L^2 Estimates for Approximations to Minimal Surfaces

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Abstract In a previous paper the authors developed a new algorithm for finding discrete approximations to (possibly unstable) disc-like minimal surfaces. Optimal convergence rates in the H^1 norm were obtained. Here we recall the key ideas and prove optimal L^2 convergence rates.

1 Introduction

Suppose Γ is a smooth curve in $\mathbb{R}^n$. We are interested in the problem of obtaining discrete approximations to (possibly unstable) disc-like minimal surfaces spanning Γ.

Let D be the unit disc in $\mathbb{R}^2$ and let

$$\mathcal{C} = \{u : D \to \mathbb{R}^n \mid \triangle u = 0,\ u|_{\partial D} \text{ is a monotone parametrisation of } \Gamma\}.$$

Denote the Dirichlet energy by

$$\mathcal{D}(u) = \frac{1}{2}\int_D |Du|^2.$$

It is well-known that if u is a critical point for $\mathcal{D}$ restricted to $\mathcal{C}$ then $u[D]$ is a minimal surface spanning Γ. Moreover, u is then conformal. Conversely, any minimal surface spanning Γ can be obtained in this manner. We will make this the basis of the numerical algorithm.

Harmonic maps are uniquely determined by their boundary values. Thus if $\Gamma = \gamma[S^1]$ is given by the parametrisation

$$\gamma : S^1 \to \Gamma,$$

[1] Lecture delivered by Hutchinson.

then instead of $\mathcal{C}$ one can equivalently consider the class

$$\overline{\mathcal{M}} = \{s : \partial D \to S^1 \mid s \text{ is monotone}\}.$$

For $s \in \overline{\mathcal{M}}$ the corresponding harmonic map spanning Γ is

$$u = \Phi(\gamma \circ s)$$

where Φ denotes harmonic extension.

The *energy functional* on $\mathcal{M}$ is defined by

$$\begin{aligned} E(s) &= \mathcal{D}\left(\Phi(\gamma \circ s)\right) \\ &= \frac{1}{2}\int_D \left|D\left(\Phi(\gamma \circ s)\right)\right|^2 \\ &= \frac{1}{16\pi}\int_{\partial D}\int_{\partial D} \frac{|(\gamma \circ s)(\phi) - (\gamma \circ s)(\phi')|^2}{\sin^2\left(\frac{\phi - \phi'}{2}\right)} d\phi\, d\phi'. \end{aligned} \tag{1}$$

The last integral is known as the *Douglas Integral*, c.f. [N2; §§310–311].

There is a three parameter family of conformal maps from the unit disc D parametrising a given simply connected smooth surface. The usual normalisation is to specify the image of three points on ∂D. Here it is theoretically more convenient, and numerically more stable, to consider maps s such that

$$\begin{aligned} \int_0^{2\pi} (s(\theta) - \theta)\; d\theta &= 0, \\ \int_0^{2\pi} (s(\theta) - \theta) \cos\theta \, d\theta &= 0, \\ \int_0^{2\pi} (s(\theta) - \theta) \sin\theta \, d\theta &= 0. \end{aligned} \tag{2}$$

Thus we define

$$\mathcal{M} = \{s \in C^0(\partial D, S^1) : s \text{ is monotone},\ s \text{ satisfies (2)},\ E(s) < \infty\}. \tag{3}$$

See [St; Section II.2].

If s is critical for E restricted to $\mathcal{M}$ we say s is *stationary* and the corresponding harmonic map $u = \Phi(\gamma \circ s)$ is called the *minimal surface* corresponding to s.

Given a fixed grid ϕ_j, $j = 1, \ldots, N$, on ∂D, with typical grid-size h (i.e. the distance between successive points is controlled above and below by multiples of h), the discrete analogue of (3) is

$$\mathcal{M}_h = \{s_h = (s_1, \ldots, s_N) : s_h \text{ is a monotone sequence of points on } S^1\}.$$

It is convenient to identify both ∂D and S^1 with the interval $[0, 2\pi)$, and we will often do this. It is also convenient to identify $s_h \in \mathcal{M}_h$ with the corresponding piecewise linear map $s_h : \partial D \to S^1$ for which $s_h(\phi_j) = s_j$ ("piecewise linear" with respect to arc length, i.e. angle variable).

The *discrete energy functional* is simply the restriction of the energy functional E to $\mathcal{M}_h$ and is defined by

$$E_h(s_h) = E(s_h) = \frac{1}{2}\int_D |D\left(\Phi(\gamma \circ s_h)\right)|^2 .$$

If s_h is critical for E_h restricted to $\mathcal{M}_h$ we say s_h is *stationary* and the corresponding harmonic map $u_h = \Phi(\gamma \circ s_h)$ is called the *semi-discrete minimal surface* corresponding to s_h.

Numerically, one approximates the Douglas functional in order to compute E_h, and one computes *discrete harmonic* approximations to u_h with boundary data $u_h(\phi_j) = \gamma(s_j)$. The numerical algorithm for finding critical points for E_h restricted to $\mathcal{M}_h$ is:

Algorithm *Given a grid ϕ_j, $j = 1, \dots, n$, on ∂D, initial values $s_h = (s_1, \dots, s_n)$ and parametrisation γ:*

1. *Compute the derivative of the approximate energy $E_h'(s_h)$.*
2. *If $|E_h'(s_h)|/|s_h| \le \epsilon$ then stop.*
3. *Compute the second derivative of the approximate energy $E_h''(s_h)$.*
4. *Solve the linear system $E_h''(s_h)d = -E_h'(s_h)$, update the solution $s_h := s_h + d$ and go to step 1.*

Here $|s_h|$ is the l^2-norm of s_h and ϵ is a given tolerance.

Suppose s_0 is stationary and u_0 is the corresponding minimal surface. In [DH] we showed, as $h \to 0$, the existence of a sequence of discrete stationary s_h and corresponding semi-discrete minimal surfaces u_h, such that

$$\|s_h - s_0\|_{H^{1/2}(\partial D)} \le ch^{3/2}, \tag{4}$$

$$\|u_h - u_0\|_{H^1(D)} \le ch^{3/2}. \tag{5}$$

In this paper we show that

$$\|s_h - s_0\|_{H^{-1/2}(\partial D)} \le ch^{5/2}, \tag{6}$$

$$\|u_h - u_0\|_{L^2(D)} \le ch^{5/2}. \tag{7}$$

The proof of (6) and (7) will use a variant of the Aubin-Nitsche technique.

The computational significance of our results is as follows. Suppose $s_h \to s_0$ in $C^0 \cap H^{1/2}(\partial D)$ as $h \to 0$, where the s_h are discrete stationary points. Then it is straightforward to prove s_0 is stationary, see [DH; Theorem 6.4]. Moreover, if s_0 is monotone and non-degenerate and $|\ln h|^{3/2}\|s_h - s_0\|_{C^0\cap H^{1/2}(\partial D)} \to 0$, then the convergence rates of (4)–(7) will apply. By non-degeneracy we mean that there are no non-zero Jacobi fields for s_0. If s_0 has no branch points (and this can be determined by observation of the approximating sequence) then non-degeneracy is generically true, see [BT].

The theoretically predicted rates of convergence typically appear after a small number of iterations, and provide strong evidence that the sequence of discrete stationary points (or corresponding sequence of semi-discrete minimal surfaces) is indeed converging towards a non-degenerate stationary point (or corresponding non-degenerate minimal surface).

For related results and further references, see [DH].

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2 Background Material

We recall the main ideas and results from [DH]. We follow the approach (in the non-discrete setting) of [St1, St2].

Assume γ is C^r where $r \geq 5$.

It is necessary to enlarge $\mathcal{M}$ as it is not linear, or even affine. We do this by first selecting a fixed member of $\mathcal{M}$, which for convenience we take to be the identity map

$$\mathrm{id}\colon \partial D \to S^1, \quad \mathrm{id}(\phi) = \phi.$$

We will consider maps

$$s = \mathrm{id} + \sigma$$

such that $\sigma \in H^{1/2}(\partial D; I\!R)$ and

$$\int_0^{2\pi} \sigma(\phi)\, d\phi = 0, \quad \int_0^{2\pi} \sigma(\phi) \cos\phi \, d\phi = 0, \quad \int_0^{2\pi} \sigma(\phi) \ \sin\phi \, d\phi = 0, \tag{8}$$

c.f. (2). Thus we define

$$\begin{aligned} H &= H^{1/2}(\partial D; I\!R) \cap \{\xi : (8) \text{ is satisfied with } \sigma \text{ replaced by } \xi\} \\ \mathcal{H} &= \mathrm{id} + H. \end{aligned}$$

The $H^{1/2}$ semi inner product is defined by

$$(\xi, \eta)_{H^{1/2}} = \int_{\partial D} \int_{\partial D} \frac{\big(\xi(\phi) - \xi(\phi')\big) \cdot \big(\eta(\phi) - \eta(\phi')\big)}{|\phi - \phi'|^2} d\phi \, d\phi' \tag{9}$$

The corresponding seminorm $|\cdot|_{H^{1/2}}$ is in fact a norm on H, by the first equality in (8) and the Poincaré inequality.

The definition of E is extended to $\mathcal{H}$ by (1).

Unfortunately E is not C^1 on $\mathcal{H}$, and so for this reason we define

$$\begin{aligned} T &= H \cap C^0(\partial D; I\!R) \\ \mathcal{T} &= \mathrm{id} + T \\ \|\xi\| &= |\xi|_{H^{1/2}(\partial D)} + \|\xi\|_{C^0}. \end{aligned}$$

In particular,

$$\mathcal{M} \subset \mathcal{T} \subset \mathcal{H}, \quad T \subset H.$$

If $s \in \mathcal{H}$ the corresponding harmonic map spanning Γ is denoted by

$$\overline{s} = \Phi(\gamma \circ s).$$

If $s \in \mathcal{M}$ is fixed and $\xi \in H$, then the corresponding vector field along $\gamma \circ s$ will be denoted by

$$\overline{\xi} = (\gamma' \circ s)\, \xi = \gamma'(s)\, \xi.$$

The harmonic extension of $\overline{\xi}$, which is an harmonic vector field over $\gamma[D]$, will also be denoted by $\overline{\xi}$.

Then one has

Proposition 2.1 *The energy functional $E : \mathcal{T} \to I\!R$ is C^{r-1}. Let $s = \mathrm{id} + \sigma$. Then*

$$\begin{aligned} dE(s)(\xi) &= \int_D D\overline{s} \cdot D\overline{\xi}, \\ d^2E(s)(\xi_1, \xi_2) &= \int_D D\overline{\xi}_1 \cdot D\overline{\xi}_2 + \int_{\partial D} \frac{\partial \overline{s}}{\partial \nu} \cdot \gamma''(s)\, \xi_1 \xi_2. \end{aligned}$$

Also

$$\begin{aligned} E(s) &\leq c\,(||\gamma||_{C^1}) \left(1 + |\sigma|^2_{H^{1/2}}\right), \\ |d^j E(s)(\xi_1, \dots, \xi_j)| &\leq c(||\gamma||_{C^{j+1}})(1 + |\sigma|^2_{H^{1/2}})||\xi_1||_T \cdot \ldots \cdot ||\xi_j||_T \quad 1 \leq j \leq r-1. \end{aligned}$$

If $\sigma \in C^0(\partial D; S^1)$ then

$$|\sigma|^2_{H^{1/2}} \leq c\,(E(s) + 1),$$

where c depends on $||\gamma^{-1}||_{C^1}$ and the modulus of continuity of σ.

The expressions for dE and d^2E are straightforward computations. For the remainder, see [St, Section II] and [DH, Proposition 4.3].

The following will be applied in case s is stationary, see Proposition 2.5.

Proposition 2.2 *If s is C^2 then $dE(s)$ and $d^2E(s)$ extend to bounded linear and bilinear operators respectively on H, and*

$$\begin{aligned} |dE(s)(\xi)| &\leq c\,(||\gamma||_{C^2}, ||s||_{C^1})\, |\xi|_{H^{1/2}}, \\ |d^2E(s)(\xi_1, \xi_2)| &\leq c\,(||\gamma||_{C^2}, ||s||_{C^1})\, |\xi_1|_{H^{1/2}} |\xi_2|_{H^{1/2}}. \end{aligned}$$

See [St, Section II] and [DH, Proposition 4.4].

Definition 2.3 The function $s \in \mathcal{M}$ is a *stationary point* for E if

$$\left.\frac{d}{dt}\right|_{t=0} E(s + t\xi) \geq 0$$

whenever $s + \xi \in \mathcal{M}$. If s is stationary for E then we say that the harmonic map $u = \Phi(\gamma \circ s)$ is a *minimal surface* or that u is a *solution of the Plateau Problem.*

One has:

Proposition 2.4 *The function $s \in \mathcal{M}$ is stationary for E with respect to monotone variations iff s is stationary in the sense of $\mathcal{T}$, i.e. iff*

$$dE(s)(\xi) = 0 \quad \forall \xi \in T. \tag{10}$$

The regularity results of [Hil], [Ja], [N1], [He] imply the regularity of stationary s.

Proposition 2.5 *If γ is $C^{r,\alpha}$ where $r \geq 1$ and $0 < \alpha < 1$, and $s \in \mathcal{M}$ is stationary for E, then s is $C^{r,\alpha}$ and $||s||_{C^{r,\alpha}}$ is controlled by $||\gamma||_{C^{r,\alpha}}$.*

Definition 2.6 Suppose s is a stationary point for E. The self-adjoint bounded linear map

$$\nabla^2 E(s) : H \to H$$

is defined by

$$\left(\nabla^2 E(s)(\xi), \eta\right)_{H^{1/2}(\partial D)} = d^2 E(s)(\xi, \eta) \quad \forall \xi, \eta \in H.$$

Definition 2.7 A stationary $s \in \mathcal{M}$ is *non-degenerate* if $d^2E(s) : H \times H \to I\!R$ is a non-degenerate bilinear operator. Equivalently, $\nabla^2 E(s)$ is invertible with bounded inverse.

We next define discrete approximations to H as follows. For each $h > 0$ let $\mathcal{G}_h$ be any grid on $\partial D \cong [0, 2\pi)$ such that

$$c^{-1}h < |I| < ch \qquad \forall I \text{ an interval of } \mathcal{G}_h,$$

where $|I|$ is the length of I and c is independent of h. If I is an interval, denote by

$$P_1(I)$$

the space of first order polynomials (in the arc length variable) defined over I.

Definition 2.8 Let

$$\begin{aligned} H_h \;=\; \Big\{ \xi_h \in C^0(\partial D, I\!R) : \xi_h \in P_1(I)\ \forall I \in \mathcal{G}_h,\ (8) \text{ is satisfied} \\ \text{with } \sigma \text{ there replaced by } \xi_h \Big\}. \end{aligned}$$

Then

$$H_h \subset T \subset H.$$

Let

$$\mathcal{H}_h = \ \{\text{id}\}\ + H_h \subset \mathcal{T} \subset \mathcal{H}$$

be the corresponding finite dimensional affine space of continuous piecewise affine maps (with respect to arc length) from ∂D to S^1.

Definition 2.9 A function $s_h \in \mathcal{H}_h$ is called a *semi-discrete stationary point* for E if

$$dE_h(s_h)(\xi_h) = 0 \quad \forall \xi_h \in H_h.$$

The associated function $u_h = \Phi(\gamma \circ s_h)$ is called a *semi-discrete minimal surface.*

The main result from [DH] is:

Theorem 2.10 (Energy Estimate) *Assume $r \geq 5$. Let $s_0 \in \mathcal{M}$ be a non-degenerate stationary point for E with associated minimal surface $u_0 = \Phi(\gamma \circ s_0)$.*

Then there exist constants h_0, ϵ_0 and c, depending on s_0, such that if $0 < h \leq h_0$ then there is a unique semi-discrete stationary point $s_h \in \mathcal{H}_h$ such that

$$|s_0 - s_h|_{H^{1/2}(\partial D)} \leq \epsilon_0 |\ln h|^{-3/2}.$$

Moreover,

$$|s_h - s_0|_{H^{1/2}(\partial D)} \leq ch^{3/2} \quad \text{and} \quad ||s_h - s_0||_{C^0(\partial D)} \leq ch^{3/2}|\ln h|^{1/2}.$$

Finally, if $u_h = \Phi(\gamma \circ s_h)$ is the corresponding semi-discrete minimal surface, then

$$||u_h - u_0||_{H^1(D)} \leq ch^{3/2} \quad \text{and} \quad ||u_h - u_0||_{C^0(D)} \leq ch^{3/2}|\ln h|^{1/2}.$$

See [DH, Theorem 6.3]. The computational significance of this is a consequence of the following result:

Theorem 2.11 *Suppose s_h is a sequence of semi-discrete stationary points and $||s_h - s_0||_T \to 0$ as $h \to 0$. Then s_0 is a stationary point for the Plateau Problem.*

If s_0 is monotone and non-degenerate and moreover $|\ln h|^{3/2}||s_h - s_0||_T \to 0$, then the convergence rates of Theorem 2.10 will apply.

See [DH Theorem 6.4].

3 Some Technical Results

In this Section we recall or prove some technical results needed for the $L^2(\partial D)$ estimate.

First note the following basic properties of the $H^{1/2}(D)$ and T norms.

Proposition 3.1

(i) *For any $\xi, \eta \in T$*

$$||\xi \cdot \eta||_T \leq c||\xi||_T ||\eta||_T.$$

(ii) *For any $\xi \in C^1(\partial D; I\!R)$ and $\eta \in H$*

$$|\xi \cdot \eta|_{H^{1/2}(\partial D)} \leq c||\xi||_{C^1} |\eta|_{H^{1/2}(\partial D)}.$$

See [St, Lemma II.2.6].

Approximations to elements of T are given by the following result.

Proposition 3.2 *There is a bounded linear map*

$$I_h : T \to H_h$$

such that

$$\begin{aligned}
||\xi - I_h\xi||_{H^{1/2}(\partial D)} &\leq ch^{3/2}||\xi||_{H^2(\partial D)}, \\
||\xi - I_h\xi||_{L^2(\partial D)} &\leq ch^{3/2}||\xi||_{H^{3/2}(\partial D)}, \\
||\xi - I_h\xi||_{H^{1/2}(\partial D)} &\leq ch||\xi||_{H^{3/2}(\partial D)}, \\
||\xi - I_h\xi||_{C^0(\partial D)} &\leq ch^2||\xi||_{C^2(\partial D)}.
\end{aligned}$$

Proof: The main point is to preserve the normalisation conditions.

The first and last inequalities are from [DH, Proposition 5.2]. The other estimates follow in the same manner from the standard estimates

$$\begin{aligned}
||\xi - \overline{I_h}\xi||_{L^2(\partial D)} &\leq ch^{3/2}||\xi||_{H^{3/2}(\partial D)}, \\
||\xi - \overline{I_h}\xi||_{H^{1/2}(\partial D)} &\leq ch||\xi||_{H^{3/2}(\partial D)},
\end{aligned}$$

where $\overline{I}_h$ is the usual interpolation operator uniquely defined by requiring

$$\overline{I}_h\xi(\phi_i) = \xi(\phi_i)$$

for all vertices ϕ_i of $\mathcal{G}_h$ and by requiring that $\overline{I}_h\xi$ be linear between vertices. ∎

The following estimate shows that the $|\cdot|_{H^{1/2}(\partial D)}$ and $|\cdot|_T$ norms are equivalent on H_h up to a factor $|\ln h|^{1/2}$. In particular, this factor is dominated by any positive power of h.

Proposition 3.3 *If $\xi_h \in H_h$ then*

$$||\xi_h||_{H^{1/2}(\partial D)} \leq ||\xi_h||_T \leq c|\ln h|^{1/2}\, ||\xi_h||_{H^{1/2}(\partial D)},$$

for $h \leq 1/2$, say.

See [DH, Proposition 5.3].

Next define $H^{-1/2}(\partial D)$ to be the dual space of $H^{1/2}(\partial D)$ with the usual operator norm. There is a natural imbedding

$$H^{1/2}(D) \hookrightarrow H^{-1/2}(\partial D)$$

given by

$$\langle \zeta, \eta \rangle = \int_{\partial D} \zeta\eta \quad \forall \eta \in H^{1/2}(D),$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing of $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$. Thus

$$||\zeta||_{H^{-1/2}(\partial D)} = \sup_{||\eta||_{H^{1/2}(\partial D)}=1} \int_{\partial D} \zeta\eta.$$

Define

$$H^{3/2}(\partial D) = \{\xi \in H^{1/2}(\partial D) : \xi' \in H^{1/2}(\partial D)\},$$

where ξ' is the distributional derivative of ξ. Define the seminorm

$$|\xi|_{H^{3/2}(\partial D)} = |\xi'|_{H^{1/2}(\partial D)}$$

and norm

$$||\xi||_{H^{3/2}(\partial D)} = |\xi|_{H^{3/2}(\partial D)} + ||\xi||_{L^2(\partial D)}.$$

We will need the simple interpolation result

$$||\eta||_{L^2(\partial D)} \leq c||\eta||^{1/2}_{H^{-1/2}(\partial D)}||\eta||^{1/2}_{H^{1/2}(\partial D)}, \tag{11}$$

which follows easily from the relevant definitions.

If

$$\eta = \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

then it is standard that $\|\eta\|^2_{H^s(\partial D)}$ is comparable to

$$\sum_{-\infty}^{\infty}(1+n^2)^s|a_n|^2. \tag{12}$$

4 The L^2 estimate

The following Theorem will be established by a variant of the Aubin-Nitsche Lemma, c.f. [C; pp 140–143].

Theorem 4.1 *With the same hypotheses and notation as in Theorem 2.10, we have in addition that*

$$||s_h - s_0||_{H^{-1/2}(\partial D)} \leq ch^{5/2}, \quad ||s_h - s_0||_{L^2(\partial D)} \leq ch^2, \quad ||u_h - u_0||_{L^2(D)} \leq ch^{5/2}.$$

Proof: In the following, constants c may depend on s_0. $I_h : T \to H_h$ is the interpolation-type operator from Proposition 3.2.

Consider an arbitrary $\xi \in H$. From the following Lemma there is a unique $\phi_\xi \in H$ solving the "adjoint problem"

$$d^2E(s_0)(\phi_\xi, \eta) = \int_{\partial D} \xi\eta \quad \forall \eta \in H.$$

Moreover $\phi_\xi \in H^{3/2}(\partial D)$ and

$$|\phi_\xi|_{H^{3/2}(\partial D)} \leq c|\xi|_{H^{1/2}(\partial D)}. \tag{13}$$

Hence

$$\begin{aligned}
\int_{\partial D} \xi(s_h - s_0) &= d^2E(s_0)(\phi_\xi, s_h - s_0) \\
&= d^2E(s_0)(\phi_\xi - I_h\phi_\xi, s_h - s_0) + \\
&\quad \Big(dE(s_0)(I_h\phi_\xi) + d^2E(s_0)(I_h\phi_\xi, s_h - s_0) - dE(s_h)(I_h\phi_\xi)\Big) \\
&= A + B.
\end{aligned}$$

But

$$\begin{aligned}
|A| &\leq c||s_h - s_0||_{H^{1/2}(\partial D)}||\phi_\xi - I_h\phi_\xi||_{H^{1/2}(\partial D)} \quad \text{by Proposition 2.2} \\
&\leq ch^{3/2}h|\phi_\xi|_{H^{3/2}(\partial D)} \quad \text{by Theorem 2.10 and Proposition 3.2} \\
&\leq ch^{5/2}|\xi|_{H^{1/2}(\partial D)} \quad \text{by (13).}
\end{aligned}$$

Also

$$\begin{aligned}
|B| &\leq c||s_h - s_0||_T^2||I_h\phi_\xi||_T \quad \text{by Proposition 2.1} \\
&\leq ch^3|\ln h|^{3/2}||I_h\phi_\xi||_{H^{1/2}(\partial D)} \quad \text{by Theorem 2.10 and Proposition 3.3} \\
&\leq ch^3|\ln h|^{3/2}|\xi|_{H^{1/2}(\partial D)} \quad \text{by Proposition 3.2 and (13).}
\end{aligned}$$

Thus

$$\int_{\partial D} \xi(s_h - s_0) \leq ch^{5/2}|\xi|_{H^{1/2}(\partial D)}$$

for arbitrary $\xi \in H^{1/2}(\partial D)$ (if $\xi \in H^{1/2}(\partial D)$ is L^2-orthogonal to H then the above integral is 0) and so

$$||s_h - s_0||_{H^{-1/2}(\partial D)} \leq ch^{5/2}.$$

The second inequality of the Theorem follows from (11) and Theorem 2.10.

To prove the third inequality of the Theorem we estimate

$$\begin{aligned}
||\gamma \circ s_h - \gamma \circ s_0||_{H^{-1/2}(\partial D;\boldsymbol{R}^n)} &\leq \Big\|\gamma' \circ s_0 \cdot (s_h - s_0)\Big\|_{H^{-1/2}(\partial D;\boldsymbol{R}^n)} + \\
&\quad \Big\|\Big(\int_0^1 \gamma'' \circ (s_0 + t(s_h - s_0))dt\Big)(s_h - s_0)^2\Big\|_{H^{-1/2}(\partial D;\boldsymbol{R}^n)} \\
&= C + D.
\end{aligned}$$

But

$$C \leq c||s_h - s_0||_{H^{-1/2}(\partial D)} \leq ch^{5/2}$$

from the operator definition of $||\cdot||_{H^{-1/2}(\partial D)}$, Proposition 3.1(ii), and the previous estimate for $||s_h - s_0||_{H^{-1/2}(\partial D)}$.

Also

$$\begin{aligned}
D &\leq c\Big\|\Big(\int_0^1 \gamma'' \circ (s_0 + t(s_h - s_0))dt\Big)(s_h - s_0)^2\Big\|_{H^{-1/2}(\partial D;\boldsymbol{R}^n)} \\
&\leq c||s_h - s_0||_{C^0(\partial D)}||s_h - s_0||_{H^{-1/2}(\partial D)} \\
&\leq ch^4|\ln h|^{1/2},
\end{aligned}$$

from Proposition 3.1 and Theorem 2.10.

Hence

$$||\gamma \circ s_h - \gamma \circ s_0||_{H^{-1/2}(\partial D;\boldsymbol{R}^n)} \le ch^{5/2}$$

and so

$$||u_h - u_0||_{L^2(D;\boldsymbol{R}^n)} \le ch^{5/2},$$

as required. ∎

The following Lemma completes the proof of the previous Theorem.

Lemma 4.2 *Assume $r \ge 5$ and s_0 is a non-degenerate stationary point for E. Suppose $\xi \in H$. Then the "adjoint" problem*

$$d^2E(s_0)(\phi_\xi, \eta) = \int_{\partial D} \xi\eta \quad \forall \eta \in H \tag{14}$$

has a unique solution $\phi_\xi \in H$. Moreover, $\phi_\xi \in H^{3/2}(\partial D)$ and

$$|\phi_\xi|_{H^{3/2}(\partial D)} \le c|\xi|_{H^{1/2}(\partial D)}.$$

The constant c depends on s_0.

Proof: In the following, constants c may depend on s_0.

Define $\tilde{\xi} \in H$ by

$$(\tilde{\xi}, \eta)_{H^{1/2}(\partial D)} = \int_{\partial D} \xi\eta \quad \forall \eta \in H.$$

Then $\tilde{\xi}$ exists and is unique by the Riesz representation theorem and

$$|\tilde{\xi}|_{H^{1/2}(\partial D)} = ||\xi||_{H^{-1/2}(\partial D)} \le c|\xi|_{H^{1/2}(\partial D)},$$

using (12).

Define

$$\phi_\xi = \left(\nabla^2 E(s_0)\right)^{-1}(\tilde{\xi}),$$

see Definition 2.6. It follows ϕ_ξ satisfies (14). Moreover, ϕ_ξ is the unique solution since $d^2E(s_0)$ is non-degenerate. Also

$$|\phi_\xi|_{H^{1/2}(\partial D)} \le c|\tilde{\xi}|_{H^{1/2}(\partial D)} \le c|\xi|_{H^{1/2}(\partial D)}. \tag{15}$$

We next estimate $|\phi_\xi|_{H^{3/2}(\partial D)}$. See [St2, Section II.5] for similar arguments.

If $\eta : \partial D \to \mathbb{R}^k$ and $h > 0$ is fixed we use the notation

$$\begin{aligned} \eta_\pm(\theta) &= \eta(\theta \pm h), \\ \partial_h \eta &= \frac{1}{h}(\eta_+ - \eta). \end{aligned}$$

If $\eta \in H$ then so are $\eta_\pm$ and $\partial_h\eta$, since the normalisation conditions (8) are preserved. Note that

$$\begin{aligned} \partial_h(\psi\eta) &= \partial_h\psi\,\eta_+ + \psi\,\partial_h\eta, \\ \partial_{-h}\partial_h(\psi\eta) &= \partial_{-h}\partial_h\psi\,\eta + \partial_h\psi(\partial_{-h}\eta)_+ + \partial_{-h}\psi(\partial_h\eta)_- + \psi\partial_{-h}\partial_h\eta. \end{aligned}$$

The operator ∂_h extends to a difference operator in the angle variable for functions defined over D. Moreover, ∂_h commutes with the harmonic extension operator by the uniqueness of harmonic extension. Also

$$\int_D D\Phi(f)\cdot D\Phi(\partial_{-h}g) = -\int_D D\Phi(\partial_h f)\cdot D\Phi(g),$$

as follows from integrating in polar coordinates and using parts.

We next substitute $\eta = \partial_{-h}\partial_h\phi_\xi$ in (14). From Proposition 2.1 and the previous formulae,

$$\begin{aligned}
&d^2E(s_0)(\phi_\xi, \partial_{-h}\partial_h\phi_\xi)\\
&= \int_D D\Big(\Phi(\gamma'(s_0)\phi_\xi)\Big)\cdot D\Big(\Phi(\gamma'(s_0)\partial_{-h}\partial_h\phi_\xi)\Big) + \int_{\partial D}\frac{\partial\overline{s_0}}{\partial\nu}\cdot\gamma''(s_0)\,\phi_\xi\partial_{-h}\partial_h\phi_\xi\\
&= -\int_D\Big|D\Big(\Phi(\partial_h(\gamma'(s_0)\phi_\xi))\Big)\Big|^2\\
&\quad -\int_D D\Big(\Phi(\gamma'(s_0)\phi_\xi)\Big)\cdot\Big(D\Big(\Phi(\partial_{-h}\partial_h\gamma'(s_0)\,\phi_\xi)\Big)\\
&\qquad + D\Big(\Phi(\partial_h\gamma'(s_0)\,(\partial_{-h}\phi_\xi)_+)\Big) + D\Big(\Phi(\partial_{-h}\gamma'(s_0)\,(\partial_h\phi_\xi)_-)\Big)\Big)\\
&\quad +\int_{\partial D}\frac{\partial\overline{s_0}}{\partial\nu}\cdot\gamma''(s_0)\,\phi_\xi\partial_{-h}\partial_h\phi_\xi\\
&= -A + B + C.
\end{aligned}$$

Hence

$$A = -\int_{\partial D}\xi\partial_{-h}\partial_h\phi_\xi + B + C. \tag{16}$$

To estimate A from below, we first *claim* for any $\eta\in H$ that

$$|\eta|^2_{H^{1/2}(\partial D)} \le c|\gamma'(s_0)\eta|^2_{H^{1/2}(\partial D;\boldsymbol{R}^n)} + c|\eta|^2_{H^{-1/2}(\partial D)}. \tag{17}$$

To see this compute

$$\begin{aligned}
&\Big|\gamma'(s_0(\theta))\eta(\theta) - \gamma'(s_0(\theta'))\eta(\theta')\Big|^2\\
&\quad = \Big|\gamma'(s_0(\theta))\Big(\eta(\theta)-\eta(\theta')\Big) + \eta(\theta')\Big(\gamma'(s_0(\theta)) - \gamma'(s_0(\theta'))\Big)\Big|^2\\
&\quad \ge c_1\Big|\eta(\theta)-\eta(\theta')\Big|^2 - |\eta(\theta')|^2\Big|\gamma'(s_0(\theta)) - \gamma'(s_0(\theta'))\Big|^2
\end{aligned}$$

where $c_1 = \inf|\gamma'(s_0(\theta))|^2 > 0$. Hence from (9),

$$|\eta|^2_{H^{1/2}(\partial D)} \le c|\gamma'(s_0)\eta|^2_{H^{1/2}(\partial D;\boldsymbol{R}^n)} + c|\eta|^2_{L^2(\partial D)}.$$

Inequality (17) now follows from (11).

Since

$$\partial_h(\gamma'(s_0)\phi_\xi) = \partial_h\gamma'(s_0)(\phi_\xi)_+ + \gamma'(s_0)\partial_h\phi_\xi,$$

it follows from (17) with $\eta = \partial_h\phi_\xi$, the definition of A and Proposition 3.1 that

$$\begin{aligned}
|\partial_h\phi_\xi|^2_{H^{1/2}(\partial D)} &\le c\Big(|\partial_h(\gamma'(s_0)\phi_\xi)|^2_{H^{1/2}(\partial D;\boldsymbol{R}^n)} + |\partial_h\gamma'(s_0)(\phi_\xi)_+|^2_{H^{1/2}(\partial D;\boldsymbol{R}^n)}\\
&\qquad + |\partial_h\phi_\xi|^2_{H^{-1/2}(\partial D)}\Big)\\
&\le c\Big(A + |\phi_\xi|^2_{H^{1/2}(\partial D)} + |\partial_h\phi_\xi|^2_{H^{-1/2}(\partial D)}\Big).
\end{aligned}$$

It follows from (12) that for any $\eta \in H$,

$$\|\partial_h \eta\|_{H^{-1/2}(\partial D)} \leq c|\eta|_{H^{1/2}(\partial D)} \tag{18}$$

Hence, from (15),

$$|\partial_h \phi_\xi|^2_{H^{1/2}(\partial D)} \leq c\left(A + |\xi|^2_{H^{1/2}(\partial D)}\right). \tag{19}$$

We next compute

$$\begin{aligned} -B &= \Big(\gamma'(s_0)\phi_\xi,\ \partial_{-h}\partial_h \gamma'(s_0)\phi_\xi + \partial_h \gamma'(s_0)(\partial_{-h}\phi_\xi)_+ \\ &\qquad\qquad + \partial_{-h}\gamma'(s_0)(\partial_h \phi_\xi)_-\Big)_{H^{1/2}(\partial D;\boldsymbol{R}^n)}. \end{aligned}$$

Hence using Proposition 3.1 and then (15),

$$\begin{aligned} |B| &\leq \epsilon|\partial_h \phi_\xi|^2_{H^{1/2}(\partial D)} + c(\epsilon)|\phi_\xi|^2_{H^{1/2}(\partial D)} \\ &\leq \epsilon|\partial_h \phi_\xi|^2_{H^{1/2}(\partial D)} + c(\epsilon)|\xi|^2_{H^{1/2}(\partial D)}. \end{aligned} \tag{20}$$

Also

$$\begin{aligned} C &\leq c\int_{\partial D} |\phi_\xi|\,|\partial_{-h}\partial_h \phi_\xi| \\ &\leq c\|\partial_{-h}\partial_h \phi_\xi\|_{H^{-1/2}(\partial D)}\,|\phi_\xi|_{H^{1/2}(\partial D)} \\ &\leq c|\partial_h \phi_\xi|_{H^{1/2}(\partial D)}\,|\phi_\xi|_{H^{1/2}(\partial D)} \qquad \text{from (18)} \\ &\leq \epsilon|\partial_h \phi_\xi|^2_{H^{1/2}(\partial D)} + c(\epsilon)|\xi|^2_{H^{1/2}(\partial D)} \qquad \text{from (15).} \end{aligned} \tag{21}$$

Finally,

$$\begin{aligned} \left|\int_{\partial D} \xi \partial_{-h}\partial_h \phi_\xi\right| &\leq \|\partial_{-h}\partial_h \phi_\xi\|_{H^{-1/2}(\partial D)}\,|\xi|_{H^{1/2}(\partial D)} \\ &\leq c|\partial_h \phi_\xi|_{H^{1/2}(\partial D)}\,|\xi|_{H^{1/2}(\partial D)} \\ &\leq \epsilon|\partial_h \phi_\xi|^2_{H^{1/2}(\partial D)} + c(\epsilon)|\xi|^2_{H^{1/2}(\partial D)}. \end{aligned} \tag{22}$$

From (19), (16), (22), (20) and (21)

$$|\partial_h \phi_\xi|^2_{H^{1/2}(\partial D)} \leq c|\xi|^2_{H^{1/2}(\partial D)}.$$

Since this is true for any $h > 0$, it follows for example from (12) that

$$|\phi_\xi{}'|^2_{H^{1/2}(\partial D)} \leq c|\xi|^2_{H^{1/2}(\partial D)},$$

and so

$$|\phi_\xi|^2_{H^{3/2}(\partial D)} \leq c|\xi|^2_{H^{1/2}(\partial D)}.$$

This completes the proof of the Lemma and hence of the Theorem. ∎

5 Numerical Results

For the sake of completeness we recall some of the numerical results from [DH].

The possibly unstable Enneper surface is given by the harmonic extension of

$$\begin{aligned}\gamma_1(\phi) &= r_0 \cos\phi - r_0^3/3 \cos 3\phi,\\ \gamma_2(\phi) &= r_0 \sin\phi + r_0^3/3 \sin 3\phi,\\ \gamma_3(\phi) &= r_0^2 \cos 2\phi,\end{aligned}$$

for $\phi \in [0, 2\pi)$. It is well known that for $0 < r_0 < 1$ there is exactly one solution of Plateau‘s problem for $\Gamma = \gamma([0, 2\pi))$ and for $1 < r_0 < \sqrt{3}$ there are two minima and one unstable minimal surface bounded by Γ.

We computed the discrete analogue for $r_0 = 0.5$, $r_0 = 1.0$ and $r_0 = 1.5$ using the fixed parametrisation

$$\gamma^* = \gamma \circ \tau, \;\; \tau(s) = s + 0.1 \cos 2s.$$

The error

$$e_h = \|s_0 - s_h\|$$

between the piecewise linear discrete solution s_h and the continuous solution $s = \tau^{-1} \circ \mathrm{id}$ was computed for various norms and for uniform grid sizes $h = 2\pi/n$. The experimental order of convergence *eoc* between two grid sizes h_1 and h_2 is given by

$$eoc = \ln \frac{e_{h_1}}{e_{h_2}} / \ln \frac{h_1}{h_2}.$$

As the following tables show, the numerical results confirm the asymptotic convergence in the $H^{1/2}(\partial D)$ and $L^2(\partial D)$ norms predicted by our results. The experimental error in the $H^{-1/2}(\partial D)$ norm behaves like $O(h^2)$ only, due to the fact that we used an integration formula for E (and its derivatives) which restricts the order of convergence to 2. The use of a higher order quadrature would lead to a much more complicated scheme and would not change the order of convergence in the other norms.

Stable Enneper Surface (r=0.5)						
n	$H^{-1/2}$-error	*eoc*	L^2-error	*eoc*	$H^{1/2}$-error	*eoc*
20	2.7913e-3		7.6477e-3		1.5378e-2	
40	6.6122e-4	2.08	1.7860e-3	2.10	4.2490e-3	1.86
80	1.6041e-4	2.04	4.2902e-4	2.06	1.2859e-3	1.72
160	2.9575e-5	2.44	8.3727e-5	2.36	4.0727e-4	1.66
320	4.1573e-6	2.83	1.5176e-5	2.46	1.3790e-4	1.56

Enneper Surface (r = 1.0)						
n	$H^{-1/2}$-error	*eoc*	L^2-error	*eoc*	$H^{1/2}$-error	*eoc*
20	3.4175e-3		9.7364e-3		1.9275e-2	
40	7.1311e-4	2.26	1.9620e-3	2.31	4.5275e-3	2.09
80	1.6602e-4	2.10	4.4812e-4	2.13	1.3098e-3	1.79
160	4.0129e-5	2.05	1.0722e-4	2.06	4.1841e-4	1.67
320	9.8698e-6	2.02	2.6237e-5	2.03	1.4062e-4	1.57

Unstable Enneper Surface (r=1.5),						
n	$H^{-1/2}$-error	*eoc*	L^2-error	*eoc*	$H^{1/2}$-error	*eoc*
20	3.8572e-3		1.1189e-2		2.2074e-2	
40	7.2308e-4	2.42	1.9958e-3	2.49	4.5825e-3	2.27
80	1.6672e-4	2.12	4.5050e-4	2.15	1.3128e-3	1.80
160	4.0202e-5	2.05	1.0747e-4	2.07	4.1864e-4	1.65
320	9.8783e-6	2.02	2.6266e-5	2.03	1.4064e-4	1.57

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