Random Fractal Measures via the Contraction Method

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Abstract

In this paper we extend the contraction mapping method to prove various existence and uniqueness properties of (self-similar) random fractal measures, and establish exponential convergence results for approximating sequences defined by means of the scaling operator. For this purpose we introduce a version of the Monge Kantorovich metric on the class of probability distributions of random measures in order to prove the relevant results in distribution. We also use a special sample space of "construction trees" on which we define the approximating sequence of random measures, and introduce a certain operator and a compound variant of the Monge Kantorovich metric in order to establish a.s. exponential convergence to the unique random fractal measure. The arguments used apply at the random measure and random measure distribution levels, and the results cannot be obtained by previous contraction arguments which applied at the individual realisation level.

1 Introduction

Fractal sets and measures are mathematical models of non-integer dimensional sets and "grey-scale" images satisfying certain scaling properties. This gives an attractive theory with many applications to turbulence, dynamical systems, computer graphics and to image and data compression; c.f. Barnsley (1988) and Peitgen and Saupe (1988) and the references there. A mathematical fractal looks the same at all scales of magnification. This is an approximation to physical fractals which appear similar to the original object only for a certain range of scales.

Mandelbrot introduced the term *fractal* and developed the connection between these ideas and a range of phenomena in the physical and biological sciences; see Mandelbrot (1982) and the references there.

A theory of (self-similar) fractal sets and measures and the notion of a scaling operator (iterated function system) was developed in Hutchinson (1981). Existence, convergence and uniqueness results were based on contraction properties of the scaling operator with respect to the Hausdorff metric for sets and the Monge Kantorovich metric for measures.

Important applications to computer graphics and connections with Markov processes were introduced in Diaconis and Shahshahani (1984) and Barnsley and Demko (1985), and later developed extensively by Barnsley and others. Falconer (1986), Graf (1987), Mauldin and Williams (1986) and Cawley and Mauldin (1992) investigated a theory of random fractal sets, and Arbeiter (1991) and Olsen (1994) developed a theory of random fractal measures. See also Zähle (1988), Patzschke and Zähle (1990) and the survey in Hutchinson (to appear).

Arbeiter (1991) derived existence, uniqueness and convergence results for random fractal measures under various conditions. Here we prove results of this type under natural and general conditions, by an extension of the contraction method used in the deterministic case. For this we introduce two metrics, at the distribution and random variable levels respectively. This leads to relatively elementary proofs, establishes exponential convergence in terms of these metrics for certain approximating sequences, and leads to quantitative estimates for the rates of convergence. Convergence of first moments is also a consequence.

All the information about the random fractal measure and its distribution is coded in the distribution $\mathcal{S}$ of the associated random scaling law. This could be useful if one needs for example, to compress images involving background patterns of a stochastic nature for which a particular realisation of the behaviour is not important.

Let (M, d_{MK}) be the metric space of Radon measures on a complete separable metric space (X, d), where d_{MK} is the Monge Kantorovich metric. A *random measure* μ is a random variable on the underlying probability space $(\Omega, \mathcal{A}, P)$ with values in M. A *random scaling operator* $\mathbf{S}$ is a random variable whose values are $2N$-tuples $(p_1, S_1, \ldots, p_N, S_N)$ of weights $p_i \geq 0$ with $\sum p_i = 1$ and of Lipschitz functions $S_i : X \to X$. A random measure μ^* is called a *random fractal measure self-similar w.r.t.* $\mathbf{S}$ if μ^* has the same distribution as $\sum_{i=1}^N p_i S_i \mu^{*(i)}$ where the $\mu^{*(i)}$ are independent copies of μ^* and $\mathbf{S}$ is independent of the $\mu^{*(i)}$. (Note the assumption on the weights implies that μ^* and $\sum_{i=1}^N p_i S_i \mu^{*(i)}$ have the same mass.) The probability distribution $\mathcal{P}^*$ of μ^* is called a *fractal measure distribution, self-similar w.r.t.* $\mathbf{S}$.

In Section 2 we develop the model example of a random Koch curve in order to motivate the subsequent considerations.

In Section 3 we define the Monge Kantorovich metric d^{**}_{MK} on the space $\mathcal{M}_1$ of probability distributions of unit mass random measures with finite expected first moment. We then apply a contraction mapping argument and obtain a simple proof of the existence and uniqueness of a random fractal measure distribution $\mathcal{P}^* \in \mathcal{M}_1$ which is self-similar w.r.t. to $\mathcal{S}$. Exponential convergence in distribution of the iterative procedure beginning from any initial unit mass measure μ_0 and repeatedly applying independent copies of the scaling operator $\mathbf{S}$ in an appropriate manner, is similarly established.

In Section 4 we utilise the metric d^*_{MK} on the space $\mathbb{M}_1$ of unit mass random measures with finite expected first moment defined over a certain natural space of "construction trees". Exponential a.s. convergence of the above iterative procedure to a random measure μ^*, independent of μ_0 and having distribution $\mathcal{P}^*$, is established by a contraction mapping argument applied to a certain "non-constructive" operator defined on $\mathbb{M}_1$.

We next discuss the connection between the arguments and results here, and those of other authors.

Contraction mapping methods for showing the existence and uniqueness of (non random) fractal sets and fractal measures were first used in Hutchinson (1981). Falconer (1986) and Graf (1987) used contraction methods to obtain random fractal sets, and Olsen (1994) used contraction methods to obtain random fractal measures. In these three cases, one effectively applies contraction arguments to each realisation of the (infinite) random process, i.e. to each branch of the Construction Tree in Section 4 used to construct the random fractal measure. Thus all maps $S_i(\omega)$ are assumed to be contractions. (Olsen, and other authors, also allow more general families of contraction maps, "random geometrically graph directed self-similar multifractals", but this involves mainly technical complications from the point of view of the mathematical development.) In the language of the present paper, they assume that $r_i < 1$ a.s. (see Olsen (1994; Sections 2.2, 2.3 and 4.2), but it should be noted that the arguments of Olsen in fact can be modified to apply under the weaker assumption that $\sum_{i=1}^N p_i r_i < 1$ a.s.

Here we make the much weaker assumption that $\mathsf{E}\left(\sum_{i=1}^N p_i r_i\right) < 1$, but this is natural in the random setting. However, it is now impossible to apply contraction mapping arguments at the individual realisation level.

Instead, we first apply a contraction argument at the distribution level for random measures, see Theorem 3.3. Since random measure distributions are themselves measures (on a space of measures) one can define a Monge Kantorovich metric, but the argument that the operator $\mathcal{S}$ is a contraction map is now quite different from the non random case. The main point is that one must use the independence of $\mu^{(i)}, \dots, \mu^{(N)}, \mathbf{S}$ from Definition 3.1 of $\mathcal{S}$ and take appropriate conditional expectations in the proof of Theorem 3.3. This establishes existence and uniqueness at the distribution level of a self-similar random fractal measure.

In Theorem 4.1 we prove a.s. convergence of the random iterative procedure, applied to any initial measure, to the random fractal measure. We only require $\mathsf{E}\left(\sum_{i=1}^N p_i r_i\right) < 1$ and moreover exponential convergence is also established. Again it is impossible to apply contraction arguments at the individual realisation level, and instead we define in (10) a certain non-constructive operator $\mathbf{S}$ on the space of *all* random measures, show the connection between $\mathbf{S}$ and the constructive iteration procedure (see (11)), and establish $\mathbf{S}$ is a contraction map. Exponential and hence a.s. convergence of the iterative procedure is a consequence.

Arbeiter (1991) also restricts considerations to contraction maps, but much of his work does extend to the case $\mathsf{E}\left(\sum_{i=1}^N p_i r_i\right) < 1$, and his results then come closest to those in this paper. He also obtains many interesting results on the dimension of random fractal sets. But he does not use contraction mapping arguments nor obtain exponential convergence, and the arguments that are used are considerably more involved than those here.

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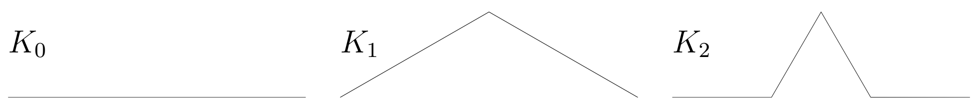
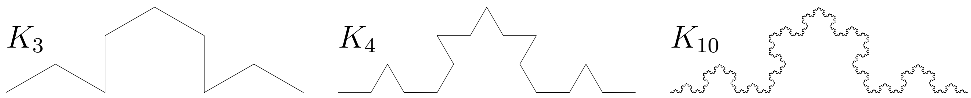
2 A model example

For motivation we consider one of the simplest non-trivial examples, a random Koch curve. Although this falls within the class of random fractal measures which can be treated by considerations at the individual realisation level, it is easily modified as discussed later at the end of this Section, to give examples which can only be analysed by fully probabilistic methods.

First recall that the deterministic (i.e. non-random) Koch curve K^* can be constructed as the limit in an appropriate sense (e.g. in the Hausdorff metric) of the following sequence of compact sets $(K_n)_{n\geq 0}$.





Each K_n has the same two boundary points $a_1 = (0,0), a_2 = (1,0)$ in $\mathbb{R}^2$ and is the union of 2^n equal length line segments. The set K_{n+1} is obtained from K_n by replacing each line segment in K_n by a connected pair of segments with the same endpoints and subtending an angle $2\pi/3$. The replacement segments point to the left or right (in the sense of the natural orientation) of the segment they replace according as n is even or odd. Equivalently,

$$K_{n+1} = S_{u1}K_n \cup S_{u2}K_n =: \mathbf{S}_u K_n$$

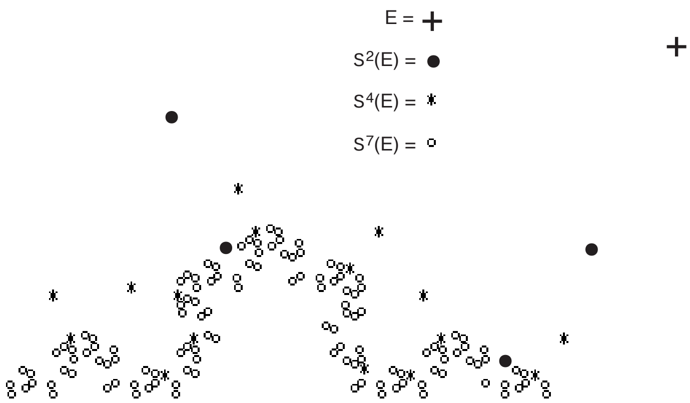
where $\mathbf{S}_u = (S_{u1}, S_{u2})$ and each $S_{ui} : \mathbb{R}^2 \to \mathbb{R}^2$ is a composition of a reflection in the x-axis, a rotation about a_i of $\pi/6$ if $i = 1$ and $-\pi/6$ if $i = 2$, and a homothety centred at a_i with Lipschitz constant $1/\sqrt{3}$. (The map S_{ui} is the unique orientation reversing isometry fixing a_i and mapping a_j $(j \neq i)$ to $(1/2, 1/(2\sqrt{3}))$.) Passing to the limit K^*, one has

$$K^* = S_{u1}K^* \cup S_{u2}K^* =: \mathbf{S}_u K^*$$

If $\mathcal{C}$ is the metric space of non-empty compact subsets of $\mathbb{R}^2$ together with the Hausdorff metric, it is easy to check that the operator $\mathbf{S}_u : \mathcal{C} \to \mathcal{C}$ defined by

$$\mathbf{S}_u K = S_{u1}K \cup S_{u2}K,$$

is a contraction map, see Hutchinson (1981). It follows that there is a *unique* compact set K^* invariant under $\mathbf{S}_u$; we say K^* is a fractal set which is *self-similar with respect to* $\mathbf{S}_u$. For any initial compact set $K_0 = E$, the sequence defined by $K_n = \mathbf{S}_u K_{n-1}$ $(= S^n(E_0)$ in the next diagram) converges to K^*.

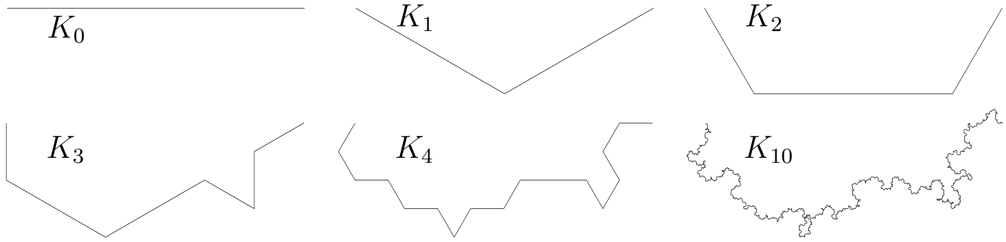


If, conversely to before, the new line segments determining K_{n+1} from K_n point right or left according as n is even or odd, then one obtains a sequence of sets obtained by reflecting the previous K_n in the x-axis. In this case

$$K_{n+1} = S_{d1}K_n \cup S_{d2}K_n =: \mathbf{S}_d K_n$$

where $\mathbf{S}_d = (S_{d1}, S_{d2})$ and S_{di} is constructed as is S_{ui} except that $\pi/6$ and $-\pi/6$ are interchanged and $(1/2, 1/(2\sqrt{3}))$ is replaced by $(1/2, -1/(2\sqrt{3}))$.

Suppose now that at each stage in the construction the new line segments point left or right with independent probabilities 1/2. For example, one experiment may give:



There are two possibilities for K_1, each with probability 1/2. Once K_1 is selected there are 4 possibilities for K_2, each with probability 1/4, etc.

We write

$$K_1 = \bigcup_i S_i K_0$$

where $\mathbf{S} := (S_1, S_2)$ is a random pair taking the two values $\mathbf{S}_u = (S_{u1}, S_{u2})$ or $\mathbf{S}_d = (S_{d1}, S_{d2})$ each with probability 1/2. In the previous diagram $\mathbf{S} = \mathbf{S}^d$. Similarly,

$$K_2 = \bigcup_{i,j} S_i \circ S^i_j\, K_0 = \bigcup_i S_i \Big(\bigcup_j S^i_j\, K_0 \Big) =: \bigcup_i S_i K_1^{(i)}$$

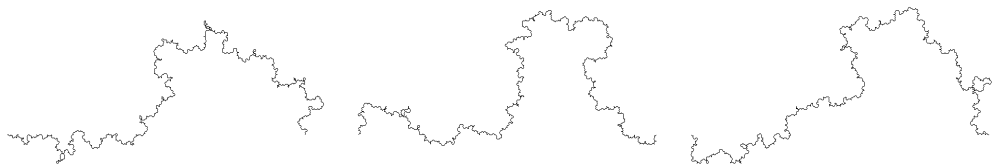
where each $\mathbf{S}^i := (S^i_1, S^i_2)$ is also a random pair taking values $\mathbf{S}_u = (S_{u1}, S_{u2})$ or $\mathbf{S}_d = (S_{d1}, S_{d2})$ with probability 1/2, and $\mathbf{S}^1, \mathbf{S}^2, \mathbf{S}$ are independent. In the previous diagram, $\mathbf{S}^1 = \mathbf{S}^u$ and $\mathbf{S}^2 = \mathbf{S}^u$ (remember that $\mathbf{S}^d$ and $\mathbf{S}^u$ reverse orientation). Note for future reference that $K_1^{(1)}, K_1^{(2)}$ have the same probability distribution as K_1 and are independent of one another and of $\mathbf{S} = (S_1, S_2)$.

Likewise,

$$K_3 = \bigcup_{i,j,k} S_i \circ S^i_j \circ S^{ij}_k K_0 = \bigcup_i S_i \Big(\bigcup_{j,k} S^i_j \circ S^{ij}_k K_0 \Big) =: \bigcup_i S_i K_2^{(i)}$$

where the $\mathbf{S}^{ij} := (S^{ij}_1, S^{ij}_2)$ take values $\mathbf{S}_u = (S_{u1}, S_{u2})$ or $\mathbf{S}_d = (S_{d1}, S_{d2})$ each with probability 1/2, and the $\mathbf{S}^{ij}, \mathbf{S}^k, \mathbf{S}$ are independent. In the previous diagram, $\mathbf{S}^{11} = \mathbf{S}^d, \mathbf{S}^{12} = \mathbf{S}^d, \mathbf{S}^{21} = \mathbf{S}^u, \mathbf{S}^{22} = \mathbf{S}^u$. Analogously to before, the $K_2^{(i)}$ have the same probability distribution as K_2 and are independent of one another and of $\mathbf{S}$. Similar remarks apply to $K_4, K_5, \ldots$.

In the following diagram, three other possible limit sets (actually, sets at the 10th stage of an approximation) are shown.



For each such (infinite) experiment ω, one obtains a limit set

$$K^* = K^*(\omega) = \bigcup_i S_i(\omega) K^{*(i)}(\omega),$$

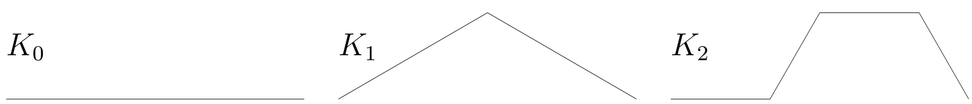
to which $K_{10} = K_{10}(\omega)$ is here a reasonable approximation. The probability distribution $\{1/2, 1/2\}$ on $\{\mathbf{S}_u, \mathbf{S}_d\}$, which we denote by $\mathcal{S}$, induces a probability distribution $\mathcal{P}^* := \operatorname{dist} K^*$ on the set of $K^*(\omega)$. As one anticipates from the previous discussion by passing to the limit,

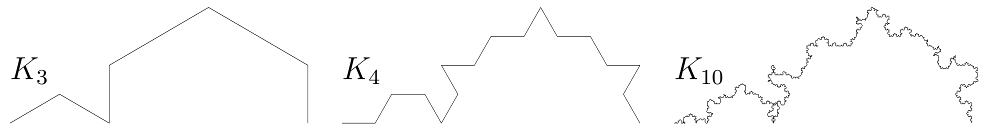
$$\mathcal{P}^* = \operatorname{dist} K^* = \operatorname{dist} \bigcup_i S_i K^{*(i)} =: \mathcal{S}\mathcal{P}^*,$$

where the $K^{*(i)}$ have probability distribution $\mathcal{P}^*$, and $\mathbf{S} = (S_1, S_2)$ has distribution $\mathcal{S}$, with $K^{*(1)}, K^{*(2)}, \mathbf{S}$ independent of one another. In this sense we say that the probability distribution $\mathcal{P}^*$ is a *random fractal distribution self-similar w.r.t.* $\mathcal{S}$. Loosely speaking, K^* is the union of two sets $S_1 K^{*(1)}$ and $S_2 K^{*(2)}$, each of which has—after applying the inverse of the appropriate component of the independent random map $\mathbf{S} = (S_1, S_2)$—the same probability distribution $\mathcal{P}^*$ as K^*.

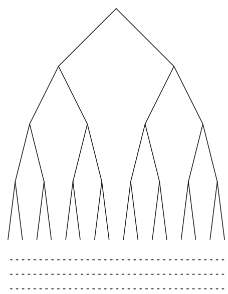
In Theorem 3.3 we establish that such a self-similar distribution $\mathcal{P}^*$ is unique and that the distributions of the K_n converge exponentially fast in an appropriate metric to $\mathcal{P}^*$. We use a contraction mapping argument on a certain space of probability distributions. (Rather than random *sets*, we consider random *measures*. This is more natural both mathematically and in applications. For example, instead of K_n consider the unit mass measure μ_n obtained by restricting to K_n a suitable normalisation of $\mathcal{H}^1$—Hausdorff one-dimensional measure.)

In the next diagram we see a sequence of realisations of the experiment where $\mathbf{S}_u = (S_{u1}, S_{u2})$ or $\mathbf{S}_d = (S_{d1}, S_{d2})$ are chosen with probabilities .9 and .1 respectively. Once again there is a unique associated self-similar random fractal distribution, to which the distributions of the K_n converge exponentially.





In addition to the results concerning probability distributions, we will see that $K_n(\omega)$ converges a.s. (and exponentially fast) to $K^*(\omega)$. Moreover, the random measure $K^*(\omega)$ (as well as the associated distribution $\mathcal{P}^*$) is independent of the starting set K_0. To establish this in general requires a more careful analysis of the the construction process. In previous diagrams we saw various steps in one realisation ω of an (infinite) construction process. Fixing K_0, ω is determined by a choice of either $\mathbf{S}_u$ or $\mathbf{S}_d$ at each node in the infinite two-fold branching tree shown below.



Identifying each ω with a tree of such choices, ω is then called a *construction tree* or *tree of scaling laws* and the sample space $(\widetilde{\Omega}, \widetilde{P})$ will be the set of all such ω together with the probability distribution induced from the probability distribution $\mathcal{S}$ on the set $\{\mathbf{S}_u, \mathbf{S}_d\}$. We define a certain "non-constructive" operator $\mathbf{S}$ on the class of all random sets (actually, measures) defined over $\widetilde{\Omega}$ with the property that $K_{n+1}(\omega) = \mathbf{S}K_n(\omega)$. A suitable metric and a contraction mapping argument then establishes the results concerning a.s. convergence.

In the previous examples one has a random scaling law $\mathbf{S} = (S_1, S_2)$ where the pair (S_1, S_2) takes one of two possible values, each with equal probability. This gives a random fractal set $K^* = K^*(\omega)$. Consider instead a more general type of random scaling law $\mathbf{S} = (p_1, S_1, p_2, S_2)$ with weights $p_1 = p_2 = 1/2$ (surely) and (S_1, S_2) having the same distribution as before. For any unit mass random measure μ in $\mathbb{R}^2$, define the random measure

$$\mathbf{S}\mu = \frac{1}{2}S_1\mu^{(1)} + \frac{1}{2}S_2\mu^{(2)}$$

where the $\mu^{(i)}$ are independent of (S_1, S_2) and are independently and identically distributed copies of μ. Beginning from any initial unit mass measure (non random or even random) this leads in an analogous manner to a random

"Koch measure" with unit mass distribution "uniformly" distributed along (realisations of) the random Koch curve.

The examples so far can all be obtained by contraction arguments applied at the individual realisation level, as in Falconer (1986), Graf (1987) and Olsen (1994). Next consider a distribution on 4-tuples (p_1, S_1, p_2, S_2) with $p_i > 0$, $\sum p_i = 1$, $S_i : \mathbb{R}^2 \to \mathbb{R}^2$, r_i the Lipschitz constant of S_i, $\mathsf{E} \sum p_i r_i < 1$ and $\mathsf{E} \sum p_i |S_i 0| < \infty$. (The S_i need not be contractions.) We establish existence, uniqueness and convergence results for a corresponding random fractal measure, although contraction arguments at the individual realisation level do not apply unless $\sum p_i r_i < 1$ a.s.

3 Construction of random fractal measures

Let (X, d) be a complete separable metric space and let $M_1 = M_1(X)$ denote the class of all Radon measures μ on $(X, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra on X, with $\mu(X) = 1$ and $\int d(x, a)\, \mathrm{d}\mu(x) < \infty$ for some $a \in X$. (In the previous section, $X = \mathbb{R}^2$.) For $\mu, \nu \in M_1$ define the *Monge Kantorovich metric* by

$$d_{MK}(\mu, \nu) \;=\; \sup \left\{ \left| \int f \,\mathrm{d}\mu - \int f \,\mathrm{d}\nu \right| \;\middle|\; f : X \to \mathbb{R}^1,\ \mathrm{Lip} f \le 1 \right\}. \qquad (1)$$

Contraction properties of d_{MK} were used in Hutchinson (1981) to establish existence and uniqueness of fractal measures.

Remarks: Note the following properties.

1. (M_1, d_{MK}) is a complete separable metric space. To show finiteness, note that $\int f \,\mathrm{d}\mu - \int f \,\mathrm{d}\nu$ is invariant under shifts $f \mapsto f + c$ and so we may assume $f(a) = 0$ in (1). Hence, if $\mathrm{Lip} f \le 1$ and $f(a) = 0$,

$$\begin{aligned} d_{MK}(\mu, \nu) &\le \int |f| \,\mathrm{d}\mu + \int |f| \,\mathrm{d}\nu \\ &\le \int d(x, a)\, \mathrm{d}\mu(x) + \int d(x, a)\, \mathrm{d}\nu(x) \;<\; \infty. \end{aligned}$$

2. $d_{MK}(\mu_n, \mu) \to 0$ if and only if
   (a) $\mu_n \xrightarrow{w} \mu$ (weak convergence) and
   (b) $\int d(x, a)\, \mathrm{d}\mu_n(x) \to \int d(x, a)\, \mathrm{d}\mu(x)$ (convergence of first moments).

3. The same definition of $d_{MK}(\mu, \nu)$ can be used for Radon measures of arbitrary mass on $(X, \mathcal{B})$. It follows that $d_{MK}(\mu, \nu) = \infty$ if the masses of μ and ν differ. Moreover,

(a) $d_{MK}(\alpha\mu, \alpha\nu) = \alpha d_{MK}(\mu, \nu)$ if $0 \le \alpha \in \mathbb{R}$,

(b) $d_{MK}(\mu_1 + \mu_2, \nu_1 + \nu_2) \le d_{MK}(\mu_1, \nu_1) + d_{MK}(\mu_2, \nu_2)$.

4. $d_{MK}(S\mu, S\nu) \le (\mathrm{Lip}S) d_{MK}(\mu, \nu)$ if $S : X \to X$, where $\mathrm{Lip}S$ is the Lipschitz constant of S.

5. $d_{MK}(\delta_a, \delta_b) = d(a, b)$ for $a, b \in X$, where δ_a is the Dirac unit mass measure concentrated at a.

□

Let $(\Omega, \mathcal{A}, P)$ be the underlying probability space. Let $\mathbb{M}$ denote the set of all random measures μ on M_1, i.e. random variables $\mu : \Omega \to M_1$, and let $\mathbb{M}_1$ denote the subset of such μ with finite expected first moment, i.e. $\mathsf{E} d_{MK}(\mu, \delta_a) < \infty$ for some (and hence any) $a \in X$. Let $\mathcal{M}$ denote the set of all probability distributions $\mathcal{P}$ of random measures $\mu \in \mathbb{M}$ and $\mathcal{M}_1$ denote the subset of probability distributions of random measures $\mu \in \mathbb{M}_1$; thus if $\mathcal{P} \in \mathcal{M}_1$ then $\int d_{MK}(\cdot, \delta_a)\, \mathrm{d}\mathcal{P} < \infty$ for some $a \in X$. (All fixed measures here have unit mass and finite first moment; the subscript "1" refers to the existence of a finite first moment condition at the individual measure, random measure or probability distribution, level.)

The scaling properties of random fractal measures are described by scaling laws.

Definition 3.1 *A* scaling law $(p_1, S_1, \ldots, p_N, S_N)$ *is a* $2N$*-tuple of real numbers* $p_i \ge 0$ *with* $\sum p_i = 1$, *and Lipschitz maps* $S_i : X \to X$. *A* random scaling law $\mathbf{S}$ *is a random variable whose values are scaling laws. The distribution of* $\mathbf{S}$ *is denoted by* $\mathcal{S}$. *The corresponding* scaling operator $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ *is defined by*

$$\mathcal{S}\mathcal{P} \stackrel{d}{=} \sum_{i=1}^{N} p_i\, S_i \mu^{(i)}, \tag{2}$$

where $\mu^{(i)} \stackrel{d}{=} \mathcal{P}$, $\mathbf{S} = (p_1, S_1, \ldots, p_N, S_N) \stackrel{d}{=} \mathcal{S}$ *and* $\mu^{(1)}, \ldots, \mu^{(N)}, \mathbf{S}$ *are independent.*

Here $S_i\mu^{(i)}$ is the image of $\mu^{(i)}$ under S_i (or the push forward measure) and $\stackrel{d}{=}$ denotes equality in distribution.

Definition 3.2 *Let* $\mathbf{S}$ *be a random scaling law with distribution* $\mathcal{S}$, *and* $\mu \in \mathbb{M}$ *be a random measure with distribution* $\mathcal{P} \in \mathcal{M}$. *Then* μ *is called a* random fractal measure self similar w.r.t. $\mathbf{S}$, *and* $\mathcal{P}$ *is called a* fractal measure distribution self-similar w.r.t. $\mathcal{S}$ *(or* $\mathbf{S}$*), if* $\mathcal{S}\mathcal{P} = \mathcal{P}$.

The following is analogous to the process for sets described informally in Section 2.

The Iterative Procedure Beginning with an initial measure $\mu_0 \in M_1$ one can iteratively apply independently distributed scaling laws with distribution $\mathcal{S}$ to obtain a sequence μ_n of random measures in $\mathbb{M}_1$, and a corresponding sequence $\mathcal{P}_n$ of distributions in $\mathcal{M}_1$, as follows.

1. Select a scaling law $\mathbf{S} = (p_1, S_1, \ldots, p_N, S_N)$ via the distribution $\mathcal{S}$ and define
$$\mu_1 \;=\; \sum_{i=1}^{N} p_i S_i \mu_0, \quad \mathcal{P}_1 \;\stackrel{d}{=}\; \mu_1,$$
2. Select $\mathbf{S}^1, \ldots, \mathbf{S}^N$ via $\mathcal{S}$ with $\mathbf{S}^i = (p_1^i, S_1^i, \ldots, p_N^i, S_N^i)$ independent of each other and of $\mathbf{S}$ and define
$$\mu_2 \;=\; \sum_{i,j} p_i p_j^i S_i \circ S_j^i \mu_0, \quad \mathcal{P}_2 \;\stackrel{d}{=}\; \mu_2,$$
3. Select $\mathbf{S}^{ij} = (p_1^{ij}, S_1^{ij}, \ldots, p_N^{ij} S_N^{ij})$ via $\mathcal{S}$ independent of one another and of $\mathbf{S}^1, \ldots, \mathbf{S}^N, \mathbf{S}$ and define
$$\mu_3 \;=\; \sum_{i,j,k} p_i p_j^i p_k^{ij} S_i \circ S_j^i \circ S_k^{ij} \mu_0, \quad \mathcal{P}_3 \;\stackrel{d}{=}\; \mu_3,$$
4. etc.

Since $\mu_{n+1} = \sum p_i S_i \mu_n^{(i)}$ where $\mu_n^{(i)} \stackrel{d}{=} \mu_n \stackrel{d}{=} \mathcal{P}_n$, $\mathbf{S} = (p_1, S_1, \ldots, p_N, S_N) \stackrel{d}{=} \mathcal{S}$, and the $\mu_n^{(i)}$ and $\mathbf{S}$ are independent, it follows that
$$\mathcal{P}_{n+1} = \mathcal{S}\mathcal{P}_n. \tag{3}$$

In the following theorem we establish the existence of a unique fractal measure distribution $\mathcal{P}^* \in \mathcal{M}_1$ which is self-similar w.r.t. $\mathcal{S}$ and show that $\mathcal{P}_n$ converges to $\mathcal{P}^*$ (independently of the initial measure μ_0). The approximation is described in terms of the Monge Kantorovich metric d_{MK}^{**} defined for $\mathcal{P}, \mathcal{Q} \in \mathcal{M}_1$ by
$$\begin{aligned} d_{MK}^{**}(\mathcal{P}, \mathcal{Q}) \;&=\; \sup\Big\{ |\mathsf{E}_{\mathcal{P}} F - \mathsf{E}_{\mathcal{Q}} F| \;\Big|\; F : M_1 \to \mathbb{R},\ \operatorname{Lip} F \le 1 \Big\} && (4)\\ &=\; \sup\Big\{ |\mathsf{E} F(\mu) - \mathsf{E} F(\nu)| \;\Big|\; F : M_1 \to \mathbb{R},\ \operatorname{Lip} F \le 1 \Big\}, && (5) \end{aligned}$$
where $\mu \stackrel{d}{=} \mathcal{P}$ and $\nu \stackrel{d}{=} \mathcal{Q}$. We use the standard notation
$$\mathsf{E}_{\mathcal{P}} F = \int F(\cdot)\, \mathrm{d}\mathcal{P} = \int_\Omega F \circ \mu \, dP = \mathsf{E} F(\mu).$$

This is analogous to (1), and in particular $(\mathcal{M}, d^{**}_{MK})$ is complete and separable and the analogue of Remark 2 is true.

For future reference note that

$$\mathsf{E}_{\mathcal{SP}}F = \int F(\cdot)\,\mathrm{d}\mathcal{SP}(\cdot) = \int_\Omega F\left(\sum p_i S_i \mu^{(i)}\right)\,\mathrm{d}P = \mathsf{E}F\left(\sum p_i S_i \mu^{(i)}\right), \quad (6)$$

where $\mu^{(i)} \stackrel{d}{=} \mathcal{P}$, $\mathbf{S} = (p_1, S_1, \ldots, p_N, S_N) \stackrel{d}{=} \mathcal{S}$, and $\mu^{(i)}, \ldots, \mu^{(N)}, \mathbf{S}$ are independent.

Theorem 3.3 *Let* $\mathbf{S} = (p_1, S_1, \ldots, p_N, S_N)$ *be a random scaling law with*

(i) $\lambda := \mathsf{E}\left(\sum_{i=1}^N p_i r_i\right) < 1$, *where* $r_i = \mathrm{Lip} S_i$,

(ii) $\Gamma_a := \mathsf{E}\sum_{i=1}^N p_i d(a, S_i a) < \infty, \quad a \in X$.

Then

(a) *the scaling operator* $\mathcal{S} : \mathcal{M}_1 \to \mathcal{M}_1$ *is a contraction w.r.t. the Monge Kantorovich metric* d^{**}_{MK} *on* $\mathcal{M}_1$,

(b) *there exists a unique fractal measure distribution* $\mathcal{P}^* \in \mathcal{M}_1$ *which is self-similar w.r.t.* $\mathcal{S}$,

(c) *the sequence of probability distributions* $(\mathcal{P}_n)_{n\geq 1}$ *converges exponentially fast w.r.t.* d^{**}_{MK} *to* $\mathcal{P}^*$.

Proof: We first need to check that $\mathcal{SP} \in \mathcal{M}_1$ for $\mathcal{P} \in \mathcal{M}_1$. The fact that $\mathcal{SP}$ has unit mass almost surely is clear.

To check the moment condition $\mathsf{E}_{\mathcal{SP}} d_{MK}(\cdot, \delta_a) < \infty$ for some $a \in X$ select $\mu^{(i)} \stackrel{d}{=} \mathcal{P}$ and $\mathbf{S} = (p_1, S_1, \ldots, p_N, S_N) \stackrel{d}{=} \mathcal{S}$ where $\mu^{(i)}, \ldots, \mu^{(N)}, \mathbf{S}$ are independent. Then using (6) and Remarks 3–5,

$$\begin{aligned}
\mathsf{E}_{\mathcal{SP}} d_{MK}(\cdot, \delta_a) &= \mathsf{E} d_{MK}\left(\sum p_i S_i \mu^{(i)}, \sum p_i \delta_a\right) \\
&\leq \mathsf{E}\sum_i p_i d_{MK}\left(S_i \mu^{(i)}, \delta_a\right) \\
&\leq \mathsf{E}\sum_i p_i \left(d_{MK}(S_i \mu^{(i)}, S_i \delta_a) + d_{MK}(S_i \delta_a, \delta_a)\right) \\
&\leq \mathsf{E}\sum_i p_i \left(r_i d_{MK}(\mu^{(i)}, \delta_a) + d(S_i a, a)\right) \\
&= \lambda \mathsf{E}_{\mathcal{P}} d_{MK}(\cdot, \delta_a) + \Gamma_a \; < \; \infty.
\end{aligned}$$

To compute $d^{**}_{MK}(\mathcal{SP}, \mathcal{SQ})$ let $F : M_1 \to \mathbb{R}$ with $\mathrm{Lip}F \le 1$. Again using (6) and Remarks 3–5,

$$\begin{aligned}
|\mathsf{E}_{\mathcal{SP}}F - \mathsf{E}_{\mathcal{SQ}}F| &= \left|\mathsf{E}F\left(\sum p_i S_i \mu^{(i)}\right) - \mathsf{E}F\left(\sum p_i S_i \nu^{(i)}\right)\right| \\
&\qquad \text{where the } \mu^{(i)} \text{ and } \mathbf{S} \text{ are independent, as are} \\
&\qquad \text{the } \nu^{(i)} \text{ and } \mathbf{S}, \text{ and } \mu^{(i)} \stackrel{d}{=} \mathcal{P},\ \nu^{(i)} \stackrel{d}{=} \mathcal{Q}, \\
&\le \sum_{j=1}^{N} \Bigg| \mathsf{E}F\Bigg(\sum_{i=1}^{j-1} p_i S_i \nu^{(i)} + p_j S_j \mu^{(j)} + \sum_{i=j+1}^{N} p_i S_i \mu^{(i)}\Bigg) \\
&\qquad - \mathsf{E}F\Bigg(\sum_{i=1}^{j-1} p_i S_i \nu^{(i)} + p_j S_j \nu^{(j)} + \sum_{i=j+1}^{N} p_i S_i \mu^{(i)}\Bigg)\Bigg| \\
&= \sum_{j=1}^{N} \left| \mathsf{E}G_j(\mu^{(j)}) - \mathsf{E}G_j(\nu^{(j)}) \right|,
\end{aligned}$$

where

$$G_j(\mu) := F\Bigg(\sum_{i=1}^{j-1} p_i S_i \nu^{(i)} + p_j S_j \mu + \sum_{i=j+1}^{N} p_i S_i \mu^{(i)}\Bigg).$$

From Remarks 3(a) and (4) we see G_j is Lipschitz with Lipschitz constant $p_j r_j$. Since $\mu^{(j)} \stackrel{d}{=} \mathcal{P}$ and $\nu^{(j)} \stackrel{d}{=} \mathcal{Q}$, on taking the conditional expectation E^* w.r.t. $\mu^{(j)}$ and $\nu^{(j)}$ given $\nu^{(i)}$ $(i \ne j)$, $\mu^{(k)}$ $(k \ne j)$ and $\mathbf{S}$, and using (5), it follows that

$$\begin{aligned}
|\mathsf{E}_{\mathcal{SP}}F - \mathsf{E}_{\mathcal{SQ}}F| &\le \sum_{j=1}^{N} \mathsf{E}\left|\mathsf{E}^* G_j(\mu^{(j)}) - \mathsf{E}^* G_j(\nu^{(j)})\right| \\
&\le \sum_{j=1}^{N} \mathsf{E} p_j r_j \, d^{**}_{MK}(\mathcal{P}, \mathcal{Q}) \\
&= \lambda d^{**}_{MK}(\mathcal{P}, \mathcal{Q}).
\end{aligned}$$

Taking the supremum over the class of F for which $\mathrm{Lip}F \le 1$, it follows that $\mathcal{S} : \mathcal{M}_1 \to \mathcal{M}_1$ is a contraction with Lipschitz constant $\lambda < 1$. This implies the existence and uniqueness of a fractal measure distribution $\mathcal{P}^* \in \mathcal{M}_1$ and exponential convergence of $\mathcal{P}_n$ to $\mathcal{P}^*$. □

4 Construction trees and a.s. convergence

The Monge Kantorovich metric d^{**}_{MK} on the set of random measures used in Section 3 describes weak convergence of the iterative sequence of random approximating measures to a random fractal measure. In this section we consider a natural probability space $\widetilde{\Omega}$, the space of construction trees, and deduce that the corresponding iterative process converges almost surely.

For this purpose, a "non-constructive" operator $\mathbf{S} : \mathbb{M} \to \mathbb{M}$ is introduced, where unless otherwise stated the underlying probability space is $\widetilde{\Omega}$. The argument for a.s. convergence to some $\mu^* \in \mathbb{M}_1$ is based on a contraction argument for $\mathbf{S}$ with respect to a compound version ${d_{MK}}^*$ of the Monge Kantorovich metric defined on $\mathbb{M}_1$ by

$$d^*_{MK}(\mu, \nu) \;=\; \mathsf{E} d_{MK}(\mu, \nu), \qquad \mu, \nu \in \mathbb{M}_1. \tag{7}$$

Note that $(\mathbb{M}_1, d^*_{MK})$ is a complete separable metric space. (The metric d^*_{MK} is "compound" in the sense that, unlike d^{**}_{MK}, it depends on the underlying probability space, and not just on the induced probability distributions.)

We now define the space of construction trees. Let $C = C_N$ denote the N-fold tree of finite sequences from $\{1, \ldots, N\}$ including the empty sequence $\emptyset$. For $\sigma = \sigma_1 \cdots \sigma_n \in C$ denote $|\sigma| = n$. If also $\tau = \tau_1 \cdots \tau_m \in C$ then $\sigma * \tau$ is the concatenated sequence $\sigma_1 \cdots \sigma_n \tau_1 \cdots \tau_m$. A *construction tree* (or tree of scaling laws) is a map $\omega : C \to \Upsilon$, where Υ is the set of scaling laws of $2N$-tuples. Let

$$\widetilde{\Omega} = \{\, w \mid w : C \to \Upsilon \,\}$$

Denote the scaling law at the node σ of ω by

$$\mathbf{S}^\sigma(\omega) \;=\; (p_1^\sigma(\omega), S_1^\sigma(\omega), \ldots, p_N^\sigma(\omega), S_N^\sigma(\omega)) \;=\; \omega(\sigma)$$

for $\sigma \in C$. Let $\widetilde{P}$ be the probability measure on $\widetilde{\Omega}$ induced by selecting iid random scaling laws $\mathbf{S}^\sigma \overset{d}{=} \mathcal{S}$ for $\sigma \in C$. Note that the distributions and independencies of the $\mathbf{S}^\sigma$ are the same as in the iterative procedure described in the previous section. In future, the underlying probability space for $\mathbb{M}$ and $\mathbb{M}_1$ will be $(\widetilde{\Omega}, \widetilde{P})$; this will not affect the final result, as noted before Theorem 4.1.

We use the notation

$$\begin{aligned}
\overline{p}^\sigma &= p_{\sigma_1} p^{\sigma_1}_{\sigma_2} p^{\sigma_1\sigma_2}_{\sigma_3} \cdots p^{\sigma_1\cdots\sigma_{p-1}}_{\sigma_p}, \\
\overline{S}^\sigma &= S_{\sigma_1} \circ S^{\sigma_1}_{\sigma_2} \circ S^{\sigma_1\sigma_2}_{\sigma_3} \circ \cdots \circ S^{\sigma_1\cdots\sigma_{p-1}}_{\sigma_p}.
\end{aligned}$$

In particular $\overline{p}^i = p_i$ and $\overline{S}^i = S_i$ for $1 \le i \le N$.

For a fixed measure $\mu_0 \in M_1$, define

$$\mu_n \;=\; \mu_n(\omega) \;=\; \sum_{|\sigma|=n} \overline{p}^\sigma(\omega) \overline{S}^\sigma(\omega) \mu_0.$$

This is just the sequence defined in Section 3 with underlying $\Omega = \widetilde{\Omega}$. In Theorem 4.1 we show that μ_n converges a.s. to a fixed point μ^* $(\in \mathbb{M}_1)$ of $\mathbf{S}$ w.r.t. weak convergence of measures. An immediate consequence will be that $\mu^* \stackrel{d}{=} \mathcal{P}^*$, the unique fractal measure distribution in $\mathcal{M}_1$ which is self-similar w.r.t. $\mathcal{S}$.

For $\omega \in \widetilde{\Omega}$ and $1 \leq i \leq N$ define $\omega^{(i)} \in \widetilde{\Omega}$ corresponding to the i-th branch of ω by

$$\omega^{(i)}(\sigma) \;=\; \omega(i * \sigma), \qquad \sigma \in C.$$

One checks that

$$\begin{aligned} \overline{p}^{i*\sigma}(\omega) &= p_i(\omega)\overline{p}^{\sigma}(\omega^{(i)}), \\ \overline{S}^{i*\sigma}(\omega) &= S_i(\omega) \circ \overline{S}^{\sigma}(\omega^{(i)}). \end{aligned} \tag{8}$$

Note that the branches $\omega^{(1)}, \ldots, \omega^{(N)}$ of ω are iid with the same distribution as ω and are independent of $(p_1(\omega), S_1(\omega), \ldots, p_N(\omega), S_N(\omega))$. More precisely, for any $\widetilde{P}$ measurable sets E, $F \subset \widetilde{\Omega}$ and $\mathcal{S}$-measurable set $B \subset \Upsilon$,

$$\begin{aligned} &\widetilde{P}(\{\, \omega \mid \omega \in E \,\}) = \widetilde{P}(\{\, \omega \mid \omega^{(i)} \in E \,\}), \\ &\{\, \omega \mid \omega^{(i)} \in E \,\} \text{ and } \{\, \omega \mid \omega^{(j)} \in E \,\} \text{ are independent for } i \neq j, \\ &\{\, \omega \mid (p_1(\omega), S_1(\omega), \ldots, p_N(\omega), S_N(\omega)) \in B \,\} \text{ and } \{\, \omega \mid \omega^{(i)} \in E \,\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{are independent.} \end{aligned} \tag{9}$$

We now define the scaling operator $\mathbf{S} : \mathbb{M} \to \mathbb{M}$ by

$$\mathbf{S}\mu(\omega) \;=\; \sum_i p_i(\omega) S_i(\omega) \mu(\omega^{(i)}). \tag{10}$$

It follows from (9) that $\mathbf{S}\mu$ is identical in distribution to the scaling operator $\mathcal{S}$ applied to $\operatorname{dist} \mu$. Moreover, from (8),

$$\begin{aligned} \mu_{n+1}(\omega) &= \sum_{i=1}^{N} \sum_{|\sigma|=n} \overline{p}^{i*\sigma} \overline{S}^{i*\sigma}(\omega) \mu_0 \\ &= \sum_{i=1}^{N} p_i(\omega) S_i(\omega) \Big(\sum_{|\sigma|=n} \overline{p}^{\sigma}(\omega^{(i)}) \overline{S}^{\sigma}(\omega^{(i)}) \mu_0 \Big) \\ &= \sum_{i=1}^{N} p_i(\omega) S_i(\omega) \mu_n(\omega^{(i)}) \;=\; \mathbf{S}\mu_n(\omega). \end{aligned} \tag{11}$$

Thus the operator $\mathbf{S}$ is defined on all of $\mathbb{M}$, and the sequence $(\mu_n(\omega))$ is obtained by iteratively applying $\mathbf{S}$ to μ_0.

We can now prove the following theorem. The conclusions are probabilistic ones, in that they concern only the (joint) distributions of the random measures involved and otherwise are independent of the choice of sample space. The proof however, and in particular the operator $\mathbf{S}$, uses the specific sample space $(\widetilde{\Omega}, \widetilde{P})$.

Theorem 4.1 *Let $\mathbf{S} = (p_1, S_1, \ldots, p_N, S_N)$ be a random scaling law with*

(i) $\lambda = \mathsf{E}\left(\sum_{i=1}^N p_i r_i\right) < 1,$

(ii) $\Gamma_a = \mathsf{E}\sum_{i=1}^N p_i d(a, S_i a) < \infty, \quad a \in X.$

Let $\mu_0 \in M_1$.

Then there exists a random measure $\mu^* \in \mathbb{M}_1$, independent of μ_0, such that $\mu^* \overset{d}{=} \mathcal{P}^*$ and such that $\mu_n \to \mu^*$ exponentially fast w.r.t. to d^*_{MK}, and in particular a.s.

Proof: We claim that $\mathbf{S} : \mathbb{M}_1 \to \mathbb{M}_1$ and $\mathbf{S}$ is a contraction mapping w.r.t. d^*_{MK}. The fact that $\mathbf{S}\mu$ has mass one a.s. is immediate. The proof that $\mathbf{S}\mu$ has finite first moment is very similar to the proof of the analogous fact in Theorem 3.3, using the independence properties in (9) and the comments which immediately precede it.

To show $\mathbf{S}$ is a contraction map, we compute, again using (9) and the comments preceding it, that

$$\begin{aligned} d^*_{MK}(\mathbf{S}\mu, \mathbf{S}\nu) &= \mathsf{E}d_{MK}\left(\sum p_i(\omega)S_i(\omega)\mu(\omega^{(i)}), \sum p_i(\omega)S_i(\omega)\nu(\omega^{(i)})\right) \\ &\le \mathsf{E}\sum_i p_i(\omega)r_i(\omega)d_{MK}\left(\mu(\omega^{(i)}), \nu(\omega^{(i)})\right) \\ &= \lambda d^*_{MK}(\mu, \nu). \end{aligned}$$

Since $\lambda < 1$ it follows that $\mathbf{S}$ is a contraction mapping on $\mathbb{M}_1$ and so has a unique fixed point μ^*. From (11) it follows that $\mu_n \to \mu^*$, exponentially fast w.r.t. to d^*_{MK}, and in particular a.s. (for example, see the argument in the following Remark (a)). Moreover, since $\mathbf{S}\mu^* = \mu^*$, taking distributions of both sides it follows that $\mathcal{S}\,\mathrm{dist}\,\mu^* = \mathrm{dist}\,\mu^*$ and hence $\mu^* \overset{d}{=} \mathcal{P}^*$ by the uniqueness result of Theorem 3.3. □

Remarks: (a) In fact the proof yields exponential a.s. convergence of $d_{MK}(\mu_n(\omega), \mu^*(\omega)) \to 0$. To see this suppose $0 < \lambda < \tau \le 1$. Then

$$\sum_{n=1}^{\infty} \widetilde{P}\left(\frac{d_{MK}(\mu_n, \mu^*)}{\tau^n} > \varepsilon\right) \le \sum_{n=1}^{\infty} \frac{c}{\varepsilon}\left(\frac{\lambda}{\tau}\right)^n < \infty.$$

This implies by the Borel-Cantelli lemma that

$$\frac{d_{MK}(\mu_n, \mu^*)}{\tau^n} \;\to\; 0 \text{ a.s.} \tag{12}$$

(b) If in the iterative procedure in Section 3 one replaces the fixed measure μ_0 by a random measure $\mu_0 \in \mathbb{M}_1$, or more generally one chooses $\mu_\sigma \in \mathbb{M}_1$ for each $\sigma \in C$ independently of the $\mathbf{S}^\tau$, then a similar result applies. For the proof, use the probability space $(\Omega, P) = (\widetilde{\Omega}, \widetilde{P}) \times (\Omega^*, P^*)$, where (Ω^*, P^*) is the probability space associated with the μ_σ.

(c) Both probability metrics introduced in this paper, d^*_{MK} and d^{**}_{MK}, are useful tools for the study of convergence of random measures with normalized mass. While d^{**}_{MK} is a suitable simple metric to study convergence in distribution, the metric d^*_{MK} is of compound type. It needs therefore a special underlying construction of the random measures (in our case the space of construction trees). Then by the structure of this new type of metric one obtains from contraction properties a.s. convergence of the iterative sequence of random measures. □

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