V-VARIABLE FRACTALS: DIMENSION RESULTS

MICHAEL BARNSLEY, JOHN E. HUTCHINSON, ÖRJAN STENFLO

Abstract

V-variable fractals, for $V = 1, 2, 3, \ldots$, interpolate between random homogeneous fractals and random recursive fractals. We compute the almost sure Hausdorff dimension of V-variable fractals satisfying the uniform open set condition. Important roles are played by the notion of a neck, leading to spatial homogeneity at various levels of magnification, and a variant of the Furstenberg Kesten theorem for products of certain random $V \times V$ matrices.

1. INTRODUCTION

In this paper we begin the study of analysis on V-variable fractals by computing the Hausdorff dimension, under the assumption of an open set condition. Important tools used in the proof are the notion of a neck, and a variant of the Furstenberg Kesten theorem for products of random matrices.

Fractal sets generated by a single iterated function system [IFS], and random fractal sets generated from a family of IFSs, are studied as mathematical models of disordered systems. In this latter setting the most commonly studied random fractals are random recursive fractals and random homogeneous fractals. In particular, Hausdorff, walk and spectral dimensions have been computed in special cases.

Let $\boldsymbol{F}$ be a family of IFSs operating on $\mathbb{R}^n$, together with a probability distribution P on $\boldsymbol{F}$. For each natural number V there is a corresponding family of V-variable random fractals; these families interpolate between the family of random homogeneous fractals ($V = 1$), and the family of random recursive fractals ($V \to \infty$), corresponding to $\boldsymbol{F}$. Each class of V-variable random fractals, with its probability distribution, has the surprising property that it can be obtained from the attractor of a *single* IFS operating on V-tuples of compact subsets of $\mathbb{R}^n$.

See [Fal03, Fal97, Ham03] for general background on fractals and random fractals, including their applications, and see [Mor46, Hut81, Fal86, Gra87, Gra91, MW86, Ham92, Ham97, Bar98, Kig01, Str06] and the references therein for the study of the various dimensions and other analytic properties of fractals. For the general development of V-variable fractals see [BHS08], and for other examples and discussion, see [BHS05, Bar06].

In order to fix ideas let $F = (\mathbb{R}^2; f_1, f_2, f_3)$ be the IFS consisting of three contraction maps on $\mathbb{R}^2$, each with contraction ratio $1/2$, and having fixed points that are the vertices of an equilateral triangle T of unit diameter. Let $G = (\mathbb{R}^2; f_1, f_2, f_3)$ be the IFS consisting of three contraction maps, each with contraction ratio $1/3$, and having the same fixed points as the corresponding maps in F. The attractors of F and G are denoted by S_F and S_G respectively, see Figure 1.

FIGURE 1. Prefractal approximations to the attractors S_F and S_G respectively.

Consider $\boldsymbol{F} = \{F, G\}$, together with the probability distribution $P = \{1/2, 1/2\}$ on $\boldsymbol{F}$. Let Ω *denote the set of labelled trees* which are rooted, 3-branching, and infinite, where the label at each

node is either F or G; see Section 2.1. A modified Sierpinski triangle K^ω can be generated from each $\omega \in \Omega$, see Figure 2.

Such fractals K^ω are examples of *recursive fractals*. If the nodes of ω at each fixed level are the same, but may vary from level to level, then K^ω is called a *homogeneous fractal*.

It will also be convenient to consider finite, level k, labelled trees for each $k \geq 0$; see Figure 2. Such trees correspond to level k prefractals.
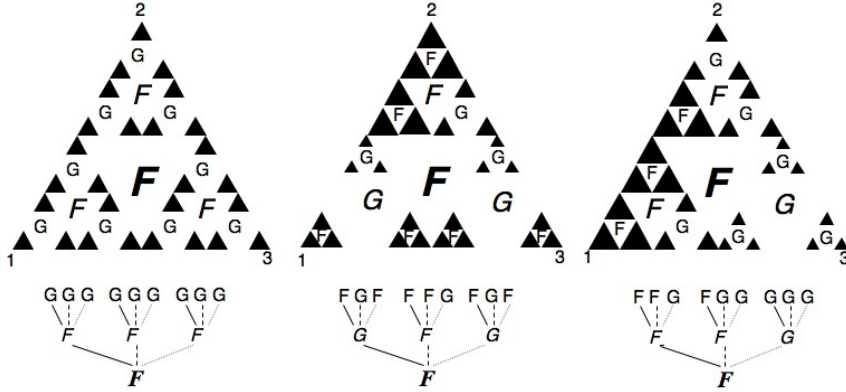

FIGURE 2. Level 3 recursive prefractals that are 1, 2 and 3-variable respectively, together with the corresponding finite labelled trees. Each vertex of the triangle T is labelled according to the function which fixes it.

Fractals such as S_F and S_G have the following properties:

(1) *Spatial self similarity*: loosely expressed, at each fixed "scale" the component parts are equivalent up to simple transformations, for example, translations in the case here.
(2) *Scale self similarity*: the equivalence class at each scale is the same.

Homogeneous fractals have spatial self similarity but, generically, do not have scale self similarity. Recursive fractals generically have neither spatial nor scale self similarity. Both classes have further statistical self similarity when we impose suitable probability distributions on their construction.

Now assume that, at each level of $\omega \in \Omega$, the subtrees rooted at that level have the property that they belong to at most V distinct isomorphism classes. In this case, ω is said to be a V-*variable labelled tree*, and the fractal K^ω is said to be a V-*variable fractal*. Such fractals have a form of partial spatial self similarity. Similarly, define V-*variable finite labelled trees* and V-*variable prefractals*, see Figure 2. Notice that homogeneous fractals and 1-variable fractals are the same.

Let Ω_V *denote the class of V-variable trees ω corresponding to $\boldsymbol{F}$, and let $\mathcal{K}_V$ denote the corresponding class of V-variable fractals K^ω.*

Prefractal approximations to recursive fractals and homogeneous fractals are built by assigning labels to nodes, beginning with the root node and working upwards through the tree. If the labels of ω are chosen in an iid manner according to P, except that in the homogeneous case all labels at each fixed level are same, then one obtains a *random recursive fractal* and a *random homogeneous fractal* respectively. For V-variable fractals with $V > 1$, one cannot directly build the appropriate labelled trees and so obtain a natural probability distribution in this manner.

However, one can do something similar by building certain V-tuples of labelled trees as follows, see Figure 3. There are natural maps Φ^a from infinite, or level k, V-tuples of labelled trees $(\omega_1, \ldots, \omega_V)$ to infinite, or level $k + 1$, V-tuples of labelled trees $(\omega'_1, \ldots, \omega'_V)$, respectively. That is,

$$(1.1) \qquad\qquad (\omega'_1, \ldots, \omega'_V) = \Phi^a(\omega_1, \ldots, \omega_V).$$

The label at the base node of each ω'_v is determined by Φ^a and is either F or G. The three subtrees of each ω'_v rooted at level one are also determined by Φ^a and taken from the set $\{\omega_1, \ldots, \omega_V\}$, possibly with repetition.

$$a_0 = \begin{bmatrix} G & 2 & 1 & 3 \\ F & 2 & 3 & 1 \\ G & 2 & 4 & 2 \\ F & 3 & 4 & 4 \end{bmatrix} \qquad a_1 = \begin{bmatrix} G & 1 & 2 & 1 \\ F & 1 & 3 & 4 \\ G & 2 & 3 & 3 \\ G & 2 & 4 & 3 \end{bmatrix}$$
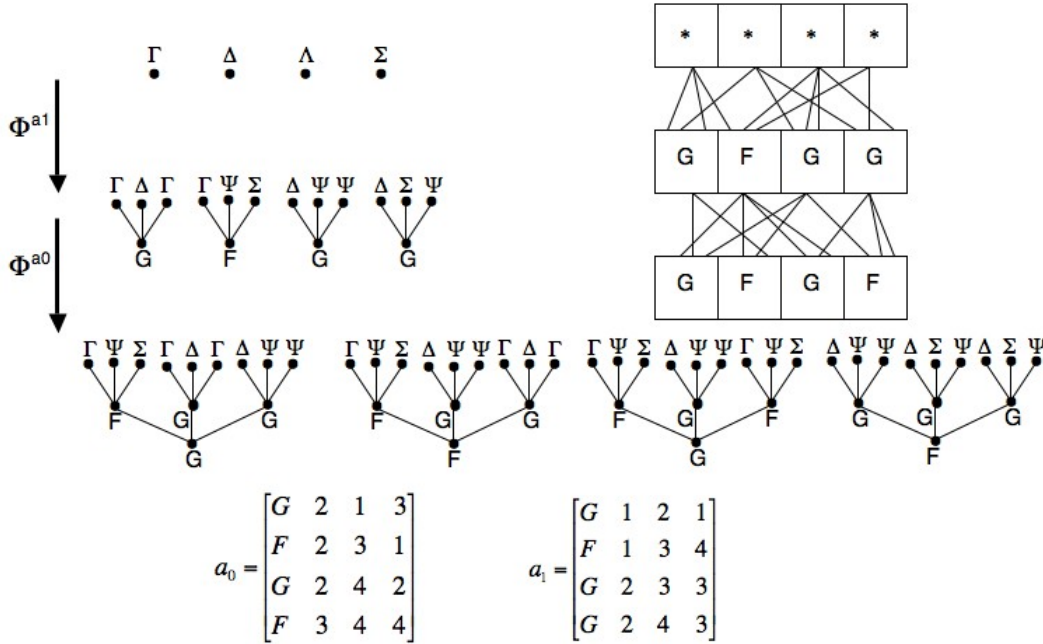
FIGURE 3. The map Φ^{a_0} is described by the array a_0. It is also described by the labels in the level 0 (bottom) boxes, and the network of lines between these boxes and the level 1 (middle) boxes. For example, row 3 of a_0 is $G, 2, 4, 2$. It contains the information that the third component of $\Phi^{a_0}(\omega_1, \omega_2, \omega_3, \omega_4)$ is the tree whose root node is labelled G, and whose three subtrees, rooted at the next level, are ω_2, ω_4 and ω_2 respectively. This information is also provided by the facts that box 3 at level 0 contains the symbol G, and the three lines from this box in the order left to right connect to boxes 2, 4 and 2 at level 1. Similar remarks apply to Φ^{a_1}.

The maps Φ^a, or equivalently the indices a, are described by $V \times (M+1)$ arrays, where $M = 3$ here, and in general M bounds the number of functions in each IFS from the family $\boldsymbol{F}$, see Figure 3. Let $\mathcal{A}_V$ *denote the set of all such indices* a, corresponding to the family $\boldsymbol{F}$. The cardinality of $\mathcal{A}_V$ in the example here is $2^V(3V)^V$.

The maps $\mathcal{F}^a$ act on V-tuples of compact subsets of $\mathbb{R}^n$ in an analogous manner, see Figure 4.

If $\boldsymbol{a} = a_0 a_1 \ldots a_k \ldots \in \mathcal{A}_V^\infty$ then

$$(1.2) \qquad \begin{aligned} (\omega_1^{\boldsymbol{a}}, \ldots, \omega_V^{\boldsymbol{a}}) &:= \lim_{k \to \infty} \Phi^{a_0} \circ \cdots \circ \Phi^{k-1}(\omega_1^0, \ldots, \omega_V^0), \\ (K_1^{\boldsymbol{a}}, \ldots, K_V^{\boldsymbol{a}}) &:= \lim_{k \to \infty} \mathcal{F}^{a_0} \circ \cdots \circ \mathcal{F}^{k-1}(K_1^0, \ldots, K_V^0). \end{aligned}$$

The limits exist and are independent of $(\omega_1^0, \ldots, \omega_V^0)$ and $(K_1^0, \ldots, K_V^0)$ respectively. Up to level k, the labelled trees $(\omega_1^{\boldsymbol{a}}, \ldots, \omega_V^{\boldsymbol{a}})$ depend only on $\Phi^{a_0}, \ldots, \Phi^{a_{k-1}}$, see Figure 4 where $k = 2$.

There are two ways of representing V-variable fractals. First, any V-variable fractal K can be represented by a labelled tree $\omega \in \Omega$. Second, and more useful here, the fractal sets $K_v^{\boldsymbol{a}}$ in (1.2) are also described by the infinite sequence $\boldsymbol{a}$. The labelled trees $\omega_v^{\boldsymbol{a}}$ and the fractal sets $K_v^{\boldsymbol{a}}$ are V-variable, and every V-variable tree and fractal set can be obtained in this manner. Not only is $(\omega_1^{\boldsymbol{a}}, \ldots, \omega_V^{\boldsymbol{a}})$ a V-tuple of V-variable trees, but it satisfies the stronger condition that, for each k, there are at most V isomorphism classes of trees rooted at level k, chosen from *all* the $\omega_1^{\boldsymbol{a}}, \ldots, \omega_V^{\boldsymbol{a}}$ taken together. A similar remark applies to $(K_1^{\boldsymbol{a}}, \ldots, K_V^{\boldsymbol{a}})$.

So far we have not required a probability distribution on the class Ω_V of V-variable trees or on the class $\mathcal{K}_V$ of V-variable fractals. However, there is a natural probability distribution P_V on $\mathcal{A}_V$ that is inherited from P. This induces a probability distribution on $\mathcal{A}_V^\infty$, thence from (1.2) on $(\Omega_V)^V \subset \Omega^V$ and $(\mathcal{K}_V)^V$, and thence on Ω_V and $\mathcal{K}_V$.
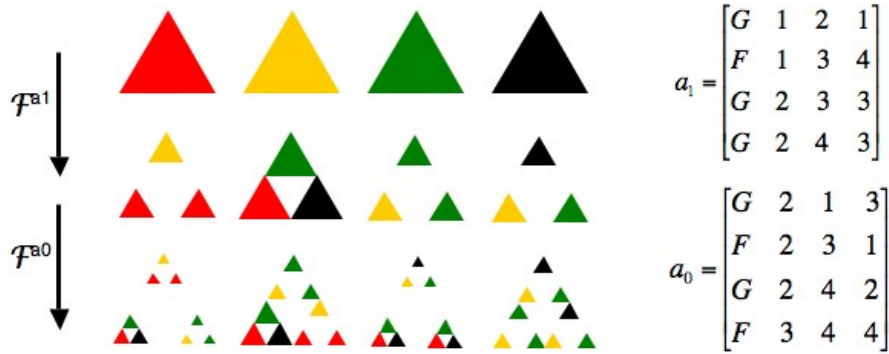
FIGURE 4. The map $\mathcal{F}^{a_1}$ followed by $\mathcal{F}^{a_0}$ act on $\mathcal{C}(\mathbb{R}^2)^4$, the space of 4-tuples of compact subsets of $\mathbb{R}^2$, in a manner which is encoded in the corresponding arrays a_0 and a_1. For example, row 3 of a_0 is $G, 2, 4, 2$ and contains the information that, for any (K_1, K_2, K_3, K_4), component 3 of $\mathcal{F}(K_1, K_2, K_3, K_4)$ will be $g_1(K_2) \cup g_2(K_4) \cup g_3(K_2)$. Note that a_0 and a_1 are the same as in Figure 3.

More precisely, all components of $a \in \mathcal{A}_V$ are chosen independently; those in the first column from $\boldsymbol{F}$ according to P and the remaining entries from $\{1, \ldots, V\}$ according to the uniform distribution. The resulting probability distribution P_V on $\mathcal{A}_V$ induces a probability distribution on $\mathcal{A}_V^\infty$, obtained by choosing the elements a_k of the sequence $\boldsymbol{a} = a_0 a_1 \ldots a_k \ldots$ in an iid manner according to P_V. The mapping $\mathcal{A}_V^\infty \to \Omega^V$ given by (1.2) now induces a probability distribution on $(\Omega_V)^V \subset \Omega^V$. The projected distribution on Ω_V in any coordinate direction is independent of the coordinate direction, and similarly for $\mathcal{K}_V$. The probability distribution on Ω_1 obtained in this manner is the same as the random homogeneous distribution. The distribution on Ω_V ($\subset \Omega$) converges to the random recursive distribution on Ω as $V \to \infty$. See [BHS08] for details.

Recall that the three fixed points of f_1, f_2 and f_3 are also the fixed points of g_1, g_2 and g_3; namely, the vertices of an equilateral triangle T of unit diameter. Suppose $K = K^\omega$ is a recursive Sierpinski triangle with labelled tree $\omega \in \Omega$. The 3^k scaled triangles in the k-level prefractal approximation provide an efficient cover of K for large k, see Figure 2 where $k = 3$, and Figure 4 where $k = 2$. In order to study the Hausdorff measure $\mathcal{H}^\alpha(K)$, it is natural to consider

$$S(\omega, k, \alpha) := \sum_{\{\boldsymbol{m} \in \omega : |\boldsymbol{m}| = k\}} |T_{\boldsymbol{m}}|^\alpha.$$

Here, on the right side, we are summing the diameters to the power α of the 3^k triangles in the level k prefractal for K, see also (3.4).

As in Figure 5, let $(K_1', \ldots, K_V') = \mathcal{F}^a(K_1, \ldots, K_V)$, where $(K_1, \ldots, K_V)$ has labelled trees $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_V)$ and $(K_1', \ldots, K_V')$ has labelled trees $\boldsymbol{\omega}' = (\omega_1', \ldots, \omega_V')$. By examining Figure 5, and setting $\boldsymbol{S}(\boldsymbol{\omega}, k, \alpha) = (S(\omega_1, k, \alpha), \ldots, S(\omega_V, k, \alpha))$, one can see that

$$(1.3) \qquad\qquad \boldsymbol{S}(\boldsymbol{\omega}', k, \alpha) = M^a(\alpha) \boldsymbol{S}(\boldsymbol{\omega}, k-1, \alpha),$$

where, in general, $M^a(\alpha)$ is the $V \times V$ matrix constructed from a as in Definition 3.2. See also Proposition 3.1. If $\boldsymbol{\omega^a} = (\omega_1^{\boldsymbol{a}}, \ldots, \omega_V^{\boldsymbol{a}})$ and $(K_1^{\boldsymbol{a}}, \ldots, K_V^{\boldsymbol{a}})$ are as in (1.2), it follows on iterating (1.3) that

$$\boldsymbol{S}(\boldsymbol{\omega^a}, k, \alpha) = M^{a_0} \circ \cdots \circ M^{a_{k-1}} \mathbf{1}.$$

The vector $\mathbf{1}$ is a vector of units, see also Proposition 3.3.

For a matrix M, let $\|M\|$ be obtained by summing the absolute values of all components. It is now plausible that there is a unique d such that $\|M^{a_0} \circ \cdots \circ M^{a_{k-1}}\|$ should diverge exponentially fast to $+\infty$ a.s. for $\alpha < d$, decay exponentially fast to 0 a.s. for $\alpha > d$, and that this d should be the a.s. Hausdorff dimension of $K_1^{\boldsymbol{a}}, \ldots, K_V^{\boldsymbol{a}}$. We show this is the case in the main theorem, see Sections 3 and 4. In Section 5 we compute some examples.

$$a = \begin{bmatrix} G & 1 & 2 & 1 \\ F & 1 & 3 & 4 \\ G & 2 & 3 & 3 \\ G & 2 & 4 & 3 \end{bmatrix} \qquad M^a(\alpha) = \begin{bmatrix} 2\left(\frac{1}{3}\right)^\alpha & \left(\frac{1}{3}\right)^\alpha & 0 & 0 \\ \left(\frac{1}{2}\right)^\alpha & 0 & \left(\frac{1}{2}\right)^\alpha & \left(\frac{1}{2}\right)^\alpha \\ 0 & \left(\frac{1}{3}\right)^\alpha & 2\left(\frac{1}{3}\right)^\alpha & 0 \\ 0 & \left(\frac{1}{3}\right)^\alpha & \left(\frac{1}{3}\right)^\alpha & \left(\frac{1}{3}\right)^\alpha \end{bmatrix}$$
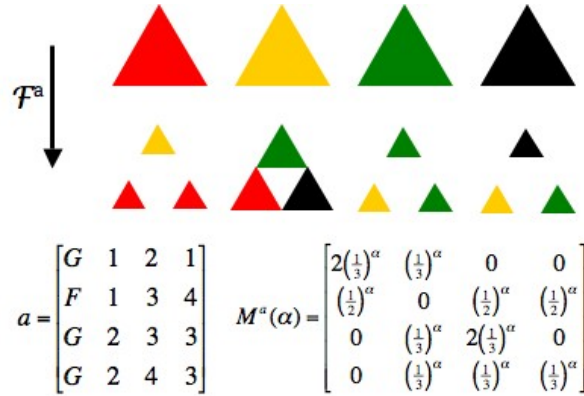
FIGURE 5. The first row represents fractals $(K_1, \ldots, K_4)$, or prefractals of level $\geq k-1$, corresponding to labelled trees $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_4)$. The second row represents $(K'_1, \ldots, K'_4) = \mathcal{F}^a(K_1, \ldots, K_4)$ with labelled trees $\boldsymbol{\omega}' = (\omega'_1, \ldots, \omega'_4)$. If the vth component of $\boldsymbol{S}(\boldsymbol{\omega}, k-1, \alpha)$ is the $k-1$ level approximation to $\mathcal{H}^\alpha(K_v)$ then, in the case here, $S(\omega'_3, k, \alpha) = \left(\frac{1}{3}\right)^\alpha S(\omega_2, k-1, \alpha) + 2\left(\frac{1}{3}\right)^\alpha S(\omega_3, k-1, \alpha)$. Similarly, $\boldsymbol{S}(\boldsymbol{\omega}', k, \alpha) = M^a(\alpha)\boldsymbol{S}(\boldsymbol{\omega}, k-1, \alpha)$.

Results for walk and spectral dimensions of V-variable fractals, and their properties, have been obtained in work by Uta Freiberg and Hutchinson.

We are interested in V-variable fractals for the following reasons, see [BHS08, BHS05].

(1) Families of V-variable fractals, together with their probability distributions, interpolate between random homogeneous fractals and random recursive fractals.
(2) The families of functions $(\Phi^{\boldsymbol{a}}, a \in \mathcal{A}_V, P_V)$ and $(\mathcal{F}^{\boldsymbol{a}}, a \in \mathcal{A}_V, P_V)$, including the probability distribution P_V on $\mathcal{A}_V$, each form a standard IFS, one acting on Ω^V and the other on $\mathcal{C}(\mathbb{R}^2)^V$. The attractors, projected in any of the V component directions, give the set of V-variable fractal trees and sets, together with their probability distributions.
(3) The chaos game for these IFSs can be used to generate a sample of V-variable fractals, whose empirical distribution will approach the probability distribution on V-variable fractals.
(4) Analogous results apply to V-variable fractal measures under weak local contractive conditions. Such conditions are natural, for example, in modelling stochastic processes where individual sample paths may be bounded, but there is no uniform bound. In this case one has an IFS operating on a non locally compact state space. But the chaos game result can be extended to this setting.
(5) By taking large V the chaos game gives a fast forward process for the generation of a sample of fractals approximating random recursive fractals and their probability distribution. This is of practical interest, since normally one builds individual examples of random recursive fractals by a computationally expensive backward process.

2. NOTATION AND ASSUMPTIONS

The *diameter* of a set A is denoted by $|A|$. The *Hausdorff measure* of A in the dimension α is denoted by $\mathcal{H}^\alpha(A)$. The *Hausdorff dimension* of A is denoted by $\dim_{\mathcal{H}}(A)$.

Fix a family $\boldsymbol{F} = \{F^\lambda : \lambda \in \Lambda\}$ of IFSs acting on $\mathbb{R}^n$, where $F^\lambda = (\mathbb{R}^n; f_1^\lambda, \ldots, f_{M_\lambda}^\lambda)$, each $f_m^\lambda : \mathbb{R}^n \to \mathbb{R}^n$ with Lipschitz constant $r_m^\lambda < 1$, and Λ may be infinite. Fix a probability distribution P on Λ, or equivalently on $\boldsymbol{F}$.

Assume

(2.1) $$0 < r_{\min} \leq r_m^\lambda \leq r_{\max} < 1, \quad M := \max_{\lambda \in \Lambda} M_\lambda < \infty.$$

We usually further assume the $f_m^\lambda : \mathbb{R}^n \to \mathbb{R}^n$ are *similitudes*. That is,

$$(2.2) \qquad |f_m^\lambda(x) - f_m^\lambda(y)| = r_m^\lambda |x - y| \quad \forall x, y \in \mathbb{R}^n.$$

In this case we will also assume that $\boldsymbol{F}$ satisfies a *uniform open set condition*. That is, there exists a non empty open set $\mathcal{O}$ such that

$$(2.3) \qquad \bigcup_{m=1}^{M_\lambda} f_m^\lambda(\mathcal{O}) \subset \mathcal{O}, \quad f_m^\lambda(\mathcal{O}) \cap f_n^\lambda(\mathcal{O}) = \emptyset \text{ if } m \neq n \text{ and } \lambda \in \Lambda.$$

We note explicitly when (2.2) and (2.3) apply.

2.1. Labelled trees.
We characterise the *set Ω of labelled trees* ω, corresponding to the family $\boldsymbol{F}$ of IFSs, as follows:

A *tree* is a set ω of finite sequences $\boldsymbol{m} = m_1 \ldots m_k$ where each $m_j \in \{1, \ldots, M\}$, together with the empty sequence $\emptyset$. The set ω is closed under the operation of taking initial segments. Sequences in ω are called *nodes*. We write $\boldsymbol{m} \in \omega$ if $\boldsymbol{m}$ is a node of ω.

The *level* $|\boldsymbol{m}|$ of the node $\boldsymbol{m} = m_1 \ldots m_k \in \omega$ is defined to be $|\boldsymbol{m}| = k$, while $|\emptyset| = 0$.

A *labelled tree*, corresponding to the family of IFSs $\boldsymbol{F}$, is a tree ω together with a map from the set of nodes of ω into Λ. This map is also denoted by ω. If $\boldsymbol{m} \in \omega$ and $\omega(\boldsymbol{m}) = \lambda$ then the nodes of ω whose immediate predecessor is $\boldsymbol{m}$ are precisely $\boldsymbol{m}1, \ldots, \boldsymbol{m}M_\lambda$. Notice that the number of edges rooted at a node equals the number of functions in the IFS given by the label at that node.

Let Ω *denote the set of all labelled trees* which correspond to $\boldsymbol{F}$. We frequently refer to a labelled tree simply as a tree.

A *V-variable labelled tree* $\omega \in \Omega$ is a labelled tree such that, for each level k, there are at most V non isomorphic subtrees of ω rooted at level k. Let $\Omega_V \subset \Omega$ *denote the set of all V-variable labelled trees* which correspond to $\boldsymbol{F}$.

2.2. Approximating fractal sets.
For $E \subset \mathbb{R}^n$, $\omega \in \Omega$ and nodes $\boldsymbol{m} = m_1 \ldots m_k \in \omega$, define

$$(2.4) \quad E_\emptyset = E^0 = E, \quad E_{m_1 \ldots m_k} = f_{m_1}^{\omega(\emptyset)} \circ f_{m_2}^{\omega(m_1)} \circ \cdots \circ f_{m_k}^{\omega(m_1 \ldots m_{k-1})}(E), \quad E^k = \bigcup_{|\boldsymbol{m}|=k} E_{m_1 \ldots m_k}.$$

When we need to make the dependence on ω explicit we will write $E_{\boldsymbol{m}}(\omega)$ or $E^k(\omega)$, for example.

For $\mathcal{O}$ as in (2.3) we have

$$
\begin{aligned}
&\mathcal{O} \supset \mathcal{O}_{m_1} \supset \cdots \supset \mathcal{O}_{m_1 \ldots m_k} \supset \cdots, \quad \overline{\mathcal{O}} \supset \overline{\mathcal{O}}_{m_1} \supset \cdots \supset \overline{\mathcal{O}}_{m_1 \ldots m_k} \supset \cdots, \\
&\mathcal{O}_{\boldsymbol{m}} \cap \mathcal{O}_{\boldsymbol{n}} = \emptyset \quad \text{if } |\boldsymbol{m}| = |\boldsymbol{n}|, \boldsymbol{m} \neq \boldsymbol{n}, \\
&\mathcal{O} \supset \mathcal{O}^1 \supset \cdots \supset \mathcal{O}^k \supset \cdots, \quad \overline{\mathcal{O}} \supset \overline{\mathcal{O}}^1 \supset \cdots \supset \overline{\mathcal{O}}^k \supset \cdots, \\
&K^\omega := \bigcap_{k \geq 0} \overline{\mathcal{O}}^k(\omega).
\end{aligned}
$$

(2.5)

The set K^ω *is the fractal set corresponding to ω.* Even if the open set condition does not apply, we obtain the same set K^ω by replacing $\overline{\mathcal{O}}$ by any non empty compact set E for which

$$(2.6) \qquad \bigcup_{m=1}^{M_\lambda} f_m^\lambda(E) \subset E \quad \forall \lambda \in \Lambda.$$

The set $\mathcal{K}_V$ of V-variable fractals sets is defined by

$$\mathcal{K}_V = \{K^\omega : \omega \in \Omega_V\}.$$

3. The Hausdorff dimension of V-variable fractals

3.1. **The maps Φ^a and $\mathcal{F}^a$.** (See the examples in Figures 3, 4, 5 and 6.) The index set $\mathcal{A}_V$ is defined to be the set of all arrays a of the form

$$
(3.1) \qquad a = \begin{bmatrix} I^a(1) & J^a(1,1) & \dots & J^a(1,M) \\ \vdots & \vdots & \ddots & \vdots \\ I^a(V) & J^a(V,1) & \dots & J^a(V,M) \end{bmatrix},
$$

where $I^a(v) \in \Lambda$, M is as in (2.1), $J^a(v,m) \in \{1,\dots,V\}$ if $1 \leq m \leq M_{I^a(v)}$ and $J^a(v,m) = 0$ if $M_{I^a(v)} < m \leq M$. See also [BHS08, Section 5.3].

The probability distribution P_V on $\mathcal{A}_V$ is obtained by selecting all elements of a independently. The elements in the first column are chosen according to P. The elements $J^a(v,m)$ for $1 \leq m \leq M_{I^a(v)}$ are chosen according to the uniform distribution on $\{1,\dots,V\}$. Any remaining elements $J^a(v,m)$ for $m > M_{I^a(v)}$ are set equal to 0.

The map $\Phi_V^a : \Omega^V \to \Omega^V$ is defined by requiring, for any V-tuple $(\omega_1,\dots,\omega_V) \in \Omega^V$, that the vth component of $\Phi^a(\omega_1,\dots,\omega_V) \in \Omega^V$ is the labelled tree whose base node is labelled $F^{I^a(v)}$, and whose mth subtree rooted at level one is $\omega_{J^a(v,m)}$ for $1 \leq m \leq M_{I^a(v)}$. Any zeros at the end of each row of a are markers so that all rows have equal length, but otherwise play no role.

Similarly, for any V-tuple of sets $(K_1,\dots,K_V)$, the map $\mathcal{F}^{a_V} : \mathcal{C}(\mathbb{R}^n) \to \mathbb{R}^n)$ is defined by

$$
(3.2) \qquad \mathcal{F}^a(K_1,\dots,K_V) = \left(\bigcup_{m=1}^{M_{I^a(1)}} f_m^{I^a(1)}(K_{J^a(1,m)}), \dots, \bigcup_{m=1}^{M_{I^a(V)}} f_m^{I^a(V)}(K_{J^a(V,m)}) \right).
$$

For $\boldsymbol{a} = a_0 \dots a_{k-1} \dots \in \mathcal{A}_V^\infty$, assuming (2.6) for some compact E, we define

$$
(3.3) \qquad \begin{aligned} (\omega_1^{\boldsymbol{a}},\dots,\omega_V^{\boldsymbol{a}}) &:= \lim_{k\to\infty} \Phi^{a_0} \circ \cdots \circ \Phi^{a_{k-1}}(\omega_1^0,\dots,\omega_V^0), \\ (K_1^{\boldsymbol{a}},\dots,K_V^{\boldsymbol{a}}) &:= \lim_{k\to\infty} \mathcal{F}^{a_0} \circ \cdots \circ \mathcal{F}^{a_{k-1}}(K_1^0,\dots,K_V^0). \end{aligned}
$$

The limits exist, and are independent of $(\omega_1^0,\dots,\omega_V^0)$ and $(K_1^0,\dots,K_V^0)$, respectively.

If $v \in \{1,\dots,V\}$ then $\omega_v^{\boldsymbol{a}} \in \Omega_V$, and every V-variable labelled tree can be obtained in this manner. However, if Ω_V^* is the set of V-tuples of labelled trees obtained in this manner then $\Omega_V^* \subsetneq (\Omega_V)^V$.

The probability distribution P_V^∞ on $\mathcal{A}_V^\infty$ is defined by selecting the a_k in an iid manner according to P_V. This induces a probability distribution on $\Omega_V^* \subset (\Omega_V)^V$ via (3.3), and thence a probability distribution ρ_V on Ω_V by projecting in any of the V coordinate directions. Similarly one obtains a probability distribution $\mathfrak{K}_V$ on $\mathcal{K}_V$. Both ρ_V and $\mathfrak{K}_V$ are independent of choice of projection direction, although the initial distributions are not product distributions. See [BHS08].

3.2. **Approximating Hausdorff measure.** Assume the open set condition (2.3), and without loss of generality assume $|\mathcal{O}| = 1$. (See Figure 2 and take $\mathcal{O}$ to be the interior of the triangle T.)

Suppose $\omega \in \Omega$ is a labelled tree with labels from $\boldsymbol{F}$. Keeping in mind (2.5), we think of the collection of sets $\{\overline{\mathcal{O}}_{\boldsymbol{m}} : \boldsymbol{m} \in \omega, |\boldsymbol{m}| = k\}$ as an "efficient" cover of K^ω for large k. In order to compute the Hausdorff dimension of K^ω, we consider the following quantities for $\alpha > 0$:

$$
r_\emptyset(\omega) := |\overline{\mathcal{O}}| = 1, \quad r_{m_1\dots m_k}(\omega) := |\overline{\mathcal{O}}_{m_1\dots m_k}| = r_{m_1}^{\omega(\emptyset)} \cdot \dots \cdot r_{m_k}^{\omega(m_1\dots m_{k-1})} \text{ for } m_1\dots m_k \in \omega,
$$

$$
(3.4) \quad S(\omega,k,\alpha) := \sum_{\{\boldsymbol{m}\in\omega:|\boldsymbol{m}|=k\}} |\overline{\mathcal{O}}_{\boldsymbol{m}}|^\alpha = \sum_{\{\boldsymbol{m}\in\omega:|\boldsymbol{m}|=k\}} (r_{\boldsymbol{m}}(\omega))^\alpha, \text{ noting } S(\omega,0,\alpha) = 1.
$$

We are interested in the behaviour of $S(\omega,k,\alpha)$ as $k \to \infty$ since, for large k, it is an approximation to the Hausdorff measure $\mathcal{H}^\alpha(K^\omega)$.

3.3. **Flow Matrices.** (See the example in Figure 5.) As remarked in the introduction, there are two ways of representing a V-variable fractal K. One can either use the corresponding labelled tree $\omega \in \Omega$, or one can use a sequence $\boldsymbol{a} \in (\mathcal{A}_V)^\infty$ as in (3.3) which generates a V-tuple containing K as a component. We use the latter in order to study $S(\omega,k,\alpha)$.

For $(\omega_1,\dots,\omega_V) \in \Omega^V$ define the following vector of real numbers:

$$
(3.5) \qquad \boldsymbol{S}\big((\omega_1,\dots,\omega_V),k,\alpha\big) = (S(\omega_1,k,\alpha),\dots,S(\omega_V,k,\alpha)).
$$

Then $\boldsymbol{S}\big((\omega_1, \ldots, \omega_V), 0, \alpha\big) = \boldsymbol{1}$, the V-vector whose components all equal 1.

Proposition 3.1. *If $(\omega_1, \ldots, \omega_V) \in \Omega^V$, and $(\omega'_1, \ldots, \omega'_V) = \Phi^a(\omega_1, \ldots, \omega_V)$, then*

$$S(\omega'_v, k, \alpha) = \sum_{w=1}^{V} \left(\sum_{\{m : J^a(v,m) = w\}} \left(r_m^{I^a(v)} \right)^\alpha \right) S(\omega_w, k-1, \alpha).$$

Proof. For any $\omega \in \Omega$ it follows from (3.4) that

$$r_{m_1 \ldots m_k}(\omega) = r_{m_1}^{\omega(\emptyset)} r_{m_2 \ldots m_k}(\omega^{(m_1)}),$$

where $\omega^{(m_1)}$ is the labelled subtree of ω rooted at node m_1 at level one. From the definition of $\Phi^a(\omega_1, \ldots, \omega_V)$ in Section 3.1, it follows that

$$r_{m_1 \ldots m_k}(\omega'_v) = r_{m_1}^{I^a(v)} r_{m_2 \ldots m_k}(\omega_{J^a(v,m_1)}).$$

Hence,

$$S(\omega'_v, k, \alpha) = \sum_{m_1 \ldots m_k \in \omega'_v} \left(r_{m_1}^{I^a(v)} \right)^\alpha \left(r_{m_2 \ldots m_k}(\omega_{J^a(v,m_1)}) \right)^\alpha \quad \text{from (3.4) and the above}$$

$$= \sum_{m_1 = 1}^{M_{I^a(v)}} \left(\left(r_{m_1}^{I^a(v)} \right)^\alpha \sum_{m_2 \ldots m_k \in \omega_{J^a(v,m_1)}} \left(r_{m_2 \ldots m_k}(\omega_{J^a(v,m_1)}) \right)^\alpha \right)$$

$$= \sum_{w=1}^{V} \sum_{\{m_1 : J^a(v,m_1) = w\}} \left(\left(r_{m_1}^{I^a(v)} \right)^\alpha \sum_{m_2 \ldots m_k \in \omega_w} \left(r_{m_2 \ldots m_k}(\omega_w) \right)^\alpha \right)$$

$$= \sum_{w=1}^{V} \left(\sum_{\{m : J^a(v,m) = w\}} \left(r_m^{I^a(v)} \right)^\alpha \right) S(\omega_w, k-1, \alpha).$$

$\square$

Motivated by this we make the following definition.

Definition 3.2. The $V \times V$ *flow matrix* $M^a = M^a(\alpha)$ for $a \in \mathcal{A}_V$, is defined by

$$M_{vw}^a = \sum_{\{m : J^a(v,m) = w\}} \left(r_m^{I^a(v)} \right)^\alpha, \quad 1 \le v, w \le V.$$

Flow matrices are the main book keeping tool for tracking the size of covers of V-variable fractals.

Proposition 3.3. *Suppose $\boldsymbol{a} \in \mathcal{A}_V^\infty$, and $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_V) = \boldsymbol{\omega}^{\boldsymbol{a}} \in \Omega^V$ has address $\boldsymbol{a}$ as in (3.3). Then,*

$$\boldsymbol{S}(\boldsymbol{\omega}^{\boldsymbol{a}}, k, \alpha) = M^{a_0} \circ \cdots \circ M^{a_{k-1}} \boldsymbol{1}, \quad i.e.\ S(\omega_v^{\boldsymbol{a}}, k, \alpha) = \sum_{w=1}^{V} \left(M^{a_0} \circ \cdots \circ M^{a_{k-1}} \right)_{vw} \text{ for } 1 \le v \le V.$$

Proof. From Proposition 3.1 and Definition 3.2, for any $\boldsymbol{\omega} \in \Omega^V$,

$$\boldsymbol{S}\big(\Phi^a(\boldsymbol{\omega}), k, \alpha\big) = M^a \, \boldsymbol{S}(\boldsymbol{\omega}, k-1, \alpha).$$

Hence,

$$\boldsymbol{S}\big(\Phi^{a_0} \circ \cdots \circ \Phi^{a_{k-1}}(\boldsymbol{\omega}), k, \alpha\big) = M^{a_0} \circ \cdots \circ M^{a_{k-1}} \, \boldsymbol{S}(\boldsymbol{\omega}, 0, \alpha)$$

$$= M^{a_0} \circ \cdots \circ M^{a_{k-1}} \, \boldsymbol{1}.$$

Since $\boldsymbol{\omega}^{\boldsymbol{a}}$ is of the form $\Phi^{a_0} \circ \cdots \circ \Phi^{a_{k-1}}(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$, the result follows. $\square$

For fixed $\boldsymbol{a} = a_0 \ldots a_{k-1} \ldots$ and $\boldsymbol{\omega}$ as in Proposition 3.3, we often write

(3.6) $$S_v(k, \alpha) = S(\omega_v, k, \alpha).$$

Note that $S_v(k, \alpha)$ depends only on $a_0, \ldots, a_{k-1}$, and not on a_j for $j \ge k$.

3.4. **Computing the Hausdorff dimension.** If A is a matrix we define the norm $\|A\| := \sum_{v,w} |A_{vw}|$. It is easily checked that this norm is submultiplicative, $\|AB\| \leq \|A\| \cdot \|B\|$.

MAIN THEOREM. *Fix $\boldsymbol{F} = \{F^\lambda : \lambda \in \Lambda\}$, a probability distribution P on Λ, an integer $V \geq 1$ and a real number $\alpha > 0$. Under the assumption (2.1),*

$$\gamma(\alpha) := \lim_{k \to \infty} \mathbb{E} \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| = \lim_{k \to \infty} \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| \ a.s.$$

In particular, the second limit is independent of $\boldsymbol{a} = a_0 \ldots a_{k-1} \ldots$ for P_V^∞ a.e. $\boldsymbol{a} \in \mathcal{A}_V^\infty$. The function $\gamma(\alpha)$ is monotonically decreasing in α, and there is a unique d such that $\gamma(d) = 0$.

Assuming (2.2), and the open set condition (2.3), $\dim_{\mathcal{H}}(K^\omega) = d$ for ρ_V a.e. $\omega \in \Omega_V$.

This theorem is proved in Section 4. For an example see Section 5.

Remark 3.4. The limit $\gamma(\alpha)$ is sometimes called a "Lyapunov exponent", since $\|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\|$ grows like $e^{k\gamma(\alpha)}$ as $k \to \infty$.

The fact $\gamma(\alpha)$ exists and is independent of $\boldsymbol{a}$, for a.e. $\boldsymbol{a} \in \mathcal{A}_V^\infty$, is a consequence of the version of the Furstenberg Kesten theorem [FK60] in [Coh88, Theorem C, p72]. Individual terms $\frac{1}{k} \log(M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha))_{vw}$ will not normally converge as $k \to \infty$. In particular, for fixed v and w, $(M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha))_{vw} = 0$ infinitely often a.s. In fact, if $J^{a_{k-1}}(u, m) \neq w$ for all u and m, which happens with positive probability, then all entries in the w column of $M^{a_{k-1}}(\alpha)$ are zero, and hence all entries in the w column of $M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)$ are also zero. However, $\lim_{k \to \infty} \frac{1}{k} \log S_v(k, \alpha)$ exists a.s. and equals $\gamma(\alpha)$, for every $v \in \{1, \ldots, V\}$, see Lemma 4.4.

Remark 3.5. Assuming the open set condition it follows, for random homogeneous fractals K^ω (the $V = 1$ case), that

$$(3.7) \qquad \dim_{\mathcal{H}}(K^\omega) \text{ is the unique } \alpha \text{ for which } \mathbb{E}_\lambda \log \sum_{m=1}^{M_\lambda} (r_m^\lambda)^\alpha = 0, \text{for a.e. } \omega.$$

See [Ham92] for the direct computation of the dimension in some particular cases. For random recursive fractals K^ω, the "$V \to \infty$" case,

$$(3.8) \qquad \dim_{\mathcal{H}}(K) \text{ is the unique } \alpha \text{ for which } \mathbb{E}_\lambda \sum_{m=1}^{M_\lambda} (r_m^\lambda)^\alpha = 1, \text{for a.e. } \omega.$$

See [Fal86, MW86, Gra87, Gra91].

4. Proof of the Main Theorem

The proof is broken into a number of lemmas.

Assumptions. We continue the assumptions from the beginning of Section 2, and the notation from Sections 2 and 3. The integer $V \geq 1$ is fixed and α is non negative.

Define

$$(4.1) \qquad R_{\min}(\alpha) = \inf_\lambda \sum_{m=1}^{M_\lambda} (r_m^\lambda)^\alpha, \quad R_{\max}(\alpha) = \sup_\lambda \sum_{m=1}^{M_\lambda} (r_m^\lambda)^\alpha.$$

The sequence $\boldsymbol{a} = a_0 \ldots a_{k-1} \ldots \in \mathcal{A}_V$ is chosen according to P_V^∞ as in Section 3.1. The V-tuple of fractal sets corresponding to $\boldsymbol{a}$ is denoted $\boldsymbol{K^a}$, and the corresponding V-tuple of labelled trees is $\boldsymbol{\omega^a}$. Let

$$(4.2) \qquad \begin{aligned} \boldsymbol{K^a} &= (K_1^{\boldsymbol{a}}, \ldots, K_V^{\boldsymbol{a}}) = (K_1, \ldots, K_V), \\ \boldsymbol{\omega^a} &= (\omega_1^{\boldsymbol{a}}, \ldots, \omega_V^{\boldsymbol{a}}) = (\omega_1, \ldots, \omega_V). \end{aligned}$$

From the next lemma, it follows there is a unique d such that as $k \to \infty$ the product $\|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\|$ grows exponentially fast to ∞ if $\alpha < d$, and decays exponentially fast to 0 if $\alpha > d$. This does not imply convergence to a non zero limit or boundedness of $\|M^{a_0}(d) \cdot \ldots \cdot M^{a_{k-1}}(d)\|$, but it does imply that any infinite growth, or decay to zero, should be slower than exponential.

Lemma 4.1. *The limit*

$$\lim_{k \to \infty} \mathbb{E} \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| =: \gamma(\alpha)$$

exists. In addition,

$$\lim_{k \to \infty} \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| = \gamma(\alpha) \ a.s.,$$

and in particular is a.s. independent of $\boldsymbol{a}$. *Moreover,*

$$\log R_{\min}(\alpha) \le \gamma(\alpha) \le \log R_{\max}(\alpha).$$

The function γ *is strictly decreasing, Lipschitz, has derivative in the interval* $[\log r_{\min}, \log r_{\max}]$, $\gamma(0) > 0$ *and* $\gamma(\alpha) \to -\infty$ *as* $\alpha \to \infty$. *In particular, there is a unique* d *such that* $\gamma(d) = 0$.

Proof. The first two claims in the lemma hold for some $\gamma(\alpha)$ with $-\infty \le \gamma(\alpha) < \infty$. This follows from the version of the Furstenberg Kesten theorem in [Coh88, Theorem C p72], since the a_k are chosen in an iid manner.

Suppose A and B are square matrices of the same size with non negative entries. Assume

$$\alpha_1 \le \sum_j A_{ij} \le \alpha_2, \ \beta_1 \le \sum_j B_{ij} \le \beta_2, \ \text{for all } i.$$

Let $C = AB$. Since $\sum_j C_{ij} = \sum_k A_{ik} \left(\sum_j B_{kj} \right)$,

$$\alpha_1 \beta_1 \le \sum_j C_{ij} \le \alpha_2 \beta_2 \text{ for all } i.$$

In particular, from Definition 3.2 and (4.1),

$$R_{\min}^k(\alpha) \le \sum_w \left(M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha) \right)_{vw} \le R_{\max}^k(\alpha) \text{ for all } v,$$

and so

$$V R_{\min}^k(\alpha) \le \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| \le V R_{\max}^k(\alpha).$$

Taking logs of both sides, and letting $k \to \infty$, gives the third claim in the lemma.

From Definition 3.2, if $0 \le \alpha < \beta$,

$$r_{\min}^{k(\beta-\alpha)} \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| \le \|M^{a_0}(\beta) \cdot \ldots \cdot M^{a_{k-1}}(\beta)\| \le r_{\max}^{k(\beta-\alpha)} \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\|.$$

Taking logs and letting $k \to \infty$,

$$(\beta - \alpha) \log r_{\min} \le \gamma(\beta) - \gamma(\alpha) \le (\beta - \alpha) \log r_{\max}.$$

Hence γ is Lipschitz, differentiable a.e., monotonically decreasing to $-\infty$, and has derivative in the range $\log r_{\min}$ to $\log r_{\max}$.

Since

$$\gamma(0) \in [\log R_{\min}(0), \log R_{\max}(0)] = [\log(\min_\lambda M_\lambda), \log M = \log(\max_\lambda M_\lambda)],$$

it follows that $\gamma(0) > 0$, and so there exists a unique d such that $\gamma(d) = 0$. $\qquad\square$

In the next following lemmas, the notion of a "neck" $a \in \mathcal{A}_V$ will play an important role, see Figure 6.

Definition 4.2. The element $a \in \mathcal{A}_V$ is a *neck* if all $J^a(v, m)$ are equal for $v \in \{1, \ldots, V\}$ and if $m \in \{1, \ldots, M_{I^a}(v)\}$.

A neck occurs at level k in $\boldsymbol{a} = a_0 \ldots a_{k-1} \ldots \in \mathcal{A}_V^\infty$, or more simply we say that k *is a neck*, if a_{k-1} is a neck.

Remark 4.3. An element a chosen according to P_V is a neck with probability at least V^{1-MV}. It follows that necks in a sequence $\boldsymbol{a} \in \mathcal{A}_V^\infty$ occur infinitely often a.s.

If a neck occurs in $\boldsymbol{a}$ at level k then all subtrees of $(\omega_1^{\boldsymbol{a}}, \ldots, \omega_V^{\boldsymbol{a}})$ rooted at level k are equal. That is, if $|\boldsymbol{m}| = |\boldsymbol{m}'| = k$ and $v, v' \in \{1, \ldots, V\}$, then $\boldsymbol{mn} \in \omega_v$ iff $\boldsymbol{m'n} \in \omega_{v'}$, and in this case $\omega_v(\boldsymbol{mn}) = \omega_{v'}(\boldsymbol{m'n})$.
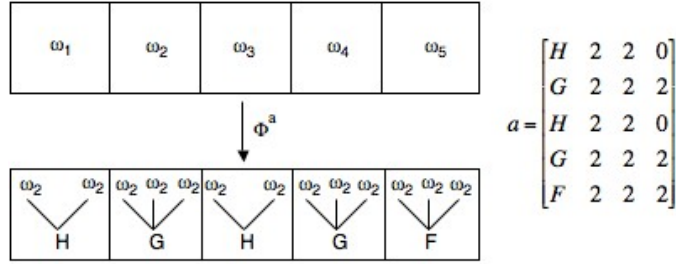
FIGURE 6. A neck a with IFSs F, G and H, and with $J^a(v,m) = 2$ or 0 for all v and m.

For the following lemma recall that $S_v(k,\alpha)$ is the sum of the elements in the vth row of $M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)$. From (3.4) we have that $S(\omega, k, \alpha)$, for large k, is an approximation to the Hausdorff measure $\mathcal{H}^\alpha(K^\omega)$. See also Remark 3.4.

Lemma 4.4. *For $\boldsymbol{a} \in \mathcal{A}_V^\infty$ and $\boldsymbol{\omega} = \boldsymbol{\omega^a} = (\omega_1, \ldots, \omega_V)$,*

$$(4.3) \qquad \lim_{k \to \infty} \frac{1}{k} \log S_v(k, \alpha) = \gamma(\alpha) \text{ a.s.}$$

Proof. Let $S_v(k, \alpha) = S(\omega_v, k, \alpha)$, as in (3.6).

Suppose the address sequence $\boldsymbol{a}$ has a *neck* at level p, where $J^{a_{p-1}}(v, m) = u$ for all $v \in \{1, \ldots, V\}$ and $m \in \{1, \ldots, M_{I^{a_{p-1}}(v)}\}$. It follows that all columns of $M^{a_{p-1}}$ are zero, except for the uth column, and hence the same is true for $M^{a_0} \circ \cdots \circ M^{a_{p-1}}$.

Suppose A and B are $V \times V$ matrices, such that all columns of A are zero except for the uth column, which we denote by $\boldsymbol{a}$. Let $\boldsymbol{b}$ be the uth row of B. Then

$$AB = [\, b_1 \boldsymbol{a} \;\; b_2 \boldsymbol{a} \;\; \ldots \;\; b_V \boldsymbol{a} \,].$$

It follows that

$$\sum_w (AB)_{vw} = a_v \sum_w b_w = \left(\sum_w A_{vw} \right) \sum_w b_w.$$

In particular, the second factor is independent of v.

Apply this to

$$(4.4) \qquad S_v(k, \alpha) = \sum_w \left(M^{a_0} \circ \cdots \circ M^{a_{k-1}} \right)_{vw},$$

see Proposition 3.3 and (3.6), with

$$A = M^{a_0} \circ \cdots \circ M^{a_{p-1}}, \quad B = M^{a_p} \circ \cdots \circ M^{a_{k-1}}.$$

It follows that

$$(4.5) \qquad S_v(k, \alpha) = S_v(p, \alpha)\, g(k, \alpha),$$

where $g(k, \alpha)$ is independent of v, and where $S_v(p, \alpha) > 0$ for all v.

From (4.4), summing over v,

$$\| M^{a_0} \circ \cdots \circ M^{a_{k-1}} \| = \sum_v S_v(k, \alpha) = g(k, \alpha) \sum_v S_v(p, \alpha).$$

Hence from Lemma 4.1,

$$\gamma(\alpha) = \lim_{k \to \infty} \frac{1}{k} \log \| M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha) \| = \lim_{k \to \infty} \log \frac{1}{k} g(k, \alpha) \quad \text{a.s.}$$

Going back to (4.5), since $S_v(p, \alpha) > 0$, it follows that

$$\lim_{k \to \infty} \frac{1}{k} \log S_v(k, \alpha) = \gamma(\alpha) \text{ a.s.}$$

$\square$

Lemma 4.5. *If $\alpha > d$ with d as in Lemma 4.1, then $\mathcal{H}^\alpha(K_v^{\boldsymbol{a}}) = 0$ for $v \in \{1, \dots, V\}$ and for a.e. $\boldsymbol{a}$. In particular, $\dim_{\mathcal{H}}(K_v^{\boldsymbol{a}}) \leq d$.*

Proof. We usually drop the reference to $\boldsymbol{a}$ and write K_v for $K_v^{\boldsymbol{a}}$, and ω_v for $\omega_v^{\boldsymbol{a}}$.

Let E be any set such that $K_1 \cup \cdots \cup K_V \subset E$, and without loss of generality suppose $|E| = 1$. For $v \in \{1, \dots, V\}$ and $\boldsymbol{m} = m_1 \dots m_k \in \omega_v$, let

$$E_{v;m_1 \dots m_k} = f_{m_1}^{\omega_v(\emptyset)} \circ f_{m_2}^{\omega_v(m_1)} \circ \cdots \circ f_{m_k}^{\omega_v(m_1 \dots m_{k-1})}(E),$$

as in (2.4). Then

$$K_v \subset \bigcup_{\{\boldsymbol{m} \in \omega_v : |\boldsymbol{m}| = k\}} E_{v;\boldsymbol{m}},$$

and

$$S_v(k, \alpha) = \sum_{\{\boldsymbol{m} \in \omega : |\boldsymbol{m}| = k\}} |E_{v;\boldsymbol{m}}|^\alpha,$$

as in (3.6) and (3.4).

Since $\gamma(\alpha) < 0$, it follows from (4.3) that

(4.6) $$\lim_{k \to \infty} S_v(k, \alpha) = \lim_{k \to \infty} e^{k(\frac{1}{k} \log S_v(k, \alpha))} = 0 \text{ a.s.}$$

Hence $\mathcal{H}^\alpha(K_v) = 0$ a.s. $\qquad \square$

Lemma 4.6. *Assume $\boldsymbol{F}$ satisfies the open set condition. If $\alpha < d$, where d is as in Lemma 4.1, then $\mathcal{H}^\alpha(K_v^{\boldsymbol{a}}) > 0$ a.s. for $1 \leq v \leq V$. In particular, $\dim_{\mathcal{H}}(K_v^{\boldsymbol{a}}) \geq d$ a.s.*

Proof. Suppose $\alpha < d$. As before, $K_v = K_v^{\boldsymbol{a}}$ and $\omega_v = \omega_v^{\boldsymbol{a}}$.

For a.e. $\boldsymbol{a}$ and each $1 \leq v \leq V$, we construct a unit mass measure μ on K_v, such that for some c,

(4.7) $$\mu(B_r(x)) \leq cr^\alpha \text{ if } r > 0, \ x \in \mathbb{R}^n.$$

It then follows by the mass distribution principle [Fal03, p60] that $\mathcal{H}^\alpha(K_v) > 0$, and so $\dim(K_v) \geq d$.

A. *Properties of Necks.* For $\boldsymbol{a} \in \mathcal{A}_V^\infty$ and $k \geq 0$, let $n^{\boldsymbol{a}}(k) = n(k)$ denote the first level $\geq k$ at which a neck occurs.

Then we claim

(4.8) $$\forall \epsilon > 0 \ \exists N > 0 \text{ such that } \forall k \big(n(k) - k \leq N + \epsilon k\big) \text{ a.s.,}$$

where N will depend on $\boldsymbol{a}$.

To see this, fix $\epsilon > 0$ and $k > 0$, and let

$$E_k = \{\boldsymbol{a} : n^{\boldsymbol{a}}(k) - k > \epsilon k\}.$$

It follows from Remark 4.3 that

$$P_V^\infty(E_k) \leq \big(1 - V^{1-MV}\big)^{\epsilon k}.$$

Since $\sum_{k \geq 1} P_V^\infty(E_k) < \infty$, it follows from the Borel Cantelli lemma that, with probability one, E_k occurs for only finitely many k, and so $n(k) - k \leq \epsilon k$ for all k sufficiently large. Hence for some N depending on $\boldsymbol{a}$ and ϵ, $n(k) - k \leq N + \epsilon k$ for all k. This proves (4.8).

B. *Construction of ν and μ.* Suppose $\boldsymbol{a}$ is as in (4.8) and consider the tree $\boldsymbol{\omega}^{\boldsymbol{a}} = (\omega_1, \dots, \omega_V)$. For fixed v, a unit measure ν will first be constructed on the *set $\widetilde{\omega}_v$ of infinite paths through ω_v*.

For $\boldsymbol{m} \in \omega_v$ the corresponding *cylinder set*, a subset of $\widetilde{\omega}_v$, is defined by

$$[\boldsymbol{m}] = \{\boldsymbol{p} \in \widetilde{\omega}_v : \boldsymbol{m} \leq \boldsymbol{p}\},$$

where $\boldsymbol{m} \leq \boldsymbol{p}$ means that $\boldsymbol{m}$ is an initial segment of $\boldsymbol{p}$.

The weight function w is defined on cylinder sets by

(4.9) $$w([\boldsymbol{m}]) = r_{v;\boldsymbol{m}}^\alpha := \left(r_{m_1}^{\omega_v(\emptyset)} \cdot \ldots \cdot r_{m_k}^{\omega_v(m_1 \dots m_{k-1})}\right)^\alpha.$$

We define a unit mass measure ν on $\widetilde{\omega}_v$ by setting, if $\boldsymbol{m} \in \omega_v$ and $|\boldsymbol{m}| = k$ is a neck,

$$(4.10) \qquad \nu([\boldsymbol{m}]) = \frac{w([\boldsymbol{m}])}{\sum\{w([\boldsymbol{m}']) : |\boldsymbol{m}'| = k\}} = \frac{r_{v;\boldsymbol{m}}^\alpha}{\sum\{r_{v;\boldsymbol{m}'}^\alpha : \boldsymbol{m} \in \omega_v, |\boldsymbol{m}| = k\}}.$$

The expression for $\nu([\boldsymbol{m}])$, in case $|\boldsymbol{m}|$ is not a neck, can be found in (4.15).

In order to show this does define a (unit mass) measure on $\widetilde{\omega}_v$, first recall that $\boldsymbol{a}$ has necks of arbitrarily large size. We will prove that if $k \leq j$ are both necks, $\boldsymbol{m} \in \omega_v$ and $|\boldsymbol{m}| = k$, then ν satisfies the consistency condition

$$(4.11) \qquad \nu([\boldsymbol{m}]) = \sum\{\nu([\boldsymbol{n}]) : \boldsymbol{m} \leq \boldsymbol{n} \in \omega_v, |\boldsymbol{n}| = j\}.$$

Here, and elsewhere, $\boldsymbol{m} \leq \boldsymbol{n}$ means $\boldsymbol{m}$ is an initial segment of the *finite* sequence $\boldsymbol{n}$.

Note that $[\boldsymbol{m}] = \bigcup\{[\boldsymbol{n}] : \boldsymbol{m} \leq \boldsymbol{n} \in \omega_v, |\boldsymbol{n}| = j\}$, and this is a union of disjoint sets. It follows from (4.11) that ν extends to a unit mass measure on the σ-algebra of subsets of $\widetilde{\omega}_v$ generated by the cylinder sets $[\boldsymbol{m}]$ for which $|\boldsymbol{m}|$ is a neck. This is just the σ-algebra generated by all cylinder sets, i.e. the class of Borel sets.

In order to prove (4.11), note that if $\boldsymbol{m} \in \omega_v$, $|\boldsymbol{m}| = k$ where k is a neck for $\boldsymbol{a}$, and $\boldsymbol{m}\boldsymbol{s} \in \omega_v$, then it follows from (4.9) and Remark 4.3 that

$$(4.12) \qquad r_{v;\boldsymbol{m}\boldsymbol{s}}^\alpha = \theta^\alpha(\boldsymbol{s})\, r_{v;\boldsymbol{m}}^\alpha,$$

where $\theta(\boldsymbol{s})$ does not depend on either $\boldsymbol{m}$ or v. Suppose now that $j \geq k$. Then from (4.9) and (4.12),

$$\sum\{w([\boldsymbol{n}]) : \boldsymbol{m} \leq \boldsymbol{n} \in \omega_v, |\boldsymbol{n}| = j\} = r_{v;\boldsymbol{m}}^\alpha \sum\{\theta^\alpha(\boldsymbol{s}) : \boldsymbol{m}\boldsymbol{s} \in \omega_v, |\boldsymbol{s}| = j - k\}$$
$$(4.13) \qquad\qquad =: \lambda(k, j, \alpha)\, r_{v;\boldsymbol{m}}^\alpha = \lambda(k, j, \alpha)\, w([\boldsymbol{m}]),$$

where λ does not depend on $\boldsymbol{m}$ or v. Replacing $\boldsymbol{m}$ by $\boldsymbol{m}'$ and $\boldsymbol{n}$ by $\boldsymbol{n}'$, and summing also over $\boldsymbol{m}'$,

$$(4.14) \qquad \sum\{w([\boldsymbol{n}']) : \boldsymbol{n}' \in \omega_v, |\boldsymbol{n}'| = j\} = \lambda(k, j, \alpha) \sum\{w([\boldsymbol{m}']) : \boldsymbol{m}' \in \omega_v, |\boldsymbol{m}'| = k\}.$$

Dividing (4.13) by (4.14) and using (4.10) gives (4.11).

If $\boldsymbol{m} \in \omega_v$ with $|\boldsymbol{m}| = k$, not necessarily a neck, and $j \geq n(k)$, then

$$\nu([\boldsymbol{m}]) = \sum\{\nu([\boldsymbol{n}]) : \boldsymbol{m} \leq \boldsymbol{n} \in \omega_v, |\boldsymbol{n}| = n(k)\} = \frac{\sum\{w([\boldsymbol{n}]) : \boldsymbol{m} \leq \boldsymbol{n} \in \omega_v, |\boldsymbol{n}| = n(k)\}}{\sum\{w([\boldsymbol{n}']) : \boldsymbol{n}' \in \omega_v, |\boldsymbol{n}'| = n(k)\}}$$
$$(4.15) \qquad = \frac{\sum\{r_{v;\boldsymbol{n}}^\alpha : \boldsymbol{m} \leq \boldsymbol{n} \in \omega_v, |\boldsymbol{n}| = n(k)\}}{\sum\{r_{v;\boldsymbol{n}'}^\alpha : \boldsymbol{n}' \in \omega_v, |\boldsymbol{n}'| = n(k)\}} = \frac{\sum\{r_{v;\boldsymbol{p}}^\alpha : \boldsymbol{m} \leq \boldsymbol{p} \in \omega_v, |\boldsymbol{p}| = j\}}{\sum\{r_{v;\boldsymbol{p}'}^\alpha : \boldsymbol{p}' \in \omega_v, |\boldsymbol{p}'| = j\}},$$

using (4.12) in the final equality.

Define the map $\widetilde{\tau} : \widetilde{\omega}_v \to K_v$ by

$$\widetilde{\tau}(p_1 p_2 \ldots p_k \ldots) = \lim_{k \to \infty} f_{p_1}^{\omega_v(\emptyset)} \circ f_{p_2}^{\omega_v(p_1)} \circ \cdots \circ f_{p_k}^{\omega_v(p_1 \ldots p_{k-1})}(x_0) \in K_v,$$

and note that the limit does not depend on x_0.

The measure ν on $\widetilde{\omega}_v$ projects to the unit mass measure μ on K_v, defined

$$(4.16) \qquad \mu(A) = \nu\{\boldsymbol{p} \in \widetilde{\omega}_v : \widetilde{\tau}(\boldsymbol{p}) \in A\}$$

for A a Borel subset of K_v.

C. *An upper estimate for ν.* Again assume $\boldsymbol{a}$ satisfies (4.8), and consider the corresponding tree $\boldsymbol{\omega}^{\boldsymbol{a}} = (\omega_1, \ldots, \omega_V)$. Fix v. We show for $\boldsymbol{m} = m_1 \ldots m_k \in \omega_v$ that

$$(4.17) \qquad \nu([\boldsymbol{m}]) \leq c_1 r_{v;\boldsymbol{m}}^\alpha,$$

for some constant c_1 depending on $\boldsymbol{a}$ but not on $\boldsymbol{m}$.

From (4.15),

$$(4.18) \qquad \nu([\boldsymbol{m}]) = \frac{\sum\{r_{v;\boldsymbol{n}}^\alpha : \boldsymbol{m} \leq \boldsymbol{n} \in \omega_v, |\boldsymbol{n}| = n(k)\}}{\sum\{r_{v;\boldsymbol{n}'}^\alpha : \boldsymbol{n}' \in \omega_v, |\boldsymbol{n}'| = n(k)\}} \leq \frac{r_{v;\boldsymbol{m}}^\alpha M^{n(k)-k}}{S_v(n(k), \alpha)}.$$

To establish this inequality use (4.9) with $\boldsymbol{m}$ there replaced by $\boldsymbol{n}$, note that each $r_j^\alpha \leq 1$ and the branching number of ω_v is bounded by M, and use the expression in (3.4) for $S_v(n(k), \alpha) = S(\omega_v, n(k), \alpha)$.

From (4.8), for any $\epsilon > 0$, there exists $N(\epsilon)$ such that

$$(4.19) \qquad\qquad M^{n(k)-k} \leq M^{N(\epsilon)} M^{k\epsilon}$$

for all k.

From (4.3), since $\alpha < 0$ and so $\gamma(\alpha) > 0$, there exists $c_2 \in \mathbb{R}$ such that for all j,

$$\log S_v(j, \alpha) \geq c_2 + \frac{j}{2}\gamma(\alpha).$$

Hence,

$$(4.20) \qquad\qquad S_v(n(k), \alpha) \geq c_3 e^{\frac{1}{2}n(k)\gamma(\alpha)} \geq c_3 e^{\frac{1}{2}k\gamma(\alpha)}$$

for some $c_3 > 0$ and all k.

Choose $\epsilon \leq \frac{1}{2}\gamma(\alpha)/\log M$ so that $e^{\frac{1}{2}k\gamma(\alpha)} \geq M^{k\epsilon}$, and then choose $N = N(\epsilon)$. Dividing (4.19) by (4.20), and using (4.18), gives (4.17) with $c_1 = M^N/c_3$.

D. *The estimate for μ.* Fix $\boldsymbol{a} \in \mathcal{A}_V^\infty$ satisfying (4.8), in which case (4.17) holds. Assume the open set condition (2.3) holds with the open set $\mathcal{O}$.

Fix $x \in \mathbb{R}^n$ and $r > 0$. With μ the measure on K_v as in (4.16), we show by a standard argument, see [Hut81, p737] or [Fal03, p131], that

$$(4.21) \qquad\qquad \mu(B_r(x)) \leq cr^\alpha.$$

Here c is independent of x and r, and $B_r(x)$ is the open ball of radius r centred at x.

First note that, for each infinite sequence $\boldsymbol{p} = m_1 m_2 \ldots m_k \ldots \in \widetilde{\omega}_v = \widetilde{\omega}_v^{\boldsymbol{a}}$, there is a least k such that

$$(4.22) \qquad\qquad r_{\min} r \leq r_{v;m_1 \ldots m_k} < r.$$

Let $Q(r) = Q^{\boldsymbol{a}}(r)$ be the set of all such $\boldsymbol{m} = m_1 \ldots m_k$. The sets $\mathcal{O}_{v;\boldsymbol{m}}$ for $\boldsymbol{m} \in Q(r)$ are disjoint from (2.5) and the definition of $Q(r)$, although the $|\boldsymbol{m}|$ are not necessarily equal. Let $Q(x, r) = Q^{\boldsymbol{a}}(x, r)$ be the set of $\boldsymbol{m} \in Q(r)$ such that $\mathcal{O}_{v;\boldsymbol{m}}$ meets $B_r(x)$.

Choose a_1 and a_2 so that $\mathcal{O}$ contains an open ball of radius a_1, and is contained in an open ball of radius a_2. Then the sets $\mathcal{O}_{v;\boldsymbol{m}}$ each contain a ball of radius $a_1 r_{v;\boldsymbol{m}}$, and hence of radius $a_1 r_{\min} r$, and they are contained in a ball of radius $a_2 r_{v;\boldsymbol{m}}$, and hence of radius $a_2 r$. It follows by a volume comparison that if $q = q(x, r)$ is the cardinality of $Q(x, r)$, since the inner balls are disjoint and are subsets of $B_{(1+2a_2)r}$, that

$$q(a_1 r_{\min} r)^n \leq (1 + 2a_2)^n r^n,$$

and so q is bounded independently of x and r.

Hence,

$$\mu(B_r(x)) = \mu(B_r(x) \cap K_v) = \nu\big(\{\boldsymbol{p} : \tau(\boldsymbol{p}) \in B_r(x) \cap K_v\}\big)$$
$$\leq \nu\left(\bigcup\{[\boldsymbol{m}] : \boldsymbol{m} \in Q(x, r)\}\right) = \sum\{\nu([\boldsymbol{m}]) : \boldsymbol{m} \in Q(x, r)\}$$
$$\leq c_1 q \sum\{r_{v;\boldsymbol{m}}^\alpha : \boldsymbol{m} \in Q(x, r)\} \leq c_1 q r^\alpha,$$

using the definition of $Q(x, r)$, the disjointedness of the $\boldsymbol{m} \in Q(x, r) \subset Q(r)$, the estimate (4.17) and (4.22). This establishes (4.7) and hence the lemma. $\qquad\square$
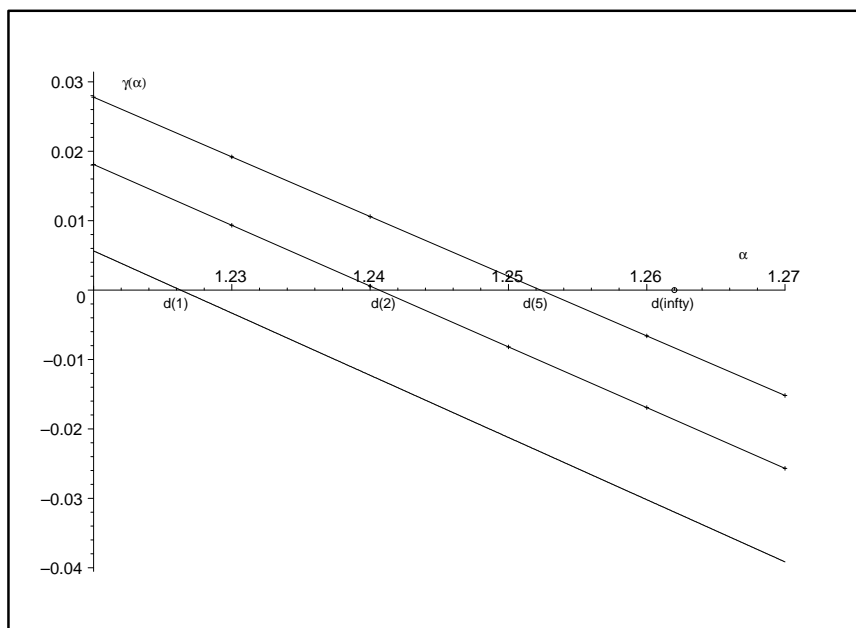
FIGURE 7. Graphs of $\gamma_V(\alpha) = \gamma(\alpha)$ for $V = 1, 2, 5$ respectively from left to right. Here $\boldsymbol{F} = \{F, G\}$ and $P = \{1/2, 1/2\}$ as in the introduction.

5. EXAMPLES

For the model problem in the introduction, with F and G each chosen with probability $1/2$, it follows from (3.7) that, for random homogeneous Sierpinski triangles, the dimension is $d(1) = 2 \log 3/(\log 2 + \log 3) \approx 1.226$. For the corresponding random recursive case, from (3.8) the dimension is the solution d of $\frac{1}{2} 3 \left(\frac{1}{2}\right)^d + \frac{1}{2} 3 \left(\frac{1}{3}\right)^d = 1$, i.e. $d(\infty) \approx 1.262$. For $V > 1$ we used Maple 10 to compute the values of $\gamma_V(\alpha)$ shown in Figure 7. These values have error at most .001 at the 95% confidence level, and from this one obtains the dimensions $d(2) \approx 1.241$, $d(5) \approx 1.252$. The computed graphs for $V > 1$ are concave up, although this does not show on the scale used.

REFERENCES

[Bar98] Martin T. Barlow, *Diffusions on fractals*, Lectures on probability theory and statistics (Saint-Flour, 1995), Lecture Notes in Math., vol. 1690, Springer, 1998, pp. 1–121.

[Bar06] Michael Fielding Barnsley, *Superfractals*, Cambridge University Press, 2006.

[BHS05] Michael Barnsley, John Hutchinson, and Örjan Stenflo, *A fractal valued random iteration algorithm and fractal hierarchy*, Fractals **13** (2005), 111–146.

[BHS08] M. F. Barnsley, John E. Hutchinson, and Örjan Stenflo, *V-variable Fractals: Fractals with Partial Self Similarity*, Advances in Mathematics (2008), to appear.

[Coh88] Joel E. Cohen, *Subadditivity, generalized products of random matrices and operations research*, SIAM Rev. **30** (1988), no. 1, 69–86.

[Fal86] Kenneth Falconer, *Random fractals*, Math. Proc. Cambridge Philos. Soc. **100** (1986), 559–582.

[Fal03] _____ , *Fractal geometry, Mathematical foundations and applications*, 2nd ed., John Wiley & Sons Ltd., Chichester, 2003.

[Fal97] _____ , *Techniques in fractal geometry*, John Wiley & Sons Ltd., Chichester, 1997.

[FK60] H. Furstenberg and H. Kesten, *Products of random matrices*, Ann. Math. Statist **31** (1960), 457–469.

[Gra87] Siegfried Graf, *Statistically self-similar fractals*, Probab. Theory Related Fields **74** (1987), 357–392.

[Gra91] _____ , *Random fractals*, Rend. Istit. Mat. Univ. Trieste **23** (1991), 81–144 (1993). School on Measure Theory and Real Analysis (Grado, 1991).

[Ham92] B. M. Hambly, *Brownian motion on a homogeneous random fractal*, Probab. Theory Related Fields **94** (1992), 1–38.

[Ham97] _____ , *Brownian motion on a random recursive Sierpinski gasket*, Ann. Probab. **25** (1997), no. 3, 1059–1102.

[Ham03] _____ , *Fractals and the modelling of self-similarity*, Stochastic processes: modelling and simulation, Handbook of Statist., vol. 21, North-Holland, 2003, pp. 371–406.

[Hut81] John E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.
[Kig01] Jun Kigami, *Analysis on fractals*, Cambridge Tracts in Mathematics, vol. 143, Cambridge University Press, 2001.
[MW86] R. Daniel Mauldin and S. C. Williams, *Random recursive constructions: asymptotic geometric and topological properties*, Trans. Amer. Math. Soc. **295** (1986), 325–346.
[Mor46] P. A. P. Moran, *Additive functions of intervals and Hausdorff measure*, Proc. Cambridge Philos. Soc. **42** (1946), 15–23.
[Ste01] Örjan Stenflo, *Markov chains in random environments and random iterated function systems*, Trans. Amer. Math. Soc. **353** (2001), 3547–3562.
[Str06] Robert S. Strichartz, *Differential equations on fractals*, Princeton University Press, 2006.