^2$-error</th>
<th>$eoc$</th>
<th>$H^{1/2}$-error</th>
<th>$eoc$</th>
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### Unstable Enneper Surface (r=1.5),

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</table>

### References


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