CONVERGENCE OF PHASE INTERFACES IN THE VAN DER WAALS-CAHN-HILLIARD THEORY

JOHN HUTCHINSON AND YOSHIHIRO TONEGAWA

ABSTRACT. We study the general asymptotic behavior of critical points, including those of non-minimal energy type, of the functional for the Van der Waals-Cahn-Hilliard theory of phase transitions. We prove that the interface is close to a hypersurface with mean curvature zero when no Lagrange multiplier is present, and with locally constant mean curvature in general. The energy density of the limiting measure has integer multiplicity almost everywhere modulo division by a surface energy constant.

1. INTRODUCTION

In this paper we study the general asymptotic behavior of critical points of the energy functional for the Van der Waals-Cahn-Hilliard theory of phase transitions. The functional in question is

$$E_\varepsilon(u) = \int_U \frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}, \tag{1.1}$$

where $u : U \subset \mathbb{R}^n \to \mathbb{R}$ is the normalized density distribution of a two-phase fluid and W is a double well potential with strict local minima at ± 1, see [26]. A similar functional also appears in the study of pattern formation (e.g. [33, 34]) for $\varepsilon \approx 0$. Critical points of the functional (1.1) satisfy

$$\varepsilon \Delta u = \varepsilon^{-1} W'(u) - \lambda, \tag{1.2}$$

where λ is the Lagrange multiplier associated with a global volume constraint of the form $\int_U u = m$.

For absolutely energy minimizing solutions with such a volume constraint, Modica [31] and Sternberg [41] used the technique of Γ-convergence [17] to show that (on passing to a subsequence) the limit of minimizers of (1.1) as $\varepsilon \to 0$ is a function with value ± 1 almost everywhere and with area minimizing interface in the appropriate class of competing functions. There have been works by many authors on various generalizations and related problems in this direction, see for example [6, 21, 28, 32, 36, 42, 43].

The focus of this paper is on general critical points which may not be absolutely energy minimizing. A good understanding of such solutions is important in the study of dynamical problems such as the Allen-Cahn equation [2] and the Cahn-Hilliard equation [12] in bounded domains, since it has been observed numerically [25] that the solution often seems to undergo patterns similar to unstable equilibria before settling down to a stable pattern. Moreover, from a purely mathematical point of view, one can show the existence of unstable mountain-pass type solutions [38] due to the non-convexity of the functional (1.1), and it is interesting to know the asymptotic limit of such solutions as $\varepsilon \to 0$ in this generality. On the other hand, Γ-convergence techniques essentially rely on energy minimality of solutions and thus do not deal with general non-minimizing solutions.

Roughly speaking, for $\lambda = 0$ and any solution of (1.2), we show that as $\varepsilon \to 0$ the interface converges in the Hausdorff distance sense to a generalized minimal hypersurface. Moreover, the energy concentrates near the hypersurface and after division by twice the surface energy constant $\sigma = \int_{-1}^{1} \sqrt{W(s)/2}\, ds$, the energy density in the limit is an integer $\mathcal{H}^{n-1}$ a.e. on the hypersurface. This integer multiplicity allows for "folding" of the interface as $\varepsilon \to 0$. When $\lambda \neq 0$, we prove that the hypersurface has locally constant mean curvature $\mathcal{H}^{n-1}$ a.e., with the situation otherwise similar to the $\lambda = 0$ case. As a corollary, we show that the additional assumption of local energy minimality implies no loss of energy in the limit, and the limit interface is a locally area minimizing hypersurface of multiplicity one.

The proof of our results depends on a local maximum bound on the discrepancy function $\xi = \frac{\varepsilon}{2}|\nabla u|^2 - \frac{1}{\varepsilon}W(u)$. For a bounded entire solution of (1.2) on $\mathbb{R}^n$ with $\lambda = 0$, Modica showed in [30] that $\xi \leq 0$. We show that ξ is locally uniformly bounded for small ε, and use this to establish a local monotonicity formula for the scaled energy density. We note that in his study of the Allen-Cahn equation with domain $\mathbb{R}^n$ ([27]), Ilmanen assumed that the initial data satisfies $\xi \leq 0$, which is preserved in time by the maximum principle, and which implies a parabolic monotonicity formula for the energy. In [40], Soner was able to drop this assumption on the initial data and showed with $|u| \leq 1$ that (essentially) ξ is bounded uniformly on $\mathbb{R}^n$ for $t > 0$. Our results also point to the way that one may localize [27, 40] to the case of bounded domains in $\mathbb{R}^n$. The question of integer multiplicity density of the limiting energy measure (modulo division by the surface energy constant) was raised in [27, Section 13] for the Allen-Cahn equation. Our result proves that, at least for the time-independent and local case, this is generally true. This paper also extends the results in [37], in which locally stable or energy minimizing solutions were analyzed. Finally, we mention the recent work by Sternberg and Zumbrun [44], which showed that the interface for stable solutions on strictly convex domains is topologically connected. There have been numerous works on associated dynamical problems in recent years, we cite [3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16, 18, 20, 27, 35, 40] and further references therein.

The organization of the paper is as follows. In Section 2, we state the assumptions, certain terminology and the main results. In Section 3, we derive the local maximum bound on the discrepancy function, and the energy monotonicity formula. We then show in Section 4 that the limiting varifold defined in Section 2 is rectifiable. Section 5 shows integer multiplicity of the limiting varifold. We discuss various additional matters in the last section.

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2. Preliminaries and main results

2.1. **Hypotheses and easy consequences.** Except where stated otherwise we take the following as the starting point of this paper. Note that we do not assume any energy minimality for the u^i.

Assumptions.

- **A:** *The function $W : \mathbb{R} \to [0, \infty)$ is C^3 and $W(\pm 1) = 0$. For some $\gamma \in (-1, 1)$, $W' < 0$ on $(\gamma, 1)$ and $W' > 0$ on $(-1, \gamma)$. For some $\alpha \in (0, 1)$ and $\kappa > 0$, $W''(x) \geq \kappa$ for all $|x| \geq \alpha$.*
- **B:** *$U \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary ∂U. A sequence of $C^3(U)$ functions $\{u^i\}_{i=1}^{\infty}$ satisfies*

$$\varepsilon_i \Delta u^i = \varepsilon_i^{-1} W'(u^i) - \lambda_i \tag{2.1}$$

on U. Here, $\lim_{i\to\infty}\varepsilon_i = 0$, and we assume there exist c_0, λ_0 and E_0 such that $\sup_U |u^i| \le c_0$, $|\lambda_i| \le \lambda_0$ and

$$\int_U \frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \le E_0$$

for all i.

Assumption **A** requires that W has a standard W-shape with non-degenerate minima at ± 1, and local maximum at γ. The uniform supremum bound on $|u^i|$ may be obtained by imposing some structural conditions on W as in Section 6.1 and [25], and also follows from the existence proof in other situations such as in Section 6.2. The regularity of u is then standard ([22]).

We next discuss a few immediate consequences of the assumptions. Let

$$\Phi(s) = \int_0^s \sqrt{W(s)/2}\, ds,$$

and define new functions

$$w^i = \Phi \circ u^i$$

for each i.

Since $|\nabla w^i| = \sqrt{W(u^i)/2}\,|\nabla u^i|$, it follows by the Cauchy-Schwartz inequality that

$$\int_U |\nabla w^i| \le \frac{1}{2}\int_U \frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \le \frac{E_0}{2}.$$

We also have $\Phi(-c_0) \le w^i \le \Phi(c_0)$. By the compactness theorem for bounded variation functions ([19]), there exists a subsequence also denoted by $\{w^i\}$ and an a.e. pointwise limit w^∞, such that

$$\lim_{i\to\infty} \int_U |w^i - w^\infty| = 0 \quad \text{and} \quad \int_U |Dw^\infty| \le \liminf_{i\to\infty} \int_U |\nabla w^i|.$$

Here, $|Dw^\infty|$ is the total variation of the vector-valued Radon measure Dw^∞.

Let Φ^{-1} be the inverse of Φ and define

$$u^\infty = \Phi^{-1}(w^\infty).$$

Then $u^i \to u^\infty$ a.e., and by the Lebesgue dominated convergence theorem

$$\int_U |u^i - u^\infty| \to 0.$$

Also by Fatou's Lemma and the energy bound, we have

$$\int_U W(u^\infty) = \int_U \lim_{i\to\infty} W(u^i) \le \liminf_{i\to\infty} \int_U W(u^i) = 0.$$

This shows that $u^\infty = \pm 1$ a.e. on U, and the sets $\{u^\infty = \pm 1\}$ have finite perimeter in U, since

$$\|\partial\{u^\infty = 1\}\|(U) = \frac{1}{2}\int_U |Du^\infty| = \frac{1}{\sigma}\int_U |Dw^\infty| \le \frac{E_0}{2\sigma},$$

where we define

$$\sigma = \int_{-1}^1 \sqrt{W(s)/2}\, ds,$$

and where $\|\partial A\|$ denotes the perimeter of A in the measure-theoretic sense (see [19]).

By the generalized Gauß-Green theorem for sets of finite perimeter ([19, page 209]), there exists an $(n-1)$-rectifiable set M^∞ (the "reduced boundary") $\subset$

$\operatorname{supp}\|\partial\{u^\infty = 1\}\|$, and an $\mathcal{H}^{n-1}$ measurable unit vector function ν^∞ defined on M^∞ (pointing into $\{u^\infty = 1\}$) such that

$$\int_{\{u^\infty=1\}} \operatorname{div} g = -\int_{M^\infty} \nu^\infty \cdot g \, d\mathcal{H}^{n-1},$$

for any $g \in C_c^1(U)$.

2.2. **The associated varifolds.** In this section we recall various definitions concerning varifolds and associate to each solution of (1.2) a varifold in a natural way. We refer to [1, 39] for a comprehensive treatment of varifolds.

Let $G(n, n-1)$ denote the Grassman manifold of unoriented $(n-1)$-dimensional planes in $\mathbb{R}^n$. We also regard $S \in G(n, n-1)$ as the orthogonal projection of $\mathbb{R}^n$ onto S, and write $S_1 \cdot S_2 = \operatorname{trace}(S_1^t \cdot S_2)$. We say V is an $(n-1)$-dimensional *varifold* in $U \subset \mathbb{R}^n$ if V is a Radon measure on $G_{n-1}(U) = U \times G(n, n-1)$. Let $V_{n-1}(U)$ denote the set of all $(n-1)$-dimensional varifolds in U. *Convergence in the varifold sense* means convergence in the usual sense of measures. For $V \in V_{n-1}(U)$, we let the *weight* $\|V\|$ be the Radon measure in U defined by

$$\|V\|(A) = V\big(\{\, (x, S) \mid x \in A,\ S \in G(n, n-1)\,\}\big)$$

for each Borel set $A \subset U$. If M is a $(n-1)$-rectifiable subset of U we define $v(M) \in V_{n-1}(U)$ by

$$v(M)(E) = \mathcal{H}^{n-1}\big(\{\, x \in U \mid (x, \operatorname{Tan}^{n-1}(\mathcal{H}^{n-1}\lfloor_M, x)) \in E\,\}\big)$$

for each Borel set $E \in G_{n-1}(U)$, where $\operatorname{Tan}^{n-1}(\mathcal{H}^{n-1}\lfloor_M, x)$ is the approximate tangent plane to M at x and so exists for $\mathcal{H}^{n-1}$ a.e. $x \in M$. We say $V \in V_{n-1}(U)$ is an $(n-1)$-dimensional *rectifiable varifold* if there exist positive real numbers $\{c_k\}_{k=1}^\infty$ and $(n-1)$-rectifiable sets $\{M_k\}_{k=1}^\infty$ such that

$$V = \sum_{k=1}^{\infty} c_k v(M_k).$$

The *density* or *multiplicity function* θ for V is given by

$$\theta(x) = \sum \{\, c_k \mid x \in M_k\,\},$$

and then $\|V\| = \theta\,\mathcal{H}^{n-1}\lfloor M$, where $M = \bigcup_k M_k$. If $\{c_k\}_{k=1}^\infty$ may be taken to be positive integers, we say V is an $(n-1)$-dimensional *integral varifold.*

For $V \in V_{n-1}(U)$, we define the *first variation* of V by

$$\delta V(g) = \int Dg(x) \cdot S \, dV(x, S)$$

for any vector field $g \in C_c^1(U; \mathbb{R}^n)$, and we say V is *stationary* if $\delta V(g) = 0$ for all such g. We also denote the total variation of δV by $\|\delta V\|$. If $\|\delta V\|$ is a Radon measure and is absolutely continuous with respect to $\|V\|$ on U, we define the *generalized mean curvature* $H(x)$ by

$$\delta V(g) = -\int g \cdot H \, d\|V\|,$$

where H is defined $\|V\|$ a.e. on U.

Finally we remark that if μ is a measure on U (e.g., $\|V\|$ or $\|\partial\{u^\infty = 1\}\|$) then by $\operatorname{supp}\mu$ we will always denote the support of μ *in* U.

We associate to each function w^i a varifold V^i defined naturally as follows ([27, 37]). By Sard's theorem, $\{w^i = t\} \subset U$ is a C^3 hypersurface for L^1 almost all t.

Define $V^i \in V_{n-1}(U)$ by

$$V^i(A) = \int_{-\infty}^{\infty} v(\{w^i = t\})(A)\, dt$$

for each Borel set $A \subset G_{n-1}(U)$. By the coarea formula ([19]), we have

$$\|V^i\|(A) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{w^i = t\} \cap A)\, dt = \int_A |\nabla w^i|$$

for each Borel set $A \subset U$. One may interpret the varifold V^i as a weighted averaging of the level sets of u^i, which is concentrated around the transition region. The first variation of V^i is given by (see [37, Section 2.1])

$$\delta V^i(g) = \int_U \left(\mathrm{div} g - \sum_{j,k=1}^{n} \frac{w^i_{x_j}}{|\nabla w^i|} \frac{w^i_{x_k}}{|\nabla w^i|} g^j_{x_k} \right) |\nabla w^i| \tag{2.2}$$

for each $g \in C^1_c(U; \mathbb{R}^n)$.

2.3. **Main results.** With the above terminology and assumptions **A** and **B**, we show the following. More detailed information is obtained in the appropriate lemmas and propositions in the paper.

Theorem 1. *Let V^i be the varifold associated with u^i (via w^i) as in Sections 2.1 and 2.2. On passing to a subsequence we can assume*

$$\lambda_i \to \lambda_\infty, \quad u^i \to u^\infty \text{ a.e.}, \quad V^i \to V \text{ in the varifold sense.}$$

Moreover,

(1) *For each $\phi \in C_c(U)$,*
$$\|V\|(\phi) = \lim_{i\to\infty} \int \phi \frac{\varepsilon_i |\nabla u^i|^2}{2} = \lim_{i\to\infty} \int \phi \frac{W(u^i)}{\varepsilon_i} = \lim_{i\to\infty} \int \phi\, |\nabla w^i|.$$

(2) $\mathrm{supp}\|\partial\{u^\infty = 1\}\| \subset \mathrm{supp}\|V\|$, *and $\{u^i\}$ converges locally uniformly to ± 1 on $U \setminus \mathrm{supp}\|V\|$.*

(3) *For each $\tilde{U} \subset\subset U$ and $0 < b < 1$, $\{|u^i| \le 1-b\} \cap \tilde{U}$ converges to $\tilde{U} \cap \mathrm{supp}\|V\|$ in the Hausdorff distance sense.*

(4) *$\sigma^{-1}V$ is an integral varifold. Moreover, the density $\theta(x) = \sigma N(x)$ of V satisfies*
$$N(x) = \begin{cases} \text{odd} & \mathcal{H}^{n-1} \text{ a.e. } x \in M^\infty, \\ \text{even} & \mathcal{H}^{n-1} \text{ a.e. } x \in \mathrm{supp}\|V\| \setminus M^\infty, \end{cases}$$
where M^∞ is the reduced boundary of $\{u^\infty = 1\}$.

(5) *The generalized mean curvature H of V is given by*
$$H(x) = \begin{cases} \frac{\lambda_\infty}{\theta(x)} \nu^\infty(x) & \mathcal{H}^{n-1} \text{ a.e. } x \in M^\infty, \\ 0 & \mathcal{H}^{n-1} \text{ a.e. } x \in \mathrm{supp}\|V\| \setminus M^\infty, \end{cases}$$
where ν^∞ is the inward normal for M^∞.

(2) follows from Proposition 4.2, (1) from Propositions 4.3 and 4.4. (3) follows from a contradiction argument using Propositions 4.2 and 4.1. (5) is established in Proposition 4.4 (although at that stage the varifold V is only known to be rectifiable) and (4) is proved in Section 5.

Heuristic interpretations may be given as follow. ¿From (1), we see that in the limit the energy is equally divided between the two terms of the energy functional (1.1). (4) suggests that folding of the interface as $\varepsilon \to 0$ occurs locally as an integer multiple of 1-D traveling wave solutions ([30]), almost everywhere in the measure-theoretic sense. This can be seen more clearly in the proof of integrality in Section 5. (5) suggests that, whenever there is a cancellation of interface in the oriented sense, the mean curvature is zero there. More generally, if N-folding occurs, the mean curvature decreases by that factor.

Without loss of generality, we may assume that $M^\infty \subset \operatorname{supp}\|\partial\{u^\infty = 1\}\|$. We were not able to prove or disprove that $\mathcal{H}^{n-1}(\operatorname{supp}\|\partial\{u^\infty = 1\}\| \setminus M^\infty) = 0$ in general. This is due to the lack of a uniform lower density estimate for the measure $\|\partial\{u^\infty = 1\}\|$ (as opposed to $\|V\|$) at $\mathcal{H}^{n-1}$ a.e. x in the closure of M^∞. On the other hand, if $N(x)$ is odd $\mathcal{H}^{n-1}$ a.e. for $x \in \operatorname{supp}\|V\|$, the result (4) shows that $\mathcal{H}^{n-1}(\operatorname{supp}\|V\| \setminus M^\infty) = 0$ and $\operatorname{supp}\|V\| = \operatorname{supp}\|\partial\{u^\infty = 1\}\|$. If $N(x) = 1$ a.e., then $\sigma^{-1}\|V\| = \|\partial\{u^\infty = 1\}\|$ and V has constant mean curvature on U by (4) and (5). This last situation corresponds to "no energy loss", since

$$\int |Dw^\infty|\,\phi = \sigma\,\|\partial\{u^\infty = 1\}\|(\phi) = \|V\|(\phi) = \lim_{i\to\infty}\int |\nabla w^i|\phi$$

for all $\phi \in C_c(U)$. The relation (5) between the Lagrange multiplier λ_∞ (or chemical potential in the two-phase fluid model) and the mean curvature of the limit interface, called the Gibbs-Thompson relation, was established by Luckhaus and Modica in [29] in the case of no energy loss.

It is well-known that the support of a rectifiable varifold with bounded mean curvature is locally a $C^{1,\alpha}$ graph on a relatively open dense subset $\mathcal{O}$ ([1]). Moreover, the multiplicity of V on $\mathcal{O}$ is locally constant, and hence the support has locally constant mean curvature by (5). Thus $\mathcal{O}$ is in fact a C^∞ submanifold. On the other hand, we do not know if $\mathcal{H}^{n-1}(\operatorname{supp}\|V\| \setminus \mathcal{O}) = 0$ in general.

If $N = 1$, $\mathcal{H}^{n-1}$ a.e. on $\operatorname{supp}\|V\|$, then the support is locally a C^∞ hypersuface of constant mean curvature, except for a closed set of $\mathcal{H}^{n-1}$ measure zero. Such a situation occurs (away from the boundary) in the locally minimizing case discussed in Theorem 2. First we need the following.

Definition. For $\tilde{U} \subset\subset U$ we say $u \in H^1(U)$ is *locally energy minimizing* on $\tilde{U}$ for E_ε if there exists a positive constant c such that $E_\varepsilon(u) \le E_\varepsilon(\tilde{u})$ for all $\tilde{u} \in H^1(U)$ satisfying $\int_U |u - \tilde{u}| < c$ and $u - \tilde{u} = 0$ on $U \setminus \tilde{U}$. We may also (depending on the problem) impose the additional volume constraint $\int_U (u - \tilde{u}) = 0$.

Note that the definition is local in both the domain and the L^1 norm, which differs from the local minimality discussed in [28]. With this, we prove

Theorem 2. *In addition to assumptions* **A** *and* **B**, *suppose* $\{u^i\}$ *are locally energy minimizing on* $\tilde{U} \subset\subset U$ *for* E_{ε_i} *(with or without volume constraint). Then* $N(x) = 1$, $\mathcal{H}^{n-1}$ *a.e. on* $\tilde{U} \cap \operatorname{supp}||V||$. $\partial\{u^\infty = 1\}$ *on* $\tilde{U}$ *has constant mean curvature* $\frac{\lambda_\infty}{\sigma}\nu^\infty$ *and no energy loss occurs on* $\tilde{U}$.

For absolutely energy minimizing solutions with a volume constraint, Modica [31] and Sternberg [41] showed that $\partial\{u^\infty = 1\}$ is an absolutely area minimizing hypersurface with the given volume constraint. For this case, with the additional assumption **A**, Theorem 1.(3) gives a new result concerning convergence of the interface in the Hausdorff distance sense. We also prove a version of the Modica-Sternberg theorem for local minimizers, which was not known before.

Theorem 3. *Suppose that* W *satisfies assumption* **A** *and* U *is a bounded open set with Lipschitz boundary. Suppose* $c > 0$, $u^i \in H^1(U)$, $\varepsilon_i \to 0$, $m \in (-|U|, |U|)$ *and* $E_0 < \infty$ *satisfy*

(1) $\int_U u^i = m$ *and* $E_{\varepsilon_i}(u^i) \le E_0$ *for all* i,

(2) $E_{\varepsilon_i}(u^i) \le E_{\varepsilon_i}(\tilde{u})$ *for all* $\tilde{u} \in H^1(U)$ *with* $\int_U |u^i - \tilde{u}| < c$ *and* $\int_U \tilde{u} = m$.

Then the assumption **B** *is satisfied. Moreover, with* u^∞ *as in Theorem 1,* $\partial\{u^\infty = 1\}$ *minimizes area locally; i.e. for any* $\tilde{u}$ *satisfying* $\tilde{u} = \pm 1$ $\mathcal{L}^n$ *a.e. on* U, $\int_U \tilde{u} = m$

and $\int_U |u^\infty - \tilde{u}| < c$ *(the same* c *in the assumption), we have*

$$||\partial\{u^\infty = 1\}||(U) \leq ||\partial\{\tilde{u} = 1\}||(U).$$

It is well-known that the support of a locally area minimizing perimeter is smooth except for a closed set of dimension at most $n-8$ [23, 24].

The reader is referred to Section 6 for additional applications of our results.

3. Local monotonicity formula

Throughout this section, in addition to assumption $\boldsymbol{A}$*, we assume that the function* $u : U \to \mathbb{R}$ *satisfies assumption* $\boldsymbol{B}$ *with* u^i *and* ε^i *there replaced by* u *and* ε *respectively. We assume* $\tilde{U}$ *is open and* $\tilde{U} \subset\subset U$.

The main result here is the local monotonicity type formula in Proposition 3.4. Note that the two integrals on the right side are positive and the remaining "error" term is controlled by r.

The first step is to establish the relationship Lemma 3.1 between the scaled energy on concentric balls of different radii. For technical reasons we do this in terms of the associated scaled energy defined on $B_r(x) \subset U$ by

$$I(r,x) = \frac{1}{r^{n-1}} \int_{B_r(x)} \frac{\varepsilon|\nabla u|^2}{2} + \frac{\widetilde{W}(u)}{\varepsilon},$$

where $\widetilde{W}(u) \equiv W(u) - \lambda\varepsilon u$. (Note that u is stationary for the non-constrained problem obtained from (1.1) with W replaced by $\widetilde{W}$, and $I(r,x)$ is the scaled energy for this new functional.)

The next step is to control from above the first integrand on the right of (3.1). This is done in Proposition 3.3, where an upper bound is established for the discrepancy function

$$\xi = \frac{\varepsilon|\nabla u|^2}{2} - \frac{W(u)}{\varepsilon}.$$

A preliminary result is the sup bound for $|u|$ in Proposition 3.2. Both these propositions are motivated by the results of Modica [30] (see also Chen [15]), where it is shown if $\lambda = 0$ that $|u| \leq 1$ and $\xi \leq 0$ for bounded entire solutions of (1.2) on all of $\mathbb{R}^n$. However, the proofs of the local results here are technically somewhat more involved.

Lemma 3.1. *If* $B_r(x) \subset\subset U$ *and* $r > s > 0$*, then*

$$I(r,x) - I(s,x) = \int_s^r \left(\frac{1}{\tau^n} \int_{B_\tau(x)} \frac{\widetilde{W}(u)}{\varepsilon} - \frac{\varepsilon|\nabla u|^2}{2} \right) d\tau + \varepsilon \int_{B_r(x)\setminus B_s(x)} \frac{((y-x)\cdot\nabla u)^2}{|y-x|^{n+1}}\, dy. \tag{3.1}$$

Proof. Multiply both sides of (1.2) by $\nabla u \cdot g$, where $g = (g^1, \cdots, g^n) \in C^1_c(U; \mathbb{R}^n)$. Then, after two integrations by parts, we obtain

$$\int_U \left(\left(\frac{\varepsilon|\nabla u|^2}{2} + \frac{\widetilde{W}(u)}{\varepsilon} \right) \mathrm{div} g - \varepsilon \sum_{i,j} u_{y_i} u_{y_j} g^j_{y_i} \right) = 0. \tag{3.2}$$

We let $x = 0$ by a suitable translation and let $g^j(y) = y_j \rho(|y|)$, where $\rho(|y|)$ is a smooth approximation to the characteristic function $\chi_{B_r(0)}$. Writing $r = |y|$, (3.2) becomes

$$\int_U \left(\left(\frac{\varepsilon|\nabla u|^2}{2} + \frac{\widetilde{W}(u)}{\varepsilon} \right)(r\rho' + n\rho) - \varepsilon\frac{\rho'}{r}(y\cdot\nabla u)^2 - \varepsilon|\nabla u|^2 \rho \right) = 0.$$

By letting $\rho \to \chi_{B_r(0)}$ and rearranging terms, we obtain

$$-(n-1)\int_{B_r}\left(\frac{\varepsilon|\nabla u|^2}{2}+\frac{\widetilde{W}(u)}{\varepsilon}\right)+r\int_{\partial B_r}\left(\frac{\varepsilon|\nabla u|^2}{2}+\frac{\widetilde{W}(u)}{\varepsilon}\right)$$
$$=\int_{B_r}\left(\frac{\widetilde{W}(u)}{\varepsilon}-\frac{\varepsilon|\nabla u|^2}{2}\right)+\frac{\varepsilon}{r}\int_{\partial B_r}(y\cdot\nabla u)^2.$$

By dividing the above expression by r^n and by integrating over the interval $[s,r]$, we obtain (3.1). □

Proposition 3.2. *There exist constants c_1 and ε_1 which depend only on λ_0, c_0, $\operatorname{dist}(\tilde{U},\partial U)$ and W such that*

$$\sup_{\tilde{U}}|u|\le 1+c_1\varepsilon \tag{3.3}$$

whenever $\varepsilon<\varepsilon_1$.

Proof. Suppose that $B_{3d}\subset U$ and consider a smooth function $\zeta\in C^\infty(B_{3d})$ such that $1+c_1\varepsilon/2\le\zeta\le 1+c_0$ on B_{3d}, $\zeta\equiv 1+c_1\varepsilon/2$ on B_d, $\zeta\equiv c_0+1$ on $B_{3d}-B_{2d}$, where c_1 will be fixed shortly. Assume that $\sup_{B_d}u\ge 1+c_1\varepsilon$ to derive a contradiction. Let g be a function defined by $g=u-\zeta$. Then, the function g satisfies $g\le -1$ on ∂B_{3d} and $\sup_{B_{3d}}g\ge c_1\varepsilon/2$. Thus g has an interior maximum point at x_0, say, and $g(x_0)\ge c_1\varepsilon/2$. Also at x_0,

$$\begin{aligned}0\ge\varepsilon\Delta g=\varepsilon(\Delta u-\Delta\zeta)&=\frac{W'(u)}{\varepsilon}-\lambda-\varepsilon\Delta\zeta\\&=\left.\frac{W'(tu+(1-t)\zeta)}{\varepsilon}\right|_{t=0}^{1}+\frac{W'(\zeta)}{\varepsilon}-\lambda-\varepsilon\Delta\zeta\\&\ge\frac{g}{\varepsilon}\int_0^1 W''(tu+(1-t)\zeta)\,dt\;-\lambda-\varepsilon\max|\Delta\zeta|.\end{aligned}$$

¿From assumption **A** and $g(x_0)\ge c_1\varepsilon/2$, it follows that $\kappa c_1/2\le\lambda+\varepsilon\max|\Delta\zeta|$. Thus, if c_1 is sufficiently large, this would be a contradiction.

The result as stated is an easy consequence. The bound on $\inf u$ is obtained similarly. □

Remark. We may obtain better estimates than Proposition 3.2, but this is sufficient for our purpose.

In particular, if $\lambda=0$ and $k>0$, by iterating the above argument one obtains $\sup_{\tilde{U}}|u|\le 1+c_1'\varepsilon^k$, where c_1' depends on k and the same quantities as c_1.

More generally, if a_ε and b_ε are chosen near -1 and $+1$ respectively so that $W'(a_\varepsilon)=W'(b_\varepsilon)=\varepsilon\lambda$, then by replacing W by $W-\varepsilon\lambda u$ in the argument, one can show $a_\varepsilon-c_1''\varepsilon^k<u<b_\varepsilon+c_1''\varepsilon^k$, where c_1'' depends on k and the same quantities as c_1.

Proposition 3.3. *There exist constants c_2 and ε_2 which depend only on λ_0, c_0, $\operatorname{dist}(\tilde{U},\partial U)$ and W such that*

$$\sup_{\tilde{U}}\left(\frac{\varepsilon|\nabla u|^2}{2}-\frac{W(u)}{\varepsilon}\right)\le c_2. \tag{3.4}$$

for all $\varepsilon<\varepsilon_2$.

Combining this proposition with Lemma 3.1 we obtain, with the notation

$$E(r,x)=r^{1-n}\int_{B_r(x)}\frac{\varepsilon|\nabla u|^2}{2}+\frac{W(u)}{\varepsilon},$$

the following monotonicity type formula.

Proposition 3.4. *Suppose $B_r(x) \subset \tilde{U}$, $r > s > 0$ and $\varepsilon \le \varepsilon_2$. Then*

$$
\begin{aligned}
E(r,x) - E(s,x) \ge& \int_s^r \left\{ \frac{1}{\tau^n} \int_{B_\tau(x)} \left(\frac{W(u)}{\varepsilon} - \frac{\varepsilon}{2}|\nabla u|^2 \right)^+ \right\} d\tau \\
&- c_3 r + \varepsilon \int_{B_r(x)\setminus B_s(x)} \frac{((y-x)\cdot\nabla u)^2}{|y-x|^{n+1}}\, dy,
\end{aligned}
\tag{3.5}
$$

where $c_3 = (4\lambda_0 + c_2)\,\omega_n$.

It remains to establish Proposition 3.3. For this, suppose $B_{3d}(0) \subset \tilde{U}$, and by scaling $x \mapsto x/\varepsilon$ consider the rescaled problem

$$
-\Delta u + W'(u) = \varepsilon\lambda \qquad \text{on } B_{3d/\varepsilon}(0).
$$

For the remainder of this section let u denote the rescaled function (which solves the rescaled problem). Without loss of generality for our purposes choose units so $d = 1$. Also for the remainder of this section we denote the rescaled discrepancy function by

$$
\xi = \frac{1}{2}|\nabla u|^2 - W(u).
$$

Proposition 3.3 then follows by rescaling back from Lemma 3.6.

It is convenient to work with the function

$$
\xi_G(x) = \frac{1}{2}|\nabla u|^2 - W(u) - G(u),
$$

where $G : \mathbb{R} \to \mathbb{R}$ will be fixed shortly. We first obtain a differential inequality for ξ_G, cf. [15, 30].

Lemma 3.5. *On $|\nabla u| > 0$,*

$$
\begin{aligned}
\Delta\xi_G - \frac{2(W'+G')\nabla u}{|\nabla u|^2}\cdot\nabla\xi_G + 2G''\xi_G& \\
\ge (G')^2 + G'W' - 2G''(W+G) + \varepsilon\lambda(W'+G')&.
\end{aligned}
\tag{3.6}
$$

Proof. Compute

$$
\begin{aligned}
\Delta\xi_G &= \sum (u_{x_ix_j})^2 + \sum u_{x_i}(\Delta u)_{x_i} - W''|\nabla u|^2 - W'\Delta u - G''|\nabla u|^2 - G'\Delta u \\
&= \sum (u_{x_ix_j})^2 - (W'+G')\Delta u - G''|\nabla u|^2 \qquad \text{(by } \Delta u = W' - \varepsilon\lambda) \\
&= \sum (u_{x_ix_j})^2 - (W'+G')(W'-\varepsilon\lambda) - 2G''(G + W + \xi_G),
\end{aligned}
$$

where the last line is by substituting $|\nabla u|^2 = 2(G + W + \xi_G)$. On the other hand,

$$
\begin{aligned}
|\nabla u|^2 \sum (u_{x_ix_j})^2 &\ge \sum_j \Big(\sum_i u_{x_i} u_{x_ix_j} \Big)^2 \\
&= \sum_j \Big((\xi_G)_{x_j} + (W'+G')u_{x_j} \Big)^2 \\
&\ge 2(W'+G')\nabla u\cdot\nabla\xi_G \;+\; (W'+G')^2|\nabla u|^2.
\end{aligned}
$$

We may then conclude (3.6). □

For the rescaled discrepancy function ξ one has the following estimate.

Lemma 3.6. *There exist constants c_4 and ε_4 which depend only on λ_0, c_0 and W such that*

$$\sup_{B_{\varepsilon^{-1}}(0)} \xi \leq c_4 \varepsilon \tag{3.7}$$

for all $\varepsilon < \varepsilon_4$.

Proof. By $\sup_{\tilde{U}} |u| \leq c_0$ and standard elliptic estimates, there exists a constant c_5 such that $\sup_{B_{3\varepsilon^{-1}-1}} |\xi| \leq c_5$. Let

$$G(u) = \varepsilon^{1/2}(1 - (u - \gamma)^2/8).$$

We will use the properties $G > 0$ on $[-1.1, 1.1]$, $G'' = -\varepsilon^{1/2}/4 < 0$ and $G'W' \geq 0$ on $[-1, 1]$. We restrict ε so that $|u| \leq 1.1$ on $B_{3\varepsilon^{-1}}$. Later in the proof we further restrict ε.

First we show

Claim 1. $\sup_{B_{2\varepsilon^{-1}}} \xi_G < \varepsilon^{1/2}$.

Proof of Claim 1. Assume, for a contradiction, that $\sup_{B_{2\varepsilon^{-1}}} \xi_G \geq \varepsilon^{1/2}$. Let $\zeta \in C_c^\infty(B_{3\varepsilon^{-1}-1})$ be such that $\zeta \equiv 1$ on $B_{2\varepsilon^{-1}}$, $|\nabla\zeta| \leq 2\varepsilon$, $|\Delta\zeta| \leq 2\varepsilon^2$, $0 \leq \zeta \leq 1$ on $B_{3\varepsilon^{-1}-1}$. Consider $\tilde{\xi} = \xi_G + c_5\zeta$. Then, on $\partial B_{3\varepsilon^{-1}-1}$, $\tilde{\xi} \leq c_5$, while $\sup_{B_{2\varepsilon^{-1}}} \tilde{\xi} \geq c_5 + \varepsilon^{1/2}$. Thus, there exists an interior maximum point x_0 of $\tilde{\xi}$ and we have the following properties:

(a.1) $|\nabla u(x_0)|^2 > 2\varepsilon^{1/2}$.

(a.2) $|\nabla \xi_G(x_0)| = c_5 |\nabla\zeta(x_0)| \leq 2c_5\varepsilon, \quad \Delta\xi_G(x_0) \leq -c_5\Delta\zeta(x_0) \leq 2c_5\varepsilon^2$.

We consider three possibilities, and show that each of them is not possible for sufficiently small ε. We denote the right-hand side of (3.6) evaluated at x_0 as (RHS) and the left-hand side as (LHS).

Case 1. When $|u(x_0)| \in \left[0, \frac{1+|\gamma|}{2}\right]$.

The term $\varepsilon\lambda(W' + G')$, which is $O(\varepsilon)$, may be negative, but the rest of the terms in (RHS) are all non-negative. Since

$$-G''W \geq \varepsilon^{1/2} \min_{|u| \in \left[0, \frac{1+|\gamma|}{2}\right]} W(u)/4,$$

for sufficiently small ε we may bound (RHS) from below by some positive constant times $\varepsilon^{1/2}$.

Using (a.1) and (a.2) above, and since $G'' < 0$ and $\xi_G(x_0) > 0$, we see that (LHS) is bounded from above by some constant times $\varepsilon^{3/4}$.

Hence for sufficiently small ε, we obtain a contradiction.

Case 2. When $|u(x_0)| \in \left[\frac{1+|\gamma|}{2}, 1\right]$.

We have

$$(G')^2 \geq \frac{(1 - |\gamma|)^2}{64}\varepsilon, \quad -2G''(W + G) > 0,$$

$$G'W' \geq \frac{(1 - |\gamma|)}{8}\varepsilon^{1/2}|W'(u(x_0))|.$$

The last term is $|\varepsilon\lambda(W' + G')| = O\left(\varepsilon|W'(u(x_0))| + \varepsilon^{3/2}\right)$, thus (RHS) may be bounded from below by $c(\varepsilon + \varepsilon^{1/2}|W'(u(x_0))|)$ for some positive constant c for sufficiently small ε.

We have

$$(\text{LHS}) \leq 2c_5\varepsilon^2 + \frac{|W'| + |G'|}{\varepsilon^{1/4}} 4c_5\varepsilon \leq 2c_5\varepsilon^2 + 4c_5\left(\varepsilon^{3/4}|W'| + \varepsilon^{5/4}\right).$$

Thus for sufficiently small ε, this is a contradiction.

Case 3. When $|u(x_0)| \in [1, 1 + c_1\varepsilon]$.
Using $|W'(u(x_0))| = O(\varepsilon)$, (RHS) is bounded from below by $c\varepsilon$ for some positive c for sufficiently small ε, and (LHS) is bounded from above by $c\varepsilon^{7/4}$. Thus again a contradiction for sufficiently small ε. This ends the proof of Claim 1.

Note that $\xi_G = \xi - G \leq \varepsilon^{1/2}$ implies

$$\sup_{B_{2\varepsilon^{-1}}(0)} \xi \leq 2\varepsilon^{1/2} \tag{3.8}$$

Next we let

$$G(u) = L\varepsilon(1 - (u - \gamma)^2/8),$$

where L will be fixed shortly. We show

Claim 2. $\sup_{B_{\varepsilon^{-1}}} \xi_G \leq L\varepsilon$ *for* L *sufficiently large.*

Proof of Claim 2. Again assume otherwise for a contradiction. Take $\zeta \in C_c^\infty(B_{2\varepsilon^{-1}})$ with $\zeta \equiv 1$ on $B_{\varepsilon^{-1}}$, $|\nabla\zeta| \leq 2\varepsilon$, $|\Delta\zeta| \leq 2\varepsilon^2$ and $0 \leq \zeta \leq 1$ on $B_{2\varepsilon^{-1}}$. Consider $\tilde{\xi} = \xi_G + 2\varepsilon^{1/2}\zeta$. On $\partial B_{2\varepsilon^{-1}}$, $\tilde{\xi} \leq 2\varepsilon^{1/2}$ by (3.8), while $\sup \tilde{\xi} \geq 2\varepsilon^{1/2} + L\varepsilon$ on $B_{\varepsilon^{-1}}$. Thus $\tilde{\xi}$ achieves an interior maximum at, say, $x_0 \in B_{2\varepsilon^{-1}}$. At x_0, a similar argument as before shows

$$|\nabla u|^2 > 2L\varepsilon, \quad |\nabla\xi_G| \leq 4\varepsilon^{3/2}, \quad \Delta\xi_G \leq 4\varepsilon^{5/2}.$$

Case 1. When $|u(x_0)| \in \left[0, \frac{1+|\gamma|}{2}\right]$.
Here (RHS) is seen to be bounded from below by $c_6 L\varepsilon$ for sufficiently large L depending only on λ and W, in a similar manner as in Claim 1. On the other hand, we have

$$\text{(LHS)} \leq 4\varepsilon^{5/2} + 2\frac{c_7 + L\varepsilon}{\sqrt{2L\varepsilon}} \cdot 4\varepsilon^{3/2}.$$

Choose L so that $\frac{8c_7}{\sqrt{2L}} < c_6 L/2$ and we may choose ε small to obtain a contradiction.

Case 2. When $|u(x_0)| \in \left[\frac{1+|\gamma|}{2}, 1\right]$.
By choosing L large depending on λ and W, we may bound (RHS) from below by $c(L^2\varepsilon^2 + L\varepsilon|W'(u(x_0))|)$ for some positive constant c. On the other hand,

$$\text{(LHS)} \leq 4\varepsilon^{5/2} + 8\varepsilon L^{-1/2}|W'| + 8L^{1/2}\varepsilon^2.$$

Thus with large enough L, this is not possible.

Case 3. When $|u(x_0)| \in [1, 1 + c_1\varepsilon]$.
Using $|W'| = O(\varepsilon)$, one sees that (RHS) is bounded from below by $cL^2\varepsilon^2$ for some positive constant c if L is sufficiently large. Since (LHS) may be bounded by $c(\varepsilon^{5/2} + L^{1/2}\varepsilon^2)$, we may exclude this case by choosing large L.

This ends the proof of Claim 2, and since $G(u) \leq L\varepsilon$ we conclude the proof of (3.7) by setting $c_4 = 2L$. □

Remark. For the Allen-Cahn equation

$$u_t = \Delta u - \frac{W'(u)}{\varepsilon^2}$$

on $U \times [0, T]$, with $\sup_{U \times [0,T]} |u| \leq c_0$, the reader may check that the preceding argument can be modified easily to prove a uniform maximum bound for $\frac{\varepsilon|\nabla u|^2}{2} - \frac{W(u)}{\varepsilon}$ on $\tilde{U} \times [t, T]$, if $\tilde{U} \subset\subset U$ and $t > 0$. This is a local version of the result in [40]; note that $|u| \leq 1$ is not needed in the proof.

4. Rectifiability of the limit varifold

In addition to assumptions **A** *and* **B** *we assume $\tilde{U}$ is open and $\tilde{U} \subset\subset U$.*

Let μ be the measure on U defined (for a suitable subsequence) by

$$\mu(\phi) = \lim_{i\to\infty} \int \left(\frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \right) \phi$$

for nonnegative $\phi \in C_c(U)$.

In Proposition 4.4 we show that the limit varifold V defined in Section 2 is rectifiable and that $\|V\|$ is the measure theoretic limit μ of the energy distribution on U (normalized by the factor $\frac{1}{2}$). In Section 5 we will see that $\sigma^{-1}V$ is an integral varifold, and so the density function θ for V equals $N\sigma$, $\mathcal{H}^{n-1}$ a.e. for some integer valued function N.

In Proposition 4.2 we note that either $u^i \to +1$ or $u^i \to -1$ uniformly on each connected compact subset of $U \setminus \mathrm{supp}\|V\|$. In particular, $\mathrm{supp}\|\partial\{u^\infty = 1\}\| \subset \mathrm{supp}\|V\|$. In Section 6.3 we give an example where equality does not hold.

Also as noted in Proposition 4.2, the terms $\frac{\varepsilon_i}{2}|\nabla u^i|^2$ and $\varepsilon_i^{-1}W(u^i)$ in the energy distribution converge uniformly to zero on compact subsets of $U \setminus \mathrm{supp}\|V\|$. Although their sum converges measure theoretically to μ, it follows from Proposition 4.3 that one has equipartition of energy in the limit in the sense that their difference (the discrepancy function) converges to zero in $L^1_{\mathrm{loc}}(U)$.

In Proposition 4.4 we also see that the generalized mean curvature vector H for V is zero on $\mathrm{supp}\|V\| \setminus M^\infty$ and equals $\frac{\lambda_\infty}{\theta(x)}\nu^\infty(x)$ on M^∞, where $M^\infty \subset \mathrm{supp}\|\partial\{u^\infty = 1\}\|$ is the reduced boundary of $\{u^\infty = 1\}$. Since $\mathrm{supp}\|V\|$ is a smooth manifold on an open dense subset by [1], and at such points H is just the classical mean curvature of the manifold, it is interesting to note the dependence of H on θ and hence on the multiplicity function N.

Proposition 4.1. *There exist constants $0 < D_1 \le D_2 < \infty$ and $r_0 > 0$ which depend only on λ_0, c_0, E_0, $\mathrm{dist}(\tilde{U}, \partial U)$ and W, such that*

$$D_1 r^{n-1} \le \mu(B_r(x)) \le D_2 r^{n-1}$$

for all $0 < r < r_0$, $x \in \mathrm{supp}\mu \cap \tilde{U}$ and $B_r(x) \subset \tilde{U}$.

Proof. The existence of D_2 is immediate from (3.5).

To establish the lower bound, let $x \in \mathrm{supp}\mu \cap \tilde{U}$.

Claim. *On passing to a subsequence there exist $x_i \in \tilde{U}$ such that $u^i(x_i) \in [-\alpha, \alpha]$ and $x_i \to x$ as $i \to \infty$.*

Proof of Claim. Suppose the converse. Then there exists some $s > 0$ such that $B_s(x) \subset \tilde{U}$ and $B_s(x) \cap \{|u^i| \le \alpha\} = \emptyset$ for all sufficiently large i. For each such i either $u^i > \alpha$ on $B_s(x)$ or $u^i < -\alpha$ on $B_s(x)$. If $u^i > \alpha$, by using the argument in Proposition 3.2 one shows (for $i \ge N$ say) that $u^i \in [1 - c\varepsilon_i, 1 + c\varepsilon_i]$ on $B_{s/2}(x)$ where c is independent of i. Similarly, if $u^i < -\alpha$, then $u^i \in [-1 - c\varepsilon_i, -1 + c\varepsilon_i]$ on $B_{s/2}(x)$. This implies $W(u^i) = O(\varepsilon_i^2)$ and thus $\frac{W(u^i)}{\varepsilon_i} \to 0$ uniformly on $B_{s/2}(x)$ as $i \to \infty$.

Also we have

$$\Delta\left(\frac{\varepsilon_i}{2}|\nabla u^i|^2\right) = \frac{W''(u^i)}{\varepsilon_i}|\nabla u^i|^2 + \varepsilon_i|\nabla^2 u^i|^2 \ge \frac{\kappa}{\varepsilon_i}|\nabla u^i|^2,$$

which implies that

$$\int \phi\, \varepsilon_i |\nabla u^i|^2 \le \frac{\varepsilon_i^2}{2\kappa} \int \phi\, \Delta(\varepsilon_i |\nabla u^i|^2) = \frac{\varepsilon_i^2}{2\kappa} \int (\Delta\phi)\, \varepsilon_i |\nabla u^i|^2 \to 0$$

as $i \to \infty$ for any nonnegative $\phi \in C_c^2(B_s(x))$. Hence we may conclude that $\mu(B_{s/2}(x)) = 0$, which is a contradiction to $x \in \mathrm{supp}\mu$. This ends the proof of the claim.

For any $x \in \tilde{U} \cap \mathrm{supp}\mu$ and $B_r(x) \subset \tilde{U}$, (3.5) shows that

$$\begin{aligned}\frac{1}{r^{n-1}}\mu(B_r(x)) &\geq \lim_{i\to\infty} \frac{1}{r^{n-1}} \int_{B_{r/2}(x_i)} \frac{\varepsilon_i|\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \\ &\geq -\frac{c_3 r}{2^{n-1}} + \lim_{i\to\infty} \frac{1}{(2\varepsilon_i)^{n-1}} \int_{B_{\varepsilon_i}(x_i)} \frac{\varepsilon_i|\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i}.\end{aligned}$$

If we let $\tilde{u}^i(y) = u^i(\varepsilon_i y + x_i)$ for $y \in B_1(0)$, we have $\tilde{u}^i(0) \in [-\alpha, \alpha]$ and $\Delta\tilde{u}^i = W'(\tilde{u}^i) - \lambda_i\varepsilon_i$. Since $|\tilde{u}^i|_{C^1(B_{1/2})} \leq c(W, \lambda_0)$, we see by considering the W term in the energy functional that there exists a constant $c_8 > 0$ depending only on W and λ_0 such that the scaled energy on $B_{\varepsilon_i}(x_i)$ is $\geq c_8$. Restrict r so that $c_3 r < c_8/2$, and we obtain $\mu(B_r(x)) \geq r^{n-1}c_8/2^n$. This shows the existence of $D_1 = c_8/2^n$. □

Proposition 4.2. *Either $u^i \to +1$ or $u^i \to -1$ uniformly on each connected compact subset of $U \setminus \mathrm{supp}\|V\|$. In particular, $\mathrm{supp}\|\partial\{u^\infty = 1\}\| \subset \mathrm{supp}\|V\|$. The terms $\frac{\varepsilon_i}{2}|\nabla u^i|^2$ and $\varepsilon_i^{-1}W(u^i)$ converge uniformly to zero on compact subsets of $U \setminus \mathrm{supp}\|V\|$.*

Proof. This follows immediately from the argument for the previous proposition. □

Let

$$\xi^i = \frac{\varepsilon_i|\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i},$$

and define (passing to a subsequence if necessary) the measure $|\xi|$ on U by

$$|\xi|(\phi) = \lim_{i\to\infty} \int |\xi^i|\phi$$

for nonnegative $\phi \in C_C(U)$. Thus $|\xi|$ is the measure theoretic limit of the absolute values of the discrepancy functions.

Proposition 4.3. *$|\xi|$ is the zero measure and so $\xi_i \to 0$ in $L^1_{loc}(U)$. Moreover, both $\frac{\varepsilon_i}{2}|\nabla u^i|^2 - |\nabla w^i|$ and $\frac{1}{\varepsilon_i}W(u^i) - |\nabla w^i|$ also converge to zero in $L^1_{loc}(U)$.*

Proof. First we claim that

$$\liminf_{r\to 0} \frac{1}{r^{n-1}}|\xi|(B_r(x)) = 0 \tag{4.1}$$

for all $x \in \mathrm{supp}|\xi| \cap \tilde{U}$. Otherwise, there would exist $x \in \mathrm{supp}|\xi| \cap \tilde{U}$, $R > 0$ and $b > 0$ such that $R \leq r_0$ and $|\xi|(B_r(x)) \geq br^{n-1}$ for all $0 < r \leq R$. Define $r_1 = \min\{b/(4c_2\omega_n), R\}$ and $r_2 = r_1 \min\{\exp[-4(4D_2 + c_3 r_1)/b], 1/2\}$ and using Proposition 4.1 and the definition of $|\xi|$ choose a large enough i such that

$$\frac{1}{r_1^{n-1}} \int_{B_{r_1}(x)} \frac{\varepsilon_i|\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \leq 2D_2, \qquad \frac{1}{\tau^{n-1}} \int_{B_\tau(x)} |\xi^i| \geq \frac{b}{2}$$

for all $r_2 \leq \tau \leq r_1$. By (3.4) and the definition of r_1,

$$\begin{aligned}\frac{1}{\tau^{n-1}} \int_{B_\tau(x)} \left(\frac{W(u^i)}{\varepsilon_i} - \frac{\varepsilon_i|\nabla u^i|^2}{2}\right)^+ &\geq \frac{1}{\tau^{n-1}} \int_{B_\tau(x)} |\xi^i| - c_2\omega_n\tau \\ &\geq \frac{b}{2} - c_2\omega_n r_1 \geq \frac{b}{4}\end{aligned}$$

for all $r_2 \leq \tau \leq r_1$. By (3.5) it follows that

$$\begin{aligned} 2D_2 &\geq \frac{1}{r_1^{n-1}} \int_{B_{r_1}(x)} \frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \\ &\geq \frac{b}{4} \int_{r_2}^{r_1} \frac{d\tau}{\tau} - c_3 r_1 = \frac{b}{4} \ln\left(\frac{r_1}{r_2}\right) - c_3 r_1 \geq 4D_2, \end{aligned}$$

which is a contradiction, and so we have proved (4.1).

Combined with Proposition 4.1 and $\mathrm{supp}|\xi| \subset \mathrm{supp}\mu$, we have

$$\liminf_{r\to 0} \frac{|\xi|(B_r(x))}{\mu(B_r(x))} = 0$$

for all $x \in \mathrm{supp}|\xi|$. A standard result in measure theory (see [19, Lemma 1 page 37]) then shows that $|\xi| = 0$.

It follows that $\xi_i \to 0$ in $L^1_{\mathrm{loc}}(U)$.

By completing the square and using $2|\nabla w^i| = \sqrt{2W(u^i)}\,|\nabla u^i|$, we see that

$$\begin{aligned} \left|\frac{\varepsilon_i}{2}|\nabla u^i|^2 + \frac{W(u^i)}{\varepsilon_i} - 2|\nabla w^i|\right| &= \left(\sqrt{\frac{\varepsilon_i}{2}}|\nabla u^i| - \sqrt{\frac{W(u^i)}{\varepsilon_i}}\right)^2 \\ &\leq \left|\frac{\varepsilon_i|\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i}\right| = |\xi^i|. \end{aligned}$$

This implies the remaining claims in the proposition. □

Proposition 4.4. *The limit varifold V satisfies $\|V\| = \frac{1}{2}\mu$ and is rectifiable.*

The first variation of V is given by

$$\delta V(g) = \frac{\lambda_\infty}{2} \int_U u^\infty \mathrm{div} g = -\lambda_\infty \int_{M^\infty} g \cdot \nu^\infty \, d\mathcal{H}^{n-1}$$

for any $g \in C^1_c(U;\mathbb{R}^n)$, where $M^\infty \subset \mathrm{supp}\|V\|$ is the reduced boundary of $\{u^\infty = 1\}$.

The generalized mean curvature vector H is given by

$$H(x) = \begin{cases} \frac{\lambda_\infty}{\theta(x)} \nu^\infty(x) & \mathcal{H}^{n-1} \text{ a.e. } x \in M^\infty \\ 0 & \mathcal{H}^{n-1} \text{ a.e. } x \in \mathrm{supp}\|V\| \setminus M^\infty, \end{cases}$$

where θ is the density function for $\|V\|$.

Proof. Since $\|V\| = \lim \|V^i\|$ and $\|V^i\| = |\nabla w^i|\, d\mathcal{L}^n$ from Section 2.1, it follows from Proposition 4.3 and the definition of μ that $\frac{1}{2}\mu = \|V\|$.

Next, rearranging terms in (3.2) and using $\frac{u^i_{x_j}}{|\nabla u^i|} = \frac{w^i_{x_j}}{|\nabla w^i|}$, we have

$$\int \left(\mathrm{div} g - \sum_{j,k} \frac{w^i_{x_j}}{|\nabla w^i|} \frac{w^i_{x_k}}{|\nabla w^i|} g^j_{x_k}\right) \varepsilon_i |\nabla u^i|^2 = \int \left(\frac{\varepsilon_i |\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i} + \lambda_i u^i\right) \mathrm{div} g,$$

for any $g \in C^1_c(U)$. ¿From (2.2) and the previous results

$$\left|\delta V^i(g) - \lambda_i \int u^i \,\mathrm{div} g\right| \leq c \sup|\nabla g| \int_{\tilde{U}} |\xi_i|,$$

if $\mathrm{supp}\, g \subset \tilde{U}$ (where c is an absolute constant; $c = 2(n + n^2) + n$ will do). With this, Proposition 4.3 and the remarks in Section 2.1, and since $V^i \to V$ in the weak sense of varifolds, we have

$$\delta V(g) = \frac{\lambda_\infty}{2} \int u^\infty \,\mathrm{div} g.$$

As discussed in Section 2.1, one may integrate by parts and so obtain the second expression for $\delta V(g)$.

Since $\mathcal{H}^{n-1}(M^\infty) \leq \|\partial\{u^\infty - 1\}\|(U) \leq E_0/2\sigma$ from Section 2.1, one has $|\delta V(g)| \leq |\lambda_\infty| \sup|g| \, E_0/2\sigma$ for all $g \in C^1_c(U)$, and so $\|\delta V\|$ is a Radon measure on U. Combined with the lower density estimate in Proposition 4.1 and Allard's rectifiability theorem [1, 5.5.(1)], we conclude that V is rectifiable.

¿From Proposition 4.1, the $(n-1)$-dimensional density of $\|V\|$ is bounded below by $\frac{D_1}{2\omega_{n-1}}$ on supp$\|V\|$ and so $\mathcal{H}^{n-1}\lfloor\text{supp}\|V\| \leq \frac{2\omega_{n-1}}{D_1}\|V\|$. Since δV is absolutely continuous with respect to $\mathcal{H}^{n-1}\lfloor M^\infty \leq \mathcal{H}^{n-1}\lfloor\text{supp}\|V\|$, it is hence also absolutely continuous with respect to $\|V\|$. From Section 2.2, it follows that

$$\delta V(g) = -\int g \cdot H \, d\|V\| = -\int_{\text{supp}\|V\|} g \cdot H \, \theta d\mathcal{H}^{n-1}.$$

(The underlying rectifiable set for $\|V\|$ can be taken as supp$\|V\|$ because of the lower density bound in Proposition 4.1). The expression for H now follows from the second expression for $\delta V(g)$ in the statement of the proposition. □

Remark. The density function θ is everywhere well-defined, either by the standard theory for varifolds with bounded mean curvature [39, Section 17], or directly using the monotonicity formula for $\|V\|$ which follows from (3.5) since $\|V\| = \frac{1}{2}\mu$. ¿From Proposition 4.1, $\frac{D_1}{2\omega_{n-1}} \leq \theta \leq \frac{D_2}{2\omega_{n-1}}$ everywhere on supp$\|V\|$. In Section 5 we see that $\sigma^{-1}V$ is an integral varifold, and so $\theta = N\sigma$ for some positive integer N, $\mathcal{H}^{n-1}$ a.e. on supp$\|V\|$.

5. Integrality of the limit varifold

In this section we show that the limit varifold V defined in Section 2 is integral, modulo division by σ, and we finish the proof of Theorem 1.

The first proposition gives a uniform smallness estimate on the energy, independent of ε, for u in the region $\{u \approx \pm 1\}$. It is convenient to work in a fixed ball of radius 3.

Proposition 5.1. *Assume* **B** *is true with u^i, ε^i and U there replaced by u, ε and $B_3(0)$ respectively, and suppose $s > 0$. Then there exist positive constants b and ε_5 depending only on λ_0, c_0, E_0, W and s, such that*

$$\int_{B_1(0)\cap\{|u|\geq 1-b\}} \frac{W(u)}{\varepsilon} \leq s$$

whenever $\varepsilon \leq \varepsilon_5$.

The following two lemmas will be used in the proof. We continue to make the same assumptions on u as in Proposition 5.1.

Define

$$Z_\alpha = \{\, x \in B_3(0) \mid u(x) \in [-\alpha, \alpha] \,\}.$$

Also define

$$e_\varepsilon = \frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}, \quad \xi_\varepsilon = \frac{\varepsilon|\nabla u|^2}{2} - \frac{W(u)}{\varepsilon}.$$

Lemma 5.2. *There exist positive constants c_{10} and ε_6 depending only on λ_0 and W, such that if $x_0 \in B_1(0)$ and*

$$u(x_0) < 1 - \varepsilon^\beta \ \text{ or } \ u(x_0) > -1 + \varepsilon^\beta, \quad \text{where } \frac{1}{c_{10}\,|\ln\varepsilon|} < \beta < \min\left\{\frac{2}{3}, \frac{1}{c_{10}\,\varepsilon|\ln\varepsilon|}\right\},$$

then

$$dist(x_0, Z_\alpha) \leq c_{10}\beta\,\varepsilon|\ln\varepsilon|,$$

provided $0 < \varepsilon < \varepsilon_6$.

Proof. Rescale $B_1(x_0)$ by ε and write $\tilde{u}(x) = u(x_0 + \varepsilon x)$. One may prove there exists an entire radial solution $\psi(x) \geq 1$ for the problem

$$\begin{aligned} \Delta\psi &= \frac{\kappa}{4}\psi \quad \text{on } \mathbb{R}^n \\ \psi(0) &= 1, \end{aligned}$$

and there exists a constant $c_{10} = c_{10}(\kappa, n)$ such that $\psi(x) > \exp(|x|/c_{10})$ if $|x| \geq 1$.

Let $r \equiv c_{10}\beta\,|\ln\varepsilon|$. Note that $1 - \varepsilon^\beta \exp(r/c_{10}) = 0 < \alpha$ and $1 \leq r \leq \varepsilon^{-1}$.

Suppose $\tilde{u}(0) < 1 - \varepsilon^\beta$ and $\inf_{B_r(0)} \tilde{u} > \alpha$ to obtain a contradiction. Define $\phi(x) = 1 - \varepsilon^\beta\psi(x)$. Then ϕ satisfies $\Delta\phi = \frac{\kappa}{4}(\phi - 1)$, and on $|x| = r$, $\phi(x) < 1 - \varepsilon^\beta \exp(r/c_{10}) < \alpha < \inf_{B_r(0)} \tilde{u}$. Hence $\tilde{u} - \phi > 0$ on $|x| = r$. Since $\tilde{u}(0) < 1 - \varepsilon^\beta = \phi(0)$, $\tilde{u} - \phi$ achieves a negative minimum value at an interior point y, say, and we have $\Delta(\tilde{u} - \phi)(y) \geq 0$ and $(\tilde{u} - \phi)(y) < 0$. We then have at y,

$$\begin{aligned} \Delta(\tilde{u} - \phi) &= W'(\tilde{u}) - \varepsilon\lambda - \frac{\kappa}{4}(\phi - 1) \\ &\leq \varepsilon\lambda_0 + W'(\phi) - \frac{\kappa}{4}(\phi - 1) \\ &\leq \varepsilon\lambda_0 - \frac{\kappa}{2}\varepsilon^\beta\psi + \frac{\kappa}{4}\varepsilon^\beta\psi, \end{aligned}$$

since $\phi(y) > \tilde{u}(y) > \alpha$, $W'' \geq \kappa$ on $[\alpha, 1]$, $W'(1) = 0$ and $1 - \phi(y) = \varepsilon^\beta\psi(y)$. Since $\psi \geq 1$ and $0 < \beta \leq 2/3$, and assuming $\varepsilon \leq 1$, we obtain

$$0 \leq \Delta(\tilde{u} - \phi) \leq \varepsilon\lambda_0 - \kappa\varepsilon^\beta/4 \leq \varepsilon^{2/3}(\varepsilon^{1/3}\lambda_0 - \kappa/4).$$

If we further restrict ε by $\varepsilon \leq \kappa/((5\lambda_0))^3$, we obtain a contradiction.

This shows the statement of the lemma after rescaling back. The sup estimate is similar. □

Lemma 5.3. *There exist positive constants c_{11} and ε_7 depending only on λ_0, c_0, E_0 and W, such that if $\varepsilon \leq r \leq 1$ then*

$$\mathcal{H}^n(\{x \in B_2(0) \,|\, \mathrm{dist}(x, Z_\alpha) < r\}) \leq c_{11} r,$$

provided $0 < \varepsilon < \varepsilon_7$.

Proof. By arguing as in the last paragraph in the proof of Proposition 4.1, there exist positive constants r_0, ε_7 and c_{12} depending only on λ_0 and W, such that $E(r, x) \geq c_{12}$ if $\varepsilon \leq r \leq r_0$ and $x \in Z_\alpha \cap B_2(0)$.

Given r with $\varepsilon \leq r \leq r_0$, let $\mathcal{B}$ be the collection of all balls with center in Z_α and radius r. Using Vitali's covering lemma, choose a pair-wise disjoint subcollection of balls $\mathcal{B}' \subset \mathcal{B}$ so that $\bigcup_{\mathcal{B}} B_r(x) \subset \bigcup_{\mathcal{B}'} B_{5r}(x)$. Since

$$\{\, x \in B_2(0) \mid \mathrm{dist}(x, Z_\alpha) < r \,\} \subset \bigcup_{\mathcal{B}'} B_{5r}(x),$$

we need only estimate $\omega_n(5r)^n\, n(\mathcal{B}')$, where $n(\mathcal{B}')$ is the number of balls in $\mathcal{B}'$. Since

$$n(\mathcal{B}')c_{12}r^{n-1} \leq \sum_{\mathcal{B}'} \int_{B_r(x)} e_\varepsilon \leq \int_{B_3(0)} e_\varepsilon \leq 3^{n-1}E_0,$$

we have $\omega_n(5r)^n\, n(\mathcal{B}') \leq r\omega_n E_0 5^n 3^{n-1}/c_{12}$.

Setting $c_{11} = \max\{\omega_n E_0 5^n 3^{n-1}/c_{12}\,,\, 2^n\omega_n/r_0\}$ gives the required result. □

Proof of Proposition 5.1. First suppose $b > 0$ satisfies $c_{10}\,|\ln b| \geq 1$ and $1 - b > \alpha$, and choose an integer $J = J(\varepsilon, b) \geq 1$ such that $\varepsilon^{1/2^{J+1}} \in (b, \sqrt{b}]$. Restrict ε so that $\varepsilon \leq \min\{\varepsilon_6, \varepsilon_7\}$ and $\varepsilon^{-1} \geq c_{10}\frac{2}{3}|\ln \varepsilon|$.

For $j = 1, \cdots, J$, define

$$A_j = \left\{ x \in B_1(0) \;\middle|\; 1 - \varepsilon^{1/2^{j+1}} \leq |u(x)| \leq 1 - \varepsilon^{1/2^j} \right\}.$$

By applying Lemma 5.2 with $\beta = 1/2^j$ for $x \in A_j$ (note that $1 \leq c_{10}\beta\,|\ln\varepsilon| \leq \varepsilon^{-1}$ is satisfied with these choices) we conclude that A_j is within $c_{10}2^{-j}\,\varepsilon|\ln\varepsilon|$ of $Z_\alpha \cap B_2(0)$. Lemma 5.3 then shows that

$$\mathcal{H}^n(A_j) \leq c_{11}c_{10}2^{-j}\,\varepsilon|\ln\varepsilon|$$

for $j = 1, \cdots, J$.

On A_j, using $|u| \geq 1 - \varepsilon^{1/2^{j+1}}$,

$$\frac{W(u)}{\varepsilon} \leq \max_{t\in[\alpha,1]} W''(t)\cdot\varepsilon^{-1}(\varepsilon^{1/2^{j+1}})^2/2 \leq c_{13}(W)\varepsilon^{2^{-j}-1}.$$

Let $Y = B_1(0) \cap \{1 - b \leq |u| \leq 1 - \sqrt{\varepsilon}\} \subset \bigcup_{j=1}^J A_j$. Since $\varepsilon^{1/2^{J+1}} < \sqrt{b}$ it now follows with $c_{14} = c_{10}c_{11}c_{12}$ (depending only on λ_0, E_0 and W) that

$$\begin{aligned}\int_Y \frac{W(u)}{\varepsilon} &\leq \sum_{j=1}^J \int_{A_j} \frac{W(u)}{\varepsilon} \leq c_{14}\,|\ln\varepsilon| \sum_{j=1}^J 2^{-j}\varepsilon^{2^{-j}} \\ &\leq c_{14}\,|\ln\varepsilon| \int_0^{J+1} 2^{-t}\varepsilon^{2^{-t}} = c_{14}\left(\varepsilon^{2^{-(J+1)}} - \varepsilon\right)/\ln 2 \leq c_{14}\sqrt{b}/\ln 2.\end{aligned}$$

We restrict b so that the last term is less than $\frac{s}{2}$.

To estimate the integral on $\{1 - \sqrt{\varepsilon} \leq |u|\}$ let

$$A_0 = \left\{ x \in B_1(0) \;\middle|\; 1 - \sqrt{\varepsilon} \leq |u(x)| \leq 1 - \varepsilon^{2/3} \right\}$$

and similarly estimate

$$\int_{A_0} \frac{W(u)}{\varepsilon} \leq c_{13}c_{11}c_{10}\frac{2}{3}\,\varepsilon|\ln\varepsilon|.$$

Finally for $\{|u| \geq 1 - \varepsilon^{2/3}\}$, using Proposition 3.2,

$$\int_{B_1(0)\cap\{1-\varepsilon^{2/3}\leq|u|\}} \frac{W(u)}{\varepsilon} \leq c_{15}(\lambda_0, c_0, W)\,\varepsilon.$$

Restricting ε again, we obtain the stated inequality. $\square$

Proposition 5.5 is analogous to [1, Theorem 6.2] in the proof of the compactness theorem for integral varifolds. We point out that we do not have a uniform control on the first variations $\|\delta V^i\|$ and thus are faced with an analogous but different situation.

Define $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$ by $T(x) = (x_1, \ldots, x_{n-1})$, and $T^\perp : \mathbb{R}^n \to \mathbb{R}$ by $T^\perp(x) = x_n$, where $x = (x_1, \ldots, x_n)$. Also define $\nu = (\nu_1, \ldots, \nu_n) = \frac{\nabla u}{|\nabla u|}$ whenever $|\nabla u| \neq 0$ and $\nu = 0$ when $|\nabla u| = 0$.

First we show

Lemma 5.4. *Suppose*

(1) $N \geq 1$ *is an integer,* Y *is a subset of* $\mathbb{R}^n$*,* $0 < R < \infty$*,* $1 < M < \infty$*,* $0 < a < \infty$*,* $0 < \varepsilon < 1$*,* $0 < \eta < 1$*,* $0 < E_0 < \infty$ *and* $-\infty \leq t_1 < t_2 \leq \infty$*.*
(2) Y *has no more than* $N+1$ *elements,* $T(y) = 0$ *for all* $y \in Y$*,* $Y \subset \{x \,|\, t_1 + a < x_n < t_2 - a\}$ *and* $|y - z| > 3a$ *for any distinct* $y, z \in Y$*.*
(3) $(M + 1)\operatorname{diameter} Y < R$*, and denote* $\tilde{R} \equiv M \operatorname{diameter} Y$*.*

(4) *On* $\{x \in \mathbb{R}^n \,|\, \mathrm{dist}(x,Y) < R\}$, *u satisfies (1.2) with* $|\lambda| \leq \eta$, $|u| \leq 2$ *and* $\xi_\varepsilon \leq \eta$.

(5) *For each* $x = (x_1, \cdots, x_n) \in Y$,

$$\int_0^R \frac{d\tau}{\tau^n} \int_{B_\tau(x) \cap \{y_n = t_j\}} |e_\varepsilon (y_n - x_n) - \varepsilon u_{x_n} (y-x) \cdot \nabla u| \, d\mathcal{H}^{n-1} y \leq \eta$$

for $j = 1, 2$.

(6) *For each* $x \in Y$ *and* $a \leq r \leq R$,

$$\int_{B_r(x)} |\xi_\varepsilon| + (1 - (\nu_n)^2) \varepsilon |\nabla u|^2 \leq \eta r^{n-1} \text{ and } \int_{B_r(x)} \varepsilon |\nabla u|^2 \leq E_0 r^{n-1}.$$

Then the following hold:

A: *There exists* $t_3 \in (t_1, t_2)$ *such that* $|x_n - t_3| \geq a$ *and*

$$\int_0^{\tilde{R}} \frac{d\tau}{\tau^n} \int_{B_\tau(x) \cap \{y_n = t_3\}} |e_\varepsilon (y_n - x_n) - \varepsilon u_{x_n} (y-x) \cdot \nabla u| \, d\mathcal{H}^{n-1} y$$
$$\leq 3(N+1) N M (\eta + E_0^{1/2} \eta^{1/2})$$

for each $x \in Y$.

B: *Denote*

$$Y_1 = Y \cap \{x \,|\, t_1 < x_n < t_3\}, \qquad Y_2 = Y \cap \{x \,|\, t_3 < x_n < t_2\},$$
$$\mathcal{S}_0 = \{x \,|\, t_1 < x_n < t_2 \text{ and } \mathrm{dist}(Y, x) < R\},$$
$$\mathcal{S}_1 = \{x \,|\, t_1 < x_n < t_3 \text{ and } \mathrm{dist}(Y_1, x) < \tilde{R}\},$$
$$\mathcal{S}_2 = \{x \,|\, t_3 < x_n < t_2 \text{ and } \mathrm{dist}(Y_2, x) < \tilde{R}\}.$$

Y_1 *and* Y_2 *are non-empty and*

$$\frac{1}{\tilde{R}^{n-1}} \left\{ \int_{\mathcal{S}_1} e_\varepsilon + \int_{\mathcal{S}_2} e_\varepsilon \right\} \leq \left(1 + \frac{1}{M}\right)^{n-1} \frac{1}{R^{n-1}} \int_{\mathcal{S}_0} e_\varepsilon + c(n) \eta (R+1)$$

holds.

Proof. Let $\rho_2(y) : \mathbb{R}^n \to \mathbb{R}$ be a smooth approximation of the characteristic function of the set $\mathcal{S} \equiv \{y \in \mathbb{R}^n \,|\, t_1 < y_n < t_2\}$ which depends only on y_n. Let $x \in Y$ (and change the coordinates so that $x = 0$) and let $\rho_1(y)$ be a smooth approximation of the characteristic function $\chi_{B_r(0)}$, where $0 < r < R$. In (3.2), we let $g^j(y) = y_j \rho_1(y) \rho_2(y)$, and after letting $\rho_1 \to \chi_{B_r(0)}$ and similarly proceeding as in Lemma 3.1, we obtain (with W replaced by W_ε in e_ε and ξ_ε only in the next two lines)

$$\frac{d}{dr} \left\{ \frac{1}{r^{n-1}} \int_{B_r} e_\varepsilon \rho_2 \right\} + \frac{1}{r^n} \int_{B_r} \xi_\varepsilon \rho_2$$
$$- \frac{\varepsilon}{r^{n+1}} \int_{\partial B_r} (y \cdot \nabla u)^2 \rho_2 - \frac{1}{r^n} \int_{B_r} \{e_\varepsilon y_n - \varepsilon u_{x_n} (y \cdot \nabla u)\} \rho_2' = 0.$$

After integrating over $[r, R]$ and letting $\rho_2 \to \chi_{\mathcal{S}}$, and then using (4) and (5), we obtain

$$\frac{1}{R^{n-1}} \int_{B_R \cap \mathcal{S}} e_\varepsilon \geq \frac{1}{r^{n-1}} \int_{B_r \cap \mathcal{S}} e_\varepsilon - c(n) \eta (R+1) \tag{5.1}$$

where $c(n)$ depends only on the dimension n. Next, choose $\tilde{y}, \tilde{z} \in Y$ such that $\tilde{z}_n - \tilde{y}_n \geq \mathrm{diameter}\, Y/N$ and that there is no element of Y in $\{x \in \mathbb{R}^n \,|\, \tilde{y}_n < x_n < \tilde{z}_n\}$. Let $\tilde{t}_1 = \tilde{y}_n + \frac{\tilde{z}_n - \tilde{y}_n}{3}$ and $\tilde{t}_2 = \tilde{z}_n - \frac{\tilde{z}_n - \tilde{y}_n}{3}$. To choose an appropriate $t \in [\tilde{t}_1, \tilde{t}_2]$ which satisfies A, we first observe, for $x \in Y$ and $y \in B_r(x)$,

$$I \equiv |e_\varepsilon (y_n - x_n) - \varepsilon u_{x_n} (y - x) \cdot \nabla u|$$
$$= \left| (-\xi_\varepsilon)(y_n - x_n) + \varepsilon |\nabla u|^2 ((y_n - x_n) - \nu_n (y-x) \cdot \nu) \right|$$

$$\leq |\xi_\varepsilon| r + \varepsilon |\nabla u|^2 r \left(1 - (\nu_n)^2 + \sqrt{1-(\nu_n)^2}\right).$$

Using (6), we compute

$$\int_{\tilde{t}_1}^{\tilde{t}_2} dt \int_0^{\tilde{R}} \frac{d\tau}{\tau^n} \int_{B_\tau(x)\cap\{y_n=t\}} I \, d\mathcal{H}^{n-1} y = \int_0^{\tilde{R}} \frac{d\tau}{\tau^n} \int_{B_\tau(x)\cap\{\tilde{t}_1<y_n<\tilde{t}_2\}} I \, dy$$

$$\leq \tilde{R}(\eta + E_0^{1/2}\eta^{1/2}).$$

Thus, we may choose $t_3 \in [\tilde{t}_1, \tilde{t}_2]$ such that

$$\int_0^{\tilde{R}} \frac{d\tau}{\tau^n} \int_{B_\tau(x)\cap\{y_n=t_3\}} I \, d\mathcal{H}^{n-1} y \leq \frac{(N+1)\tilde{R}(\eta + E_0^{1/2}\eta^{1/2})}{\tilde{t}_2 - \tilde{t}_2}$$

for all $x \in Y$. Since $\tilde{t}_2 - \tilde{t}_1 \geq \operatorname{diameter} Y/3N$, we have $\tilde{R}/(\tilde{t}_2 - \tilde{t}_1) \leq 3MN$, and we obtain A. Define $\mathcal{S}_1$ and $\mathcal{S}_2$ as in B. For any $x \in Y$, we have $\mathcal{S}_1 \cup \mathcal{S}_2 \subset B_{(\tilde{R}+\operatorname{diam} Y)}(x) \cap \mathcal{S}$, thus

$$\frac{1}{\tilde{R}^{n-1}} \left\{ \int_{\mathcal{S}_1} e_\varepsilon + \int_{\mathcal{S}_2} e_\varepsilon \right\} \leq \frac{1}{\tilde{R}^{n-1}} \int_{B_{(\tilde{R}+\operatorname{diam} Y)}(x)\cap\mathcal{S}} e_\varepsilon$$

$$\leq \left(1 + \frac{1}{M}\right)^{n-1} \left\{ \frac{1}{R^{n-1}} \int_{B_R(x)\cap\mathcal{S}} e_\varepsilon + c(n)s(R+1) \right\}.$$

We used (5.1) in the last inequality. Finally, noting that $B_R(x) \cap \mathcal{S} \subset \mathcal{S}_0$, we obtain B. □

Starting with $t_1 = -\infty$ and $t_2 = \infty$, we inductively use Lemma 5.4 to separate $\mathbb{R}^n$ into stacked horizontal domains until we separate each element of Y. By choosing M large and then choosing η suitably small, we obtain

Proposition 5.5. *Corresponding to each R, E_0, s and N such that $0 < R < \infty$, $0 < E_0 < \infty$, $0 < s < 1$ and N is a positive integer, there exists $\eta > 0$ with the following property:*

Assume the following:

1. *$Y \subset \mathbb{R}^n$ has no more than $N+1$ elements, $T(y) = 0$ for all $y \in Y$, $a > 0$, $|y - z| > 3a$ for all $y, z \in Y$ and $\operatorname{diameter} Y \leq \eta R$.*
2. *On $\{x \in \mathbb{R}^n \,|\, \operatorname{dist}(x, Y) < R\}$, u satisfies (1.2) with $|\lambda| \leq \eta$, $|u| \leq 2$ and $\xi_\varepsilon \leq \eta$.*
3. *For each $y \in Y$ and $a \leq r \leq R$,*

$$\int_{B_r(y)} |\xi_\varepsilon| + (1-(\nu_n)^2)\varepsilon|\nabla u|^2 \, dy \leq \eta r^{n-1}, \quad \int_{B_r(y)} \varepsilon|\nabla u|^2 \leq E_0 r^{n-1}.$$

Then we have

$$\sum_{y\in Y} \frac{1}{a^{n-1}} \int_{B_a(y)} e_\varepsilon \leq s + \frac{1+s}{R^{n-1}} \int_{\{x \,|\, \operatorname{dist}(Y,x)<R\}} e_\varepsilon.$$

The next proposition deals with the "ε-scale", and shows that the smallness of the discrepancy measure and tilt excess imply that the solution is close to the homogeneous traveling wave solution in 1-D.

Proposition 5.6. *Given $0 < s < 1$ and $0 < b < 1$, there exist $0 < \eta < 1$ and $1 < L < \infty$ (which also depend on W) with the following property:*
Assume $0 < \varepsilon < 1$ and u satisfies (1.2) and $\xi_\varepsilon \leq \eta$ on $B_{4\varepsilon L}(0)$ with $|\lambda| \leq \eta$, $|u(0)| \leq 1 - b$, and

$$\int_{B_{4\varepsilon L}(0)} \left(|\xi_\varepsilon| + (1-(\nu_n)^2)\varepsilon|\nabla u|^2\right) \leq \eta(4\varepsilon L)^{n-1}. \tag{5.2}$$

Then, we have $T^{-1}(0) \cap \{x \in B_{3L\varepsilon}(0) \,|\, u(x) = u(0)\} = \{0\}$ *and*

$$\left| \frac{1}{\omega_{n-1}(L\varepsilon)^{n-1}} \int_{B_{L\varepsilon}(0)} e_\varepsilon - 2\sigma \right| \le s. \tag{5.3}$$

Proof. We rescale the domain by ε for convenience. Let $q : \mathbb{R} \to (-1, 1)$ be the unique solution of the ODE

$$\begin{cases} q'(t) = \sqrt{2W(q(t))} & \text{for } \ t \in \mathbb{R}, \\ q(0) = u(0). \end{cases}$$

We note that

$$\int_{-\infty}^{\infty} \frac{1}{2}|q'(t)|^2 \, dt = \int_{-\infty}^{\infty} \sqrt{\frac{W(q(t))}{2}} q'(t) \, dt = \int_{-1}^{1} \sqrt{\frac{W(s)}{2}} \, ds = \sigma.$$

We also identify q on $\mathbb{R}^n$ by $q(x_1, \cdots, x_n) = q(x_n)$. For given b and s, we fix a large enough $L > 1$ so that

$$\left| \frac{1}{\omega_{n-1}L^{n-1}} \int_{B_L(0)} \left(\frac{1}{2}|\nabla q|^2 + W(q) \right) - 2\sigma \right| \le \frac{s}{2} \tag{5.4}$$

whenever $|q(0)| \le 1 - b$. Next, using the point-wise assumption $\frac{1}{2}|\nabla u|^2 - W(u) \le \eta$ on $B_{4L}(0)$ and $|u(0)| \le 1 - b$, we restrict η so that $|u| \le 1 - \tilde{b}$ on $B_{4L}(0)$ for some $\tilde{b} = \tilde{b}(W, b, s) > 0$. Now define a function $z(x) : B_{4L}(0) \to \mathbb{R}$ by setting

$$z(x) = q^{-1}(u(x)),$$

where $q^{-1} : (-1, 1) \to \mathbb{R}$ is the inverse function of q. Since $|u| \le 1 - \tilde{b}$, z is well-defined and $q'(z(x)) \ge \min_{|t| \le 1-\tilde{b}} \sqrt{2W(t)}$ for $x \in B_{4L}(0)$. Moreover, since we may use the equation (1.2) to estimate $\|u\|_{C^2(B_{3L}(0))}$, $\|z\|_{C^2(B_{3L}(0))}$ is uniformly bounded depending only on W, b and s. Thus, with

$$\frac{1}{2}|\nabla u|^2 - W(u) = \frac{1}{2}(q'(z))^2(|\nabla z|^2 - 1),$$

$$|\nabla u|^2(1 - (\nu_n)^2) = (q'(z))^2(|\nabla z|^2 - (z_{x_n})^2)$$

and the inequality (5.2), we may obtain (with either $+$ or $-$)

$$\|z(x) \pm x_n\|_{C^1(B_{3L}(0))} \le c(b, \delta, W)\eta^{1/(n+1)}.$$

This shows that $u(x)$ is C^1 close to $q(x_n)$ on $B_{3L}(0)$. Combined with (5.4), by choosing η sufficiently small, we obtain (5.3). Also $u_{x_n} = q'(z)z_{x_n} \ne 0$ on $B_{3L}(0)$ implies the first assertion. □

Proof of Theorem 1. Recall that parts 1–3, 5 of Theorem 1 have already been established, and that the limit varifold V has density uniformly bounded from above and below.

Since V is rectifiable, V has a weak tangent plane $\mathcal{H}^{n-1}$ a.e. on $\mathrm{supp}\|V\|$. Fix such a point and choose coordinates so that the point is the origin and the weak tangent plane is $T = \{x \in \mathbb{R}^n \,|\, x_n = 0\}$. With the notation $\Phi_r(x) = x/r$ this means there exists a sequence $r_i \to 0$ such that $(\Phi_{r_i})_\# V \to \theta \, v(T)$ in the varifold sense, where θ is the density of V at the origin and $(\Phi_r)_\#$ is the usual push-forward. By passing to a further subsequence we may assume $\lim_{i\to\infty}(\Phi_{r_i})_\# V^i = \theta \, v(T)$ and $\varepsilon_i/r_i \to 0$. Let $\tilde{u}^i(x) = u^i(r_i x)$ and observe that $\tilde{\varepsilon}_i \Delta \tilde{u}^i = \tilde{\varepsilon}_i^{-1} W'(\tilde{u}^i) - \tilde{\lambda}_i$ with $\tilde{\varepsilon}_i = \varepsilon_i/r_i \to 0$ and $\tilde{\lambda}_i = r_i\lambda_i \to 0$. In the following we omit $\sim$, and also write V^i for the varifold associated to $\tilde{u}^i$.

For all sufficiently large i, Proposition 3.3 shows that

$$\sup_{B_3(0)} \left(\frac{\varepsilon_i |\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i} \right) \leq O(r_i) \to 0. \tag{5.5}$$

By Proposition 4.3 and the first claim in Proposition 4.4 we also know that

$$\lim_{i\to\infty} \int_{B_R(0)} \left| \frac{\varepsilon_i |\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i} \right| = 0, \tag{5.6}$$

and the three Radon measures, $\frac{\varepsilon_i}{2}|\nabla u^i|^2 \, d\mathcal{L}^n$, $\varepsilon_i^{-1} W(u^i)\, d\mathcal{L}^n$ and $|\nabla w^i| \, d\mathcal{L}^n$ converge to the same limit $\theta \, \|v(T)\|$. Since $V^i \to \theta \, v(T)$ in the varifold sense, we also have

$$\lim_{i\to\infty} \int_{B_3(0)} \left(1 - \nu_n^2\right) \frac{\varepsilon_i}{2} |\nabla u^i|^2 = \lim_{i\to\infty} \int_{B_3(0)} \left(1-\nu_n^2\right) |\nabla w^i| = 0. \tag{5.7}$$

Suppose N is the smallest positive integer greater than $\sigma^{-1}\theta$. Fix an arbitrary small $s > 0$. Use Proposition 5.1 to choose $b > 0$, and then with (5.5) we have

$$\int_{B_3(0)\cap\{|u^i|\geq 1-b\}} \left(\frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \right) \leq s \tag{5.8}$$

for all sufficiently large i. With these choices of s, b and $R = 1$, we choose η and L via Proposition 5.5 and 5.6 (the smaller η should be chosen). For all large i, we define

$$\begin{aligned} G_i = {} & B_2(0) \cap \left\{ |u^i| \leq 1 - b \right\} \cap \\ & \left\{ x \mid \int_{B_r(x)} \left| \frac{\varepsilon_i |\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i} \right| + \left(1-\nu_n^2\right) \varepsilon_i |\nabla u^i|^2 \leq \eta r^{n-1} \text{ if } 4\varepsilon_i L \leq r \leq 1 \right\}. \end{aligned}$$

By the Besicovitch covering theorem and Proposition 3.4, one shows that

$$\begin{aligned} & \|V^i\| \big(B_2(0) \cap \{|u| \leq 1-b\} \setminus G_i\big) + \mathcal{L}^{n-1}\big(T\big(B_2(0) \cap \{|u| \leq 1-b\} \setminus G_i\big)\big) \\ & c(s, W, n)\eta^{-1} \int_{B_3(0)} \left| \frac{\varepsilon_i |\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i} \right| + \left(1-\nu_n^2\right) \varepsilon_i |\nabla u^i|^2, \end{aligned} \tag{5.9}$$

which goes to 0 as $i \to \infty$ by (5.6) and (5.7). Also $\operatorname{dist}(T, G_i) \to 0$ as $i \to \infty$, again using Proposition 3.4.

For any $x \in B_1^{n-1}(0) := (\mathbb{R}^{n-1} \times \{0\}) \cap B_1(0)$ and $|t| \leq 1-b$, we let $Y = T^{-1}(x) \cap G_i \cap \{u^i = t\}$ and apply Proposition 5.5, where we set $a = L\varepsilon_i$. By Proposition 5.6, each element of Y is separated by at least $3L\varepsilon_i$, and all the assumptions are satisfied for sufficiently large i. We prove that Y does not contain more than $N - 1$ elements for any $x \in B_1^{n-1}(0)$ as follows. Since

$$\sup_{x\in B_1^{n-1}(0)} \frac{1}{\omega_{n-1}} \int_{B_1(x)} \left(\frac{\varepsilon_i}{2} |\nabla u^i|^2 + \frac{W(u^i)}{\varepsilon_i} \right) \leq 2\theta + s$$

for large i, Y having more than $N-1$ elements would imply, by Proposition 5.5, that

$$2\sigma N \leq (N+1)s + (1+s)(2\theta + s).$$

This would be a contradiction to $\theta\sigma^{-1} < N$ for sufficiently small s depending only on N.

Finally,

$$\omega_{n-1}\theta = \lim_{i\to\infty} \int_{B_1(0)} |\nabla w^i| \leq \lim_{i\to\infty} \int_{B_1(0)\cap\{|u^i|\leq 1-b\}\cap G_i} |\nabla u^i| \sqrt{W(u^i)/2} + s$$

by (5.8) and (5.9). By the co-area formula, $\lim_{i\to\infty}\|T_\# V^i\| = \|V\|$ and the above discussion then implies

$$\begin{aligned}\omega_{n-1}\theta &\le \lim_{i\to\infty}\int_{-1+b}^{1-b}\big\|T_\#\big(v(\{u^i=t\}\cap G_i)\big)\big\|\big(B_1^{n-1}(0)\big)\sqrt{W(t)/2}\,dt + s \\ &\le \omega_{n-1}(N-1)\int_{-1+b}^{1-b}\sqrt{W(t)/2}\,dt + s \le \omega_{n-1}(N-1)\sigma + s.\end{aligned} \tag{5.9}$$

Since s is arbitrary, we have $\theta = (N-1)\sigma$.

To conclude the proof, we note that u^i converges locally uniformly to $+1$ on one side of T and -1 on the other at $\mathcal{H}^{n-1}$ a.e. $x\in M^\infty$, and to the same value on $\mathcal{H}^{n-1}$ a.e. $x\in U\setminus M^\infty$. Note that we may choose $x_i\in B_1^{n-1}(0)$ and $|t|\le 1-b$ such that $x_i\notin T(\{|u|\le 1-b\}\cap B_2(0)\setminus G_i)$ and $T^{-1}(x_i)\cap G_i\cap\{u^i=t\}$ has precisely $N-1$ elements, by (5.9) and (5.10). Thus $T^{-1}(x_i)\cap\{u^i=t\}$ has precisely $N-1$ elements. This immediately implies the oddness or evenness depending on the sign of u^i away from T, and thus on whether the origin is in M^∞ or not. □

Proof of Theorem 2. Assume that $\{u^i\}$ are locally energy minimizing in $\tilde U$. We show that the limiting varifold V has no density multiplicity, i.e. $\sigma^{-1}V$ is the unit density varifold on $\tilde U$ in this case. This can be done by a "cut and paste" argument and we only sketch the key points of the proof. The details could be filled easily, while they are somewhat cumbersome to write down explicitly.

Let $x\in \operatorname{supp}\|V\|\cap\tilde U$ be a point with a weak tangent plane. As was done in the proof of the integrality, consider suitable rescalings and translations so that $x=0$, $V^i\to(N\sigma)v(T)$, where $T=\mathbb{R}^{n-1}\times\{0\}$ and $N\sigma=$ density of $\|V\|$ at x. By the argument in Theorem 4.1, we may assume that u^i converges locally uniformly to either ±1 on $\{x_n>0\}$ and $\{x_n<0\}$. If u^i converges to $+1$ on both $\{x_n>0\}$ and $\{x_n<0\}$, then we can squash the function u^i to $+1$ on $B_1(0)$ and reduce the energy by a definite amount. When one imposes a volume constraint, one can drill a small hole to correct the error produced by the squashing at a nearby point. Thus this would contradict the local energy minimality. Hence u^i converges to $+1$ on one side and -1 on the other. If $N\ge 2$, then replace u^i on $B_1(0)$ by a 1-D traveling wave solution $q(x_n/\varepsilon_i)$ and bridge it with u^i restricted to $\tilde U\setminus B_{1+\varepsilon}(0)$ by a suitable Lipschitz function. Since u^i converges uniformly away from $\{x_n=0\}$ on both sides, the error of bridging may be made as small as we like. This again would reduce the energy by a definite amount. Thus $N=1$, and we have $\sigma^{-1}\|V\|=\|\partial\{u^\infty=1\}\|$ on $\tilde U$. □

Proof of Theorem 3. First, by assumption **A** and a comparison argument (see [25]), there exists $c=c(W)$ such that $||u^i||_{L^\infty}\le c$, and by the standard elliptic theory [22], $\varepsilon_i\Delta u^i=\varepsilon_i^{-1}W'(u^i)-\lambda_i$ for some $\lambda_i\in\mathbb{R}$, $u^i\in C^3(\bar U)$ and $\frac{\partial u^i}{\partial n}=0$ on ∂U. One may also prove (see [15, Lemma 3.4]) that there exists a constant $c=c(m,W,U)$ such that $|\lambda_i|\le cE_0$ for all sufficiently large i. Thus assumption B is also satisfied. By Theorem 1 and 2, $\operatorname{supp}||\partial\{u^\infty=1\}||$ is a constant mean curvature hypersurface in the generalized sense.

For the local area minimality, we use a contradiction argument. Assume that there exists a function $\tilde u$ with $\tilde u=\pm1$, L^n a.e. on U, $\int_U|u^\infty-\tilde u|<c$, $\int_U\tilde u=m$ and $||\partial\{\tilde u=1\}||(U)<||\partial\{u^\infty=1\}||(U)$. Let $\tilde c=\int_U|u^\infty-\tilde u|$, and we redefine $\tilde u$ so that

$$||\partial\{\tilde u=1\}||(U)=\inf_{u\in\mathcal{A}}||\partial\{u=1\}||(U),$$

where $\mathcal{A}=\{u\ |\ u=\pm1,\ L^n \text{ a.e. on } U,\ \int_U|u-u^\infty|\le\tilde c,\ \int_U u=m\}$. This is possible by the lower semicontinuity of the perimeter functional. In the following we construct comparison functions in precisely the same way as in [31], but one crucial additional property we have here is the fact that $\{u^\infty = 1\}$ and $\{u^\infty = -1\}$ both contain some open balls (in fact, $\{u^\infty = 1\} \setminus \partial\{u^\infty = 1\}$ is an open set). Note that, without Theorem 1, we may not exclude the possibility of having $\mathrm{supp}||\partial\{u^\infty = 1\}|| = U$ in general.

Using the minimality of $\tilde{u}$, one may again choose open balls in $\{\tilde{u} = 1\}$ and $\{\tilde{u} = -1\}$, since $\partial\{\tilde{u} = 1\}$ is a generalized constant mean curvature hypersurface on open subdomains of $\{u^\infty = 1\}$ and $\{u^\infty = -1\}$. By [31, Lemma 1], there exists a sequence of functions $\{\tilde{u}^j\}_{j=1}^\infty$ such that $\tilde{u}^j = \pm 1$, L^n a.e., $\int_U \tilde{u}^j = m$, $\int_U |\tilde{u}^j - \tilde{u}| \to 0$ and $||\partial\{\tilde{u}^j = 1\}||(U) \to ||\partial\{\tilde{u} = 1\}||(U)$ as $j \to \infty$ and $\partial\{\tilde{u}^j = 1\}$ is a smooth hypersurface with smooth boundary in ∂U. Here, open balls are used to "adjust" the volume constraint of $\tilde{u}^j$. We fix a large j so that $\int |u^\infty - \tilde{u}^j| < c$ and

$$||\partial\{\tilde{u}^j = 1\}||(U) < ||\partial\{u^\infty = 1\}||(U).$$

By [31, Proposition 2], there exists a family of Lipschitz functions $\{\tilde{u}_\varepsilon\}_{\varepsilon>0}$ such that $\int_U |\tilde{u}_\varepsilon - \tilde{u}^j| \to 0$ as $\varepsilon \to 0$, $\int_U \tilde{u}_\varepsilon = m$ and

$$\frac{1}{2\sigma} \limsup_{\varepsilon\to 0+} E_\varepsilon(\tilde{u}_\varepsilon) \le ||\partial\{\tilde{u}^j = 1\}||(U).$$

Since $u^i \to u^\infty$ in L^1, $\int_U |\tilde{u}_{\varepsilon_i} - u^i| < c$ for sufficiently large i. Finally, $w^i \to w^\infty$ in L^1 implies

$$||\partial\{u^\infty = 1\}||(U) = \frac{1}{\sigma}\int_U |Dw^\infty| \le \frac{1}{\sigma}\liminf_{i\to\infty}\int_U |\nabla w^i| \le \frac{1}{2\sigma}\liminf_{i\to\infty} E_{\varepsilon_i}(u^i).$$

The three inequalities above for sufficiently large i give a contradiction to the local minimality of u^i. □

6. Additional remarks

6.1. **General critical points.** In addition to assumption **A**, assume that there exist constants $2 < k \le \frac{2n}{n-2}$ and $c > 0$ such that $c|x|^k \le W(x) \le c^{-1}|x|^k$ and $c|x|^{k-1} \le |W'(x)| \le c^{-1}|x|^{k-1}$ for all sufficiently large $|x|$. Assume that ∂U is smooth here. We say that $u \in H^1(U)$ is a critical point of $\mathcal{E}$ with volume constraint $\int_U u = m$ ($m \in (-|U|, |U|)$) if

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}(u + t\tilde{u}) = 0$$

for all $\tilde{u} \in H^1(U)$ with $\int_U \tilde{u} = 0$. For such u, there exists $\lambda \in \mathbb{R}$ such that

$$-\varepsilon \int \nabla u \cdot \nabla \phi = \int \left(\frac{W'(u)}{\varepsilon} - \lambda\right)\phi$$

for any $\phi \in H^1(U)$. Using $W'(u)u > 0$ for $|u| > 1$ and $k > 2$, one can prove that $u \in L^p(U)$ for any $p < \infty$, and thus by the standard elliptic theory, $u \in C^3(\bar{U})$, $\frac{\partial u}{\partial n} = 0$ on ∂U. The previously cited [15, Lemma 3.4] shows that $|\lambda| \le c(U, m, W)E_\varepsilon(u)$ for all sufficiently small ε. Assume from the outset that E_0 is given and that $E_\varepsilon(u) \le E_0$. The rescaling $x \to \frac{x}{\varepsilon}$ changes the equation to $\Delta u = W' - \lambda\varepsilon$. Multiplying $|u|^{p-1}u$ to the equation for various $p > 1$ and using $k > 2$ again, one can prove that there exists $c = c(m, W, U, E_0)$ such that $||u||_{L^\infty} \le c$ for all small ε. Thus, if $\{u^i\} \subset H^1(U)$ is a sequence of critical points with fixed volume constraint, $\varepsilon_i \to 0$ and $E_{\varepsilon_i}(u^i) \le E_0$, then Theorem 1 applies.

As an interesting application, we recall the Cahn-Hilliard equation

$$\begin{cases} u_t = \Delta f & \text{on } U \times (0,\infty), \\ f = -\varepsilon \Delta u + \frac{W'(u)}{\varepsilon}, & \\ \frac{\partial u}{\partial n} = \frac{\partial f}{\partial n} = 0 & \text{on } \partial U \times (0,\infty), \\ u = u_0 & \text{on } U \times \{0\}. \end{cases}$$

The equation is used to model various phase separation phenomena in a melted alloy with two stable phases. The solution has the properties

$$\frac{d}{dt}\left(\int_U u\right) = 0 \ \text{ and } \ \frac{d}{dt}\int_U \left(\frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}\right) = -\int_U |\nabla f|^2 \leq 0.$$

The time-independent solution satisfies $\Delta f = 0$ on U and $\frac{\partial f}{\partial n} = 0$ on ∂U, which implies $f \equiv \lambda \in \mathbb{R}$. Thus, our result gives the description of the asymptotic behaviors of finite energy equilibriums for this problem.

6.2. **Mountain-pass solutions.** Given a bounded smooth domain U, we note that there always exists a family of unstable critical points $\{u_\varepsilon\}$ of E_ε with Neumann data, $|u| \leq 1$ and $E_\varepsilon(u_\varepsilon) \leq c(U)$ for all $0 < \varepsilon << 1$. The proof is a simple application of the well-known Mountain-pass Lemma [38] (consider all the paths connecting $u \equiv 1$ and $u \equiv -1$ in $H^1(U)$) and is left to the interested reader. Obviously, Theorem 1 applies to such family of solutions.

6.3. **Interface with multiplicities.** Here we briefly discuss an example where the limit of the measure corresponding to solutions for $\varepsilon \Delta u = \varepsilon^{-1} W'(u) - \lambda$, $\lambda \neq 0$, is a flat multiplicity 2 hypersurface. This example shows that, even if λ_∞ is not zero, one can have a portion of supp$||V||$ which is multiplicity 2 and $H = 0$.

Let $U = \mathbb{R}$ and $\lambda = 1$. We find a 1-dimensional solution (after a rescaling) for $u'' = W'(u) - \varepsilon$, which has two interfaces separated by distance of order $O(|\ln \varepsilon|)$. Let $W_\varepsilon(x) = W(x) - \varepsilon x$. Let a_ε be the critical point of W_ε which is close to -1. Since $W_\varepsilon(a_\varepsilon) > 0$ and $W_\varepsilon(1) < 0$, we have two solutions for $W_\varepsilon(a_\varepsilon) = W_\varepsilon(x)$ near 1 for all small ε, and we let b_ε to be the one closer to the origin. When $\varepsilon \approx 0$, we have $W_\varepsilon(a_\varepsilon) \approx \varepsilon$ and $W_\varepsilon(x) \approx \frac{1}{2}W''(1)(x-1)^2 - \varepsilon$ around $x \approx 1$. Thus b_ε satisfies $\varepsilon \approx \frac{1}{2}W''(1)(b_\varepsilon - 1)^2 - \varepsilon$, and we may conclude that there exists some $c = c(W)$ such that

$$1 - b_\varepsilon \geq c\sqrt{\varepsilon}$$

for all sufficiently small ε.

We then solve an ODE

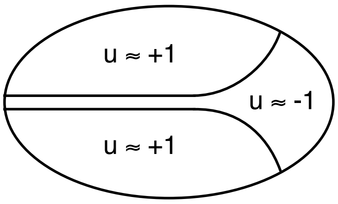
$$\begin{cases} u_\varepsilon'' = W_\varepsilon'(u_\varepsilon) & \text{on } x > 0, \\ u_\varepsilon'(0) = 0, & \\ u_\varepsilon(0) = b_\varepsilon. & \end{cases}$$

Multiply u_ε' to the equation and integrate over $[0, x]$ to obtain

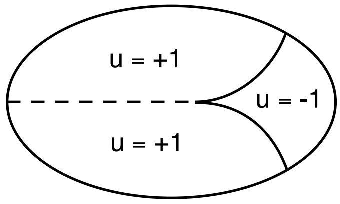
$$\frac{1}{2}(u_\varepsilon'(x))^2 = W_\varepsilon(u_\varepsilon(x)) - W_\varepsilon(b_\varepsilon).$$

This shows that $u_\varepsilon'(x) < 0$ for all $x > 0$ and $u_\varepsilon(x) \to a_\varepsilon$ as $x \to \infty$. By reflecting u_ε at the origin $x = 0$, we obtain a solution of $u_\varepsilon'' = W_\varepsilon'(u_\varepsilon)$ on $\mathbb{R}$ with $u_\varepsilon(0) = b_\varepsilon$ and $u_\varepsilon(x) \to a_\varepsilon$ as $|x| \to \infty$. Since $1 - b_\varepsilon \geq c\sqrt{\varepsilon}$, the comparison argument in Lemma 5.2 shows that the interface (which is characterized here by, say, $u_\varepsilon = 0$) is away from the origin by a distance of order $O(|\ln \varepsilon|)$. As $\varepsilon \to 0$, the two interfaces "escape" to the infinity at speed $O(|\ln \varepsilon|)$, and each of their profiles becomes closer to that of the solution of $u'' = W'(u)$ as $\varepsilon \to 0$. After rescaling back, we have a solution of $\varepsilon u_\varepsilon'' = \frac{W'(u_\varepsilon)}{\varepsilon} - 1$, the distance of the two interfaces is $O(\varepsilon|\ln \varepsilon|) \to 0$, and $\frac{1}{2\sigma}E_\varepsilon(u_\varepsilon) \to 2$. On $\mathbb{R}^n$, by homogeneously extending u_ε to $\mathbb{R}^{n-1}$ direction, we have a sequence of solutions which converges to a density 2 flat interface. As an oriented current, the limit has $\partial\{u^\infty = 1\} = \emptyset$, which agrees with our result.

It would be interesting to know whether one may have a limiting situation of the type indicated below. The dotted line indicates a multiplicity two portion of the interface, which disappears in the limit.



Phase separation for $\varepsilon \approx 0$



Phase separation as $\varepsilon \to 0$ showing "virtual" interface

The occurrence of higher odd multiplicity when $\lambda_\infty \neq 0$ is rather unlikely, even though we were not able to exclude the possibility. Such portion appears to be an interface with a "wrong" bending direction and we think such a portion should be of $\mathcal{H}^{n-1}$ measure 0.

For related multiplicity results for the Allen-Cahn equation, we cite [11] for the existence of higher multiplicity solutions.

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