Spherical codes with good separation, discrepancy and energy

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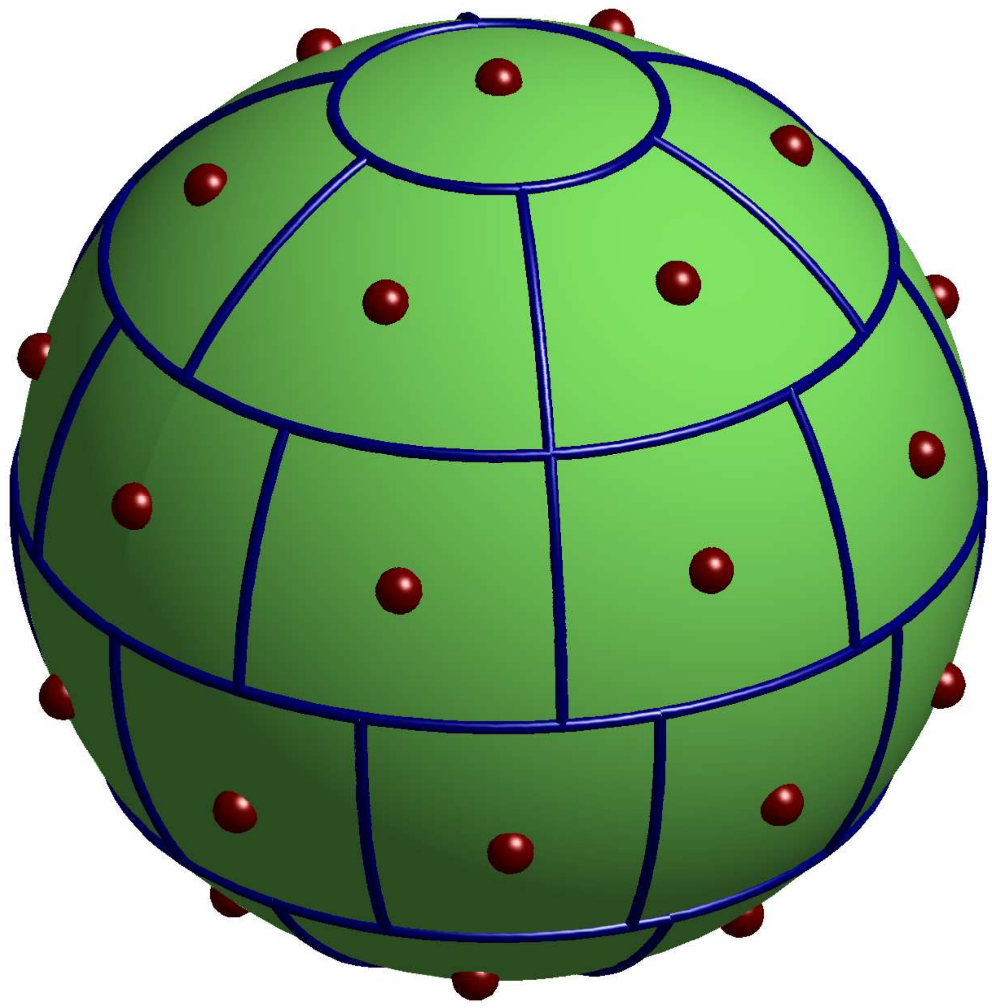
Outline of talk

EQ codes: The Recursive Zonal Equal Area spherical codes, $\mathbf{EQP}(d, \mathcal{N}) \subset \mathbb{S}^d$, with $|\mathbf{EQP}(d, \mathcal{N})| = \mathcal{N}$.

- Overview of properties of the EQ codes
- Some precedents
- Definitions: spherical polar coordinates, partitions, diameter bounds
- The Recursive Zonal Equal Area (EQ) partition
- Details of properties of the EQ codes
- Separation and discrepancy bounds imply energy bounds

The spherical code EQP(2,33) on $\mathbb{S}^2 \subset \mathbb{R}^3$



Geometric properties of the EQ codes

For $\mathbf{EQP}(d, \mathcal{N})$

Good:

- Centre points of regions of diameter $\leqslant \mathbf{O}(\mathcal{N}^{-1/d})$,
- Mesh norm (covering radius) $\leqslant \mathbf{O}(\mathcal{N}^{-1/d})$,
- Minimum distance and packing radius $\geqslant \mathbf{O}(\mathcal{N}^{-1/d})$.

Bad:

- Mesh ratio $\geqslant \mathbf{O}(\sqrt{d})$,
- Packing density $\leqslant \frac{\pi^{d/2}}{2^d\ \Gamma(d/2+1)}$.

Approximation properties of the EQ codes

For $\mathrm{EQP}(d, \mathcal{N})$

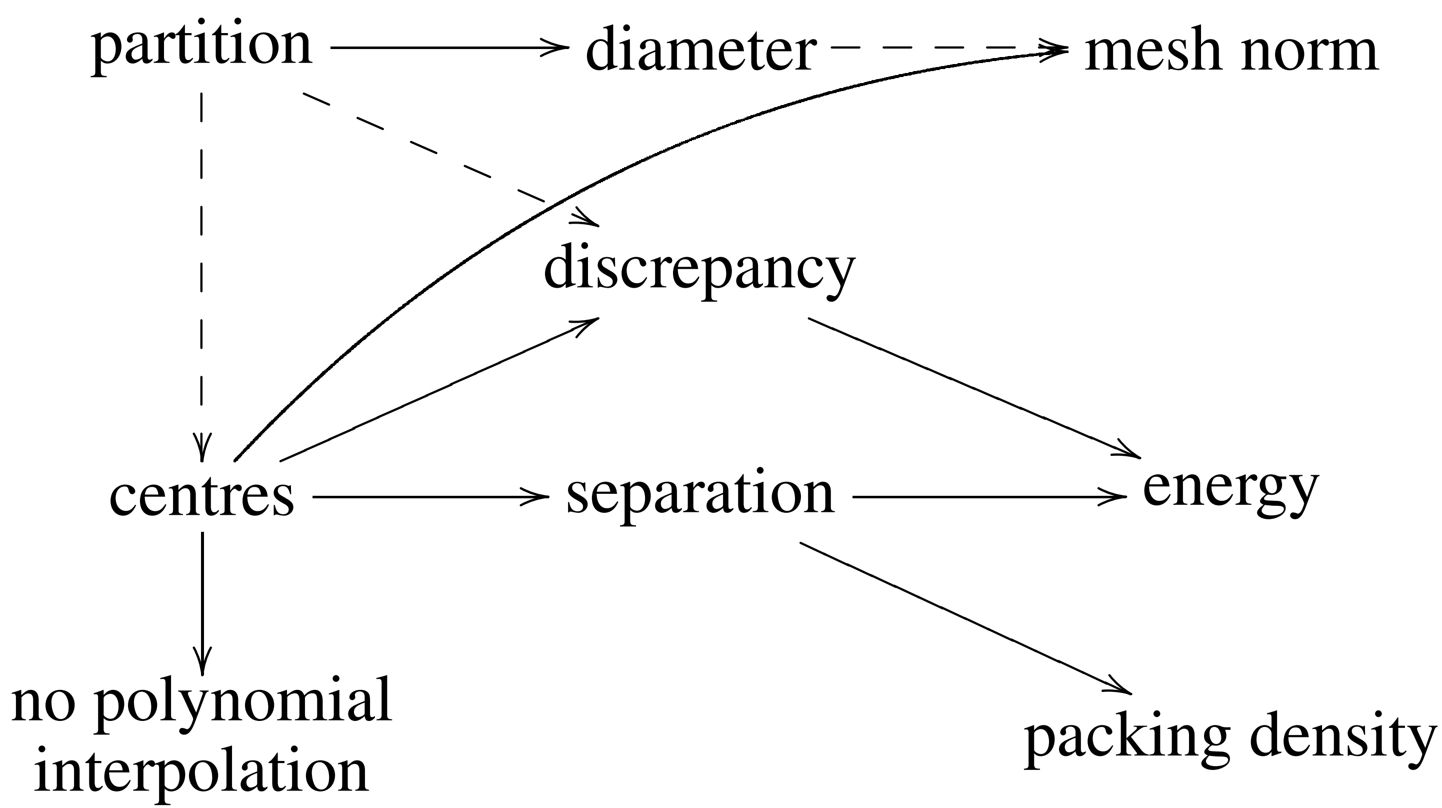
Not so bad?

- Normalized spherical cap discrepancy $\leqslant \mathrm{O}(\mathcal{N}^{-1/d})$,
- Excess normalized s-energy $\leqslant \mathrm{O}(\mathcal{N}^{s/d^2 - 1/d})$, for $0 < s < d$.

Ugly:

- Cannot be used for polynomial interpolation:
  not a fundamental system
  - proven for large enough $\mathcal{N}$, conjectured for small $\mathcal{N}$.

Relationships between properties of EQ codes



Some precedents

The $\mathbf{EQ}$ partition is based on Zhou's (1995) construction for $\mathbb{S}^2$ as modified by Saff, and on Sloan's sketch of a partition of $\mathbb{S}^3$ (2003).

Separation without equidistribution: Hamkins (1996) and Hamkins and Zeger (1997) constructed $\mathbb{S}^d$ codes with asymptotically optimal packing density.

Equidistibution without separation: Many constructions for $\mathbb{S}^2$, eg. mapped Hammersley, Halton, (t, s) etc. sequences.

Feige and Schechtman (2002) constructed a diameter bounded equal area partition of $\mathbb{S}^d$. Put one point in each region.

Equal-area partitions of $\mathbb{S}^d \subset \mathbb{R}^d$

An *equal area partition* of $\mathbb{S}^d \subset \mathbb{R}^d$ is a nonempty finite set $\mathcal{P}$ of Lebesgue measurable subsets of $\mathbb{S}^d$, such that

$$\bigcup_{R \in \mathcal{P}} R = \mathbb{S}^d,$$

and for each $R \in \mathcal{P}$,

$$\sigma(R) = \frac{\sigma(\mathbb{S}^d)}{|\mathcal{P}|},$$

where σ is the Lebesgue area measure on $\mathbb{S}^d$.

Diameter bounded sets of partitions

The *diameter* of a region $\boldsymbol{R} \subset \mathbb{R}^{\boldsymbol{d}+1}$ is defined by

$$\operatorname{diam} \boldsymbol{R} := \sup\{\|\mathrm{x} - \mathrm{y}\| \mid \mathrm{x}, \mathrm{y} \in \boldsymbol{R}\}.$$

A set $\boldsymbol{\Xi}$ of partitions of $\mathbb{S}^{\boldsymbol{d}} \subset \mathbb{R}^{\boldsymbol{d}+1}$ is *diameter-bounded* with *diameter bound* $\boldsymbol{K} \in \mathbb{R}_{+}$ if for all $\boldsymbol{\mathcal{P}} \in \boldsymbol{\Xi}$, for each $\boldsymbol{R} \in \boldsymbol{\mathcal{P}}$,

$$\operatorname{diam} \boldsymbol{R} \leqslant \boldsymbol{K} \, |\boldsymbol{\mathcal{P}}|^{-1/\boldsymbol{d}}.$$

Key properties of the EQ partition of $\mathbb{S}^d$

$\mathbf{EQ}(d, \mathcal{N})$ is the *recursive zonal equal area* partition of $\mathbb{S}^d$ into $\mathcal{N}$ regions.

The set of partitions $\mathbf{EQ}(d) := \{\mathbf{EQ}(d, \mathcal{N}) \mid \mathcal{N} \in \mathbb{N}_+\}$.

The $\mathbf{EQ}$ partition satisfies:

Theorem 1. *For* $d \geqslant 1$, $\mathcal{N} \geqslant 1$, $\mathbf{EQ}(d, \mathcal{N})$ *is an equal-area partition.*

Theorem 2. *For* $d \geqslant 1$, $\mathbf{EQ}(d)$ *is diameter-bounded.*

Spherical polar coordinates on $\mathbb{S}^d$

Spherical polar coordinates describe $\mathbf{x} \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$ by one longitude, $\xi_1 \in \mathbb{R}$ (modulo 2π), and $d-1$ colatitudes, $\xi_j \in [0, \pi]$, for $j \in \{2, \ldots, d\}$.

The spherical polar to Cartesian coordinate map $\odot : \mathbb{R} \times [0, \pi]^{d-1} \to \mathbb{S}^d \subset \mathbb{R}^{d+1}$ is

$$\odot(\xi_1, \xi_2, \ldots, \xi_d) = (x_1, x_2, \ldots, x_{d+1}),$$

$$\text{where} \quad x_1 := \cos \xi_1 \prod_{j=2}^{d} \sin \xi_j, \quad x_2 := \prod_{j=1}^{d} \sin \xi_j,$$

$$x_k := \cos \xi_{k-1} \prod_{j=k}^{d} \sin \xi_j, \quad k \in \{3, \ldots, d+1\}.$$

Spherical caps, zones, and collars

The *spherical cap* $\boldsymbol{S}(\mathrm{p}, \boldsymbol{\theta}) \subset \mathbb{S}^{\boldsymbol{d}}$ is

$$\boldsymbol{S}(\mathrm{p}, \boldsymbol{\theta}) := \left\{\mathrm{q} \in \mathbb{S}^{\boldsymbol{d}} \mid \mathbf{p} \cdot \mathbf{q} \geqslant \cos(\boldsymbol{\theta})\right\}.$$

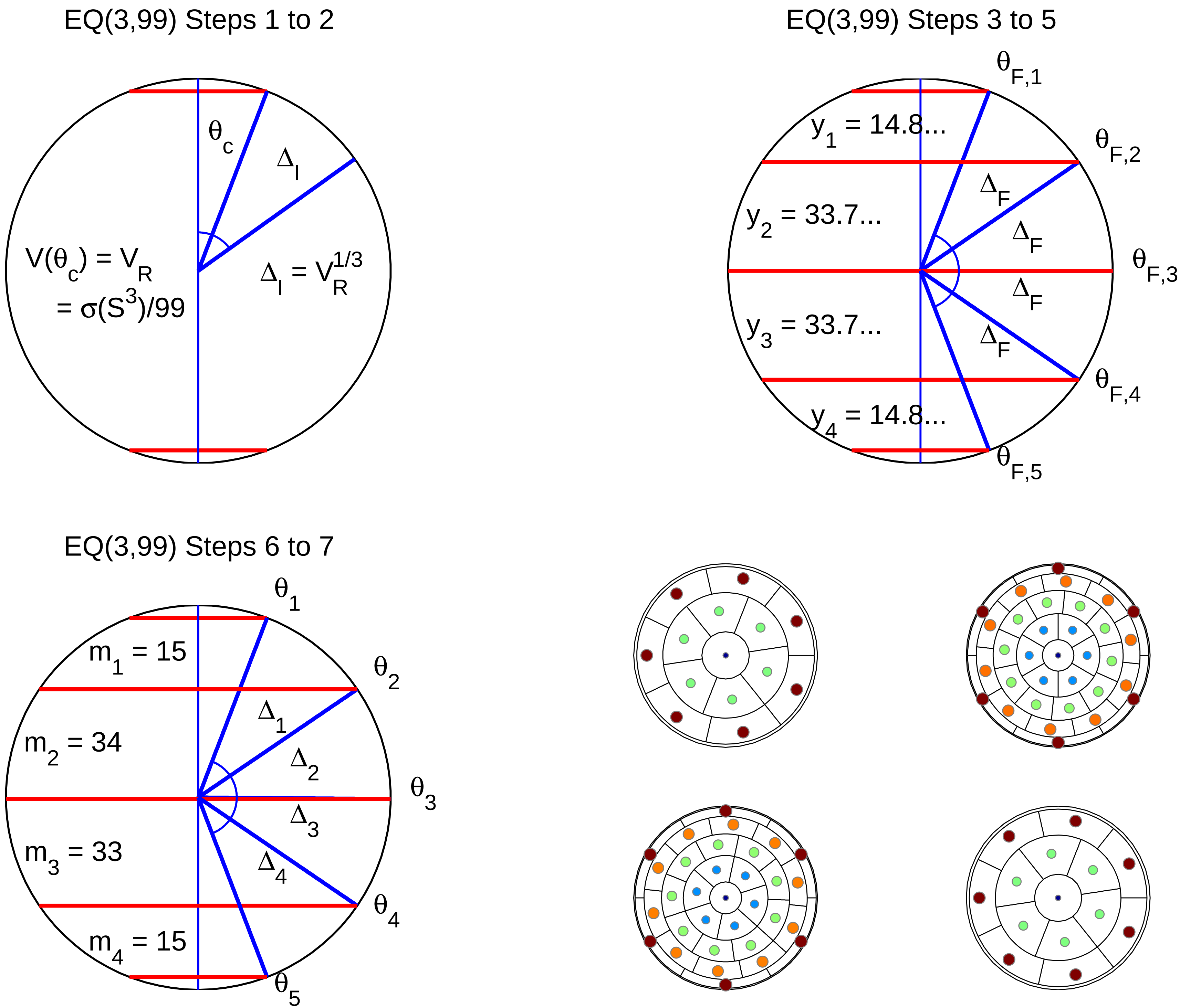
For $\boldsymbol{d} > \mathbf{1}$, a *zone* can be described by

$$\boldsymbol{Z}(\boldsymbol{\tau}, \boldsymbol{\beta}) := \left\{\odot(\boldsymbol{\xi}_{\mathbf{1}}, \ldots, \boldsymbol{\xi}_{\boldsymbol{d}}) \in \mathbb{S}^{\boldsymbol{d}} \mid \boldsymbol{\xi}_{\boldsymbol{d}} \in [\boldsymbol{\tau}, \boldsymbol{\beta}]\right\},$$

where $\mathbf{0} \leqslant \boldsymbol{\tau} < \boldsymbol{\beta} \leqslant \boldsymbol{\pi}$.

$\boldsymbol{Z}(\mathbf{0}, \boldsymbol{\beta})$ is a North polar cap and $\boldsymbol{Z}(\boldsymbol{\tau}, \boldsymbol{\pi})$ is a South polar cap.

If $\mathbf{0} < \boldsymbol{\tau} < \boldsymbol{\beta} < \boldsymbol{\pi}$, $\boldsymbol{Z}(\boldsymbol{\tau}, \boldsymbol{\beta})$ is a *collar*.


EQ(3,99) Steps 1 to 2
θ_c
Δ_I
V(θ_c) = V_R
= σ(S^3)/99
Δ_I = V_R^{1/3}
EQ(3,99) Steps 3 to 5
θ_{F,1}
y_1 = 14.8...
θ_{F,2}
Δ_F
y_2 = 33.7...
Δ_F
θ_{F,3}
Δ_F
y_3 = 33.7...
Δ_F
θ_{F,4}
y_4 = 14.8...
θ_{F,5}
EQ(3,99) Steps 6 to 7
θ_1
m_1 = 15
θ_2
Δ_1
m_2 = 34
Δ_2
θ_3
Δ_3
m_3 = 33
Δ_4
θ_4
m_4 = 15
θ_5

Centre points of regions of $\mathrm{EQ}(d, \mathcal{N})$

The placement of the centre point $a = \odot(\alpha)$ of a region

$$R = \odot\left([\tau_1, \beta_1] \times \ldots \times [\tau_d, \beta_d]\right) \text{ is}$$

$$\alpha_1 := \begin{cases} 0, & \beta_1 = \tau_1 \pmod{2\pi}, \\ (\tau_1 + \beta_1)/2 \pmod{2\pi}, & \text{otherwise}, \end{cases}$$

and for $j > 1$,

$$\alpha_j := \begin{cases} 0, & \tau_j = 0, \\ \pi, & \beta_j = \pi, \\ (\tau_j + \beta_j)/2, & \text{otherwise}. \end{cases}$$

Mesh norm (covering radius)

The *mesh norm* of $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_{\mathcal{N}}\} \subset \mathbb{S}^d$ is

$$\text{mesh norm } X := \sup_{\mathbf{y} \in \mathbb{S}^d} \min_{\mathbf{x} \in X} \cos^{-1}(\mathbf{x} \cdot \mathbf{y}).$$

Since $\mathrm{EQ}(d)$ is diameter bounded,

$$\text{mesh norm } \mathrm{EQP}(d, \mathcal{N}) \leqslant \mathrm{O}(\mathcal{N}^{-1/d}).$$

Minimum distance and packing radius

The *minimum distance* of $\boldsymbol{X} := \{\mathbf{x}_1, \ldots, \mathbf{x}_{\mathcal{N}}\} \subset \mathbb{S}^d$ is

$$\text{min dist } \boldsymbol{X} := \min_{\mathbf{x} \neq \mathbf{y} \in \boldsymbol{X}} \|\mathbf{x} - \mathbf{y}\|,$$

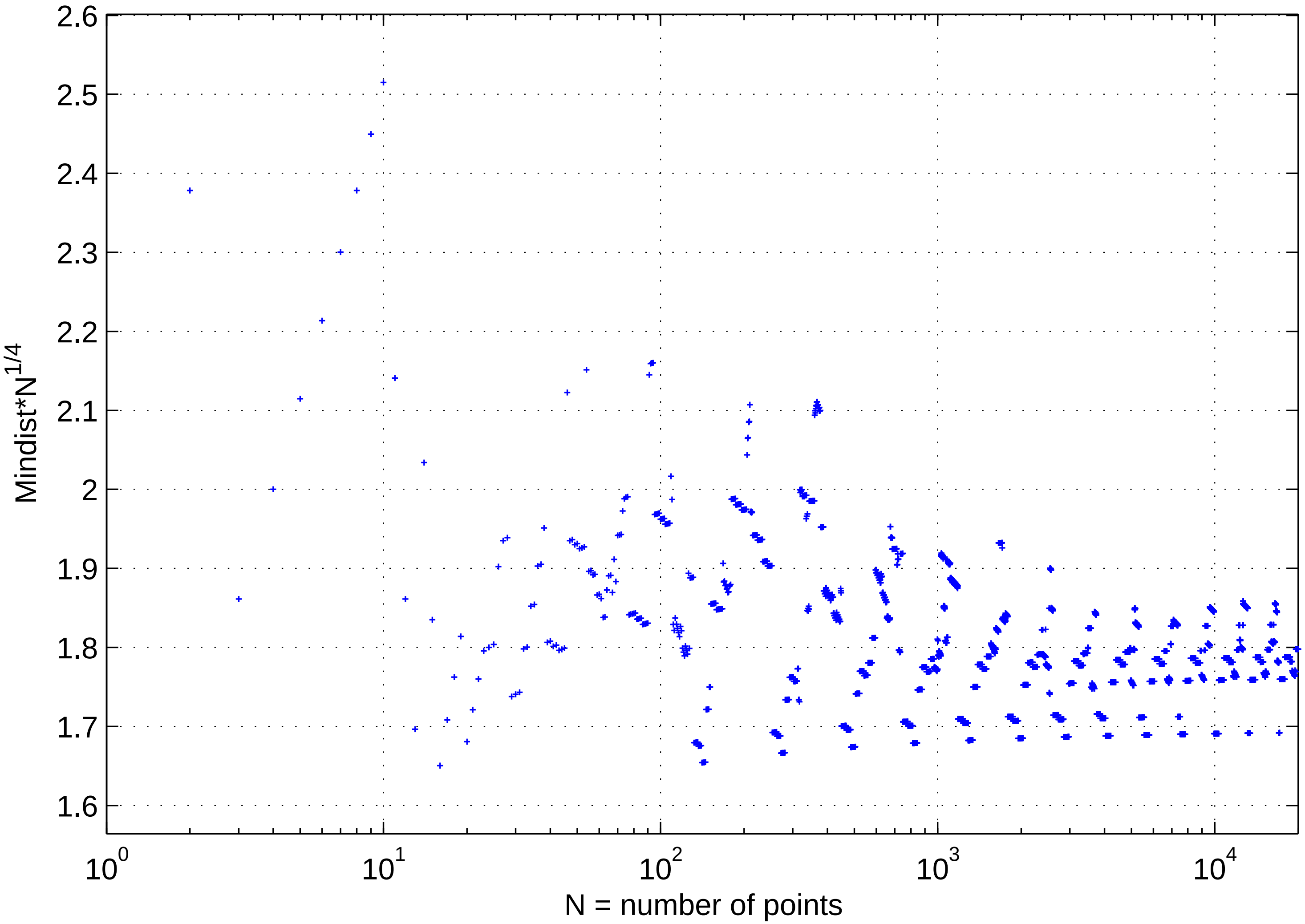
and the *packing radius* of $\boldsymbol{X}$ is

$$\text{prad } \boldsymbol{X} := \min_{\mathbf{x} \neq \mathbf{y} \in \boldsymbol{X}} \cos^{-1}(\mathbf{x} \cdot \mathbf{y})/2.$$

It can be shown that $\quad \text{min dist } \mathrm{EQP}(d, \mathcal{N}) \geqslant \mathrm{O}(\mathcal{N}^{-1/d}),$

and therefore $\quad \text{prad } \mathrm{EQP}(d, \mathcal{N}) \geqslant \mathrm{O}(\mathcal{N}^{-1/d}).$

Minimum distance of EQP(4) codes



Mesh ratio and packing density

The *mesh ratio* of $X := \{x_1, \ldots, x_{\mathcal{N}}\} \subset \mathbb{S}^d$ is

$$\text{mesh ratio } X := \text{mesh norm } X \;/\; \text{prad } X.$$

The *packing density* of X is

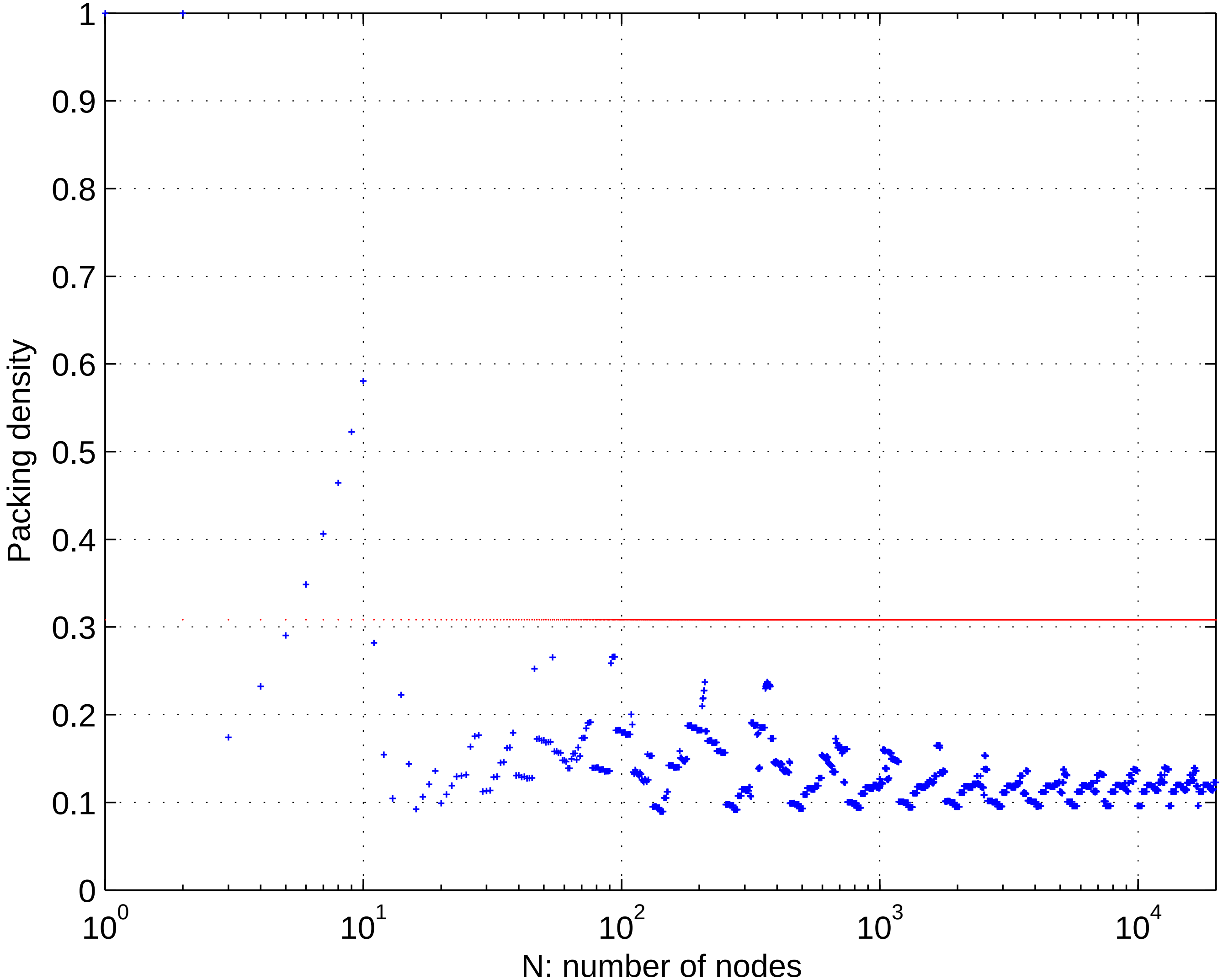
$$\text{pdens } X := \mathcal{N} \sigma\big(S\big(x, \text{prad } X\big)\big) / \, \sigma(\mathbb{S}^d).$$

Regions of $\mathrm{EQ}(d, \mathcal{N})$ near equators $\longrightarrow$ cubic as $\mathcal{N} \longrightarrow \infty$, so

mesh ratio $\mathrm{EQP}(d, \mathcal{N}) \geqslant \mathrm{O}(\sqrt{d})$, and

$$\text{pdens } \mathrm{EQP}(d, \mathcal{N}) \leqslant \frac{\pi^{d/2}}{2^d \; \Gamma(d/2 + 1)} \quad \text{(asymptotically).}$$

Packing density of EQP(4) codes



Normalized spherical cap discrepancy

We use the probability measure $\overset{*}{\sigma} := \sigma / \sigma(\mathbb{S}^d)$.

For $X := \{x_1, \ldots, x_{\mathcal{N}}\} \subset \mathbb{S}^d$ the *normalized spherical cap discrepancy* is

$$\operatorname{disc} X := \sup_{y \in \mathbb{S}^d} \sup_{\theta \in [0,\pi]} \left| \frac{|X \cap S(y,\theta)|}{\mathcal{N}} - \overset{*}{\sigma}\big(S(y,\theta)\big) \right|.$$

It can be shown that

$$\operatorname{disc} \operatorname{EQP}(d, \mathcal{N}) \leqslant \operatorname{O}(\mathcal{N}^{-1/d}).$$

Normalized s-energy

For $X := \{x_1, \ldots, x_{\mathcal{N}}\} \subset \mathbb{S}^d$ the *normalized* s*-energy* is

$$E_s(X) := \mathcal{N}^{-2} \sum_{i=1}^{\mathcal{N}} \sum_{x_i \neq x_j \in X} \|x_i - x_j\|^{-s},$$

and the *normalized energy double integral* is

$$I_s := \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x - y\|^{-s} \, d\overset{*}{\sigma}(x) d\overset{*}{\sigma}(y).$$

It can be shown that, for $0 < s < d$,

$$E_s\big(\mathrm{EQP}(d, \mathcal{N})\big) \leqslant I_s + \mathrm{O}(\mathcal{N}^{s/d^2 - 1/d}).$$

Separation and discrepancy imply energy

Theorem 3.

Let $(X_1, X_2, \ldots \in \mathbb{N})$ *be a sequence of* $\mathbb{S}^d$ *codes for which there exist* $c_1, c_2 > 0$ *such that each* $X_\mathcal{N} = \{\mathrm{x}_{\mathcal{N},1}, \ldots, \mathrm{x}_{\mathcal{N},\mathcal{N}}\}$ *satisfies*

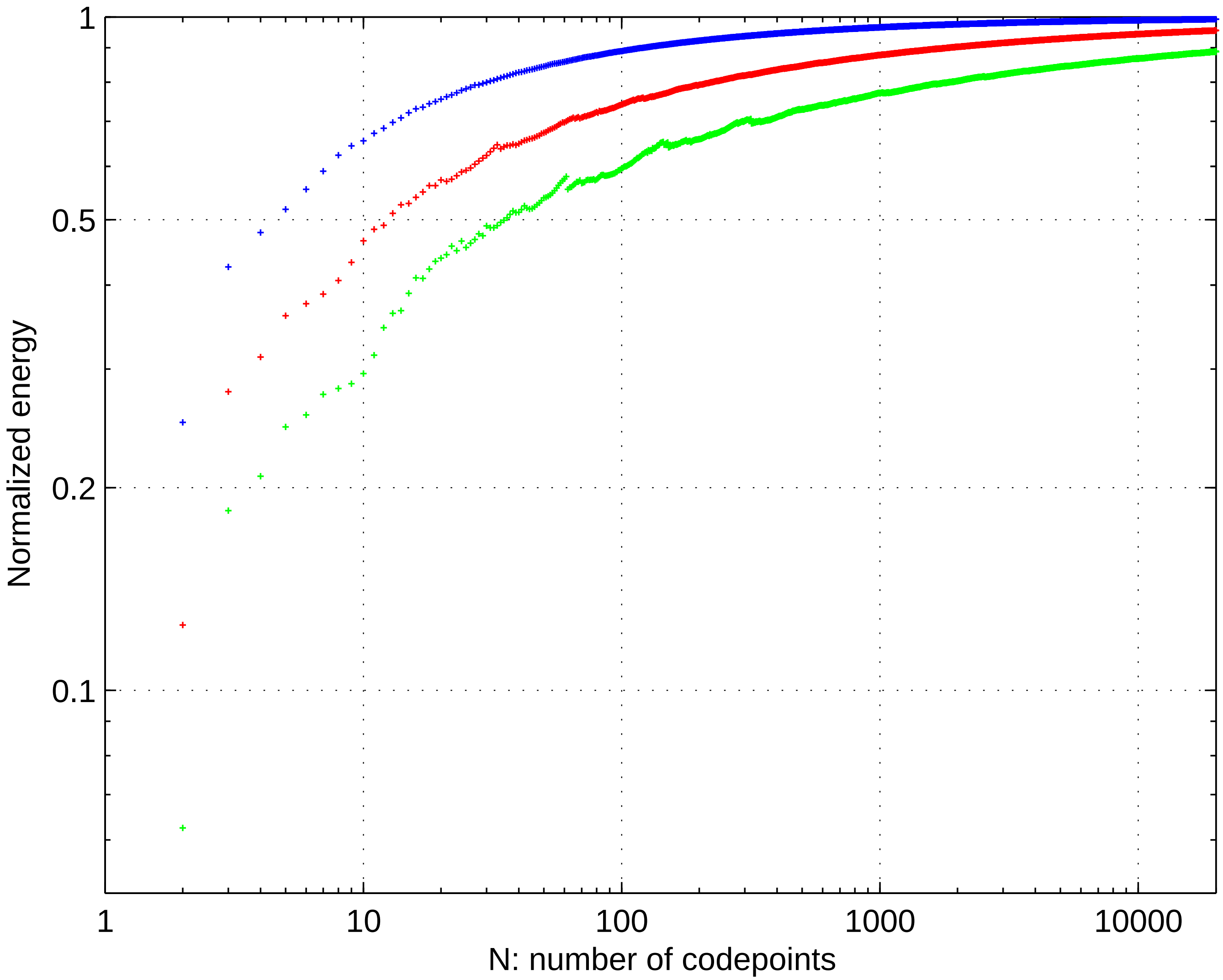
$$\|\mathrm{x}_{\mathcal{N},i} - \mathrm{x}_{\mathcal{N},j}\| > c_1\, \mathcal{N}^{-1/d}, \quad (i \neq j)$$

$$\textit{disc}\ X_\mathcal{N} \leqslant c_2\, \mathcal{N}^{-q}.$$

Then for the normalized s *energy for* $0 < s < d$*, we have for some* $c_3 \geqslant 0$*,*

$$E_s(X_\mathcal{N}) \leqslant I_s + c_3\, \mathcal{N}^{(s/d-1)q}.$$

$d - 1$ energy of EQP(2), EQP(3), EQP(4)



For EQSP Matlab code

See SourceForge web page for EQSP:

Recursive Zonal Equal Area Sphere Partitioning Toolbox:

`http://eqsp.sourceforge.net`