Approximating functions in Clifford algebras

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Topics

- Clifford algebras
- Matrix functions
- Square root
- Logarithm
- General eigenvalues

Start with a group of signed integer sets

Generators: $\{k\}$ where $k \in \mathbb{Z}^*$.

Relations: Element -1 in the centre.

$$(-1)^2 = 1,$$
$$(-1)\{k\} = \{k\}(-1) \quad \text{(for all } k\text{)},$$
$$\{k\}^2 = \begin{cases} -1 & (k < 0), \\ 1 & (k > 0), \end{cases}$$
$$\{j\}\{k\} = (-1)\{k\}\{j\} \quad (j \neq k).$$

Canonical ordering:

$$\{j, k, \ell\} := \{j\}\{k\}\{\ell\} \quad (j < k < \ell), \text{ etc.}$$

Product of signed sets is signed XOR.

(Braden 1985; Lam and Smith 1989; Lounesto 1997; Dorst 2001)

Extend to a real linear algebra

Vector space: Real linear combination of $\mathbb{Z}^*$ sets.

$$v = \sum_{S \subset \mathbb{Z}^*} v_S S.$$

Multiplication: Extends group multiplication.

$$vw = \sum_{S \subset \mathbb{Z}^*} v_S S \sum_{T \subset \mathbb{Z}^*} w_T T$$
$$= \sum_{S \subset \mathbb{Z}^*} \sum_{T \subset \mathbb{Z}^*} v_S w_T ST.$$

Clifford algebra $\mathbb{R}_{(p,q)}$ uses subsets of $\{-q, \ldots, p\}^*$.

(Braden 1985; Lam and Smith 1989; Wene 1992; Lounesto 1997; Dorst 2001; Ashdown)

Some examples of Clifford algebras

$$\mathbb{R}_{(0,0)} \equiv \mathbb{R}.$$

$$\mathbb{R}_{(0,1)} \equiv \mathbb{R} + \mathbb{R}\{-1\} \equiv \mathbb{C}.$$

$$\mathbb{R}_{(1,0)} \equiv \mathbb{R} + \mathbb{R}\{1\} \equiv {}^{2}\mathbb{R}.$$

$$\mathbb{R}_{(1,1)} \equiv \mathbb{R} + \mathbb{R}\{-1\} + \mathbb{R}\{1\} + \mathbb{R}\{-1, 1\} \equiv \mathbb{R}(2).$$

$$\mathbb{R}_{(0,2)} \equiv \mathbb{R} + \mathbb{R}\{-2\} + \mathbb{R}\{-1\} + \mathbb{R}\{-2, -1\} \equiv \mathbb{H}.$$

Real representations

Representation is a **linear map**.

$$\mathbb{R}_{(0,1)} \equiv \mathbb{C}: \quad \rho(x + y\{-1\}) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

$$\mathbb{R}_{(1,0)} \equiv {}^2\mathbb{R}: \quad \rho(x + y\{1\}) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$$

$$\mathbb{R}_{(0,2)} \equiv \mathbb{H}:$$

$$\rho(\, w + x\{-2\} + y\{-1\} + z\{-2,-1\}\,) = \begin{bmatrix} w & -y & -x & z \\ y & w & -z & -x \\ x & -z & w & -y \\ z & x & y & w \end{bmatrix}$$

$2^n \times 2^n$ real matrices for some n.

(Cartan and Study 1908; Porteous 1969; Lounesto 1997)

Real chessboard

p ↓ \ q →	0	1	2	3	4	5	6	7	8
0	1	2	4	8	8	8	8	16	16
1	2	2	4	8	16	16	16	16	32
2	2	4	4	8	16	32	32	32	32
3	4	4	8	8	16	32	64	64	64
4	8	8	8	16	16	32	64	128	128
5	16	16	16	16	32	32	64	128	256
6	16	32	32	32	32	64	64	128	256
7	16	32	64	64	64	64	128	128	256
8	16	32	64	128	128	128	128	256	256

(Cartan and Study 1908; Porteous 1969; Lounesto 1997)

Inner product and norm

Inner product: Normalized Frobenius inner product.

For $v, w \in \mathbb{R}_{(p,q)}$, $\rho(v) = V$, $\rho(w) = W \in \mathbb{R}(2^n)$,

$$\langle v, w \rangle := \sum_{T \subset \{-q,\ldots,p\}^*} v_T w_T$$

$$= 2^{-n} \sum_{j=1}^{2^n} \sum_{k=1}^{2^n} V_{j,k} W_{j,k}.$$

Norm: Induced by inner product.

$$\|v\| := \sqrt{\langle v, v \rangle}.$$

(Gilbert and Murray 1991)

Definition of matrix functions

For a function f analytic in $\Omega \subset \mathbb{C}$,

$$f(X) := \frac{1}{2\pi i} \int_{\partial\Omega} f(z)\,(zI - X)^{-1}\,dz,$$

where the spectrum $\sigma(X) \subset \Omega$.

For f analytic on an open disk $D \supset \sigma(X)$ with $0 \in D$,

$$f(X) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} X^k.$$

For invertible Y, $\quad f(YXY^{-1}) = Y f(X) Y^{-1}$.

(Rinehart 1955; Golub and van Loan 1983, 1996; Horn and Johnson 1994)

Padé approximation

For function f with power series

$$f(z) = \sum_{k=0}^{\infty} f_k z^k,$$

the (m, n) **Padé approximant** is the ratio

$$\frac{a_m(z)}{b_n(z)},$$

of polynomials a_m, b_n of degree m, n such that

$$|f(z)\, b_n(z) - a_m(z)| = \mathrm{O}(z^{m+1}).$$

(Zeilberger 2002)

Padé square root

For $(|z| \leq 1, z \neq 1)$:

$$\sqrt{1-z} = 1 - \frac{1}{2}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \frac{5}{128}z^4 - \ldots$$

For "small" $\|I - X\|$, use (n, n) Padé approximant

$$\sqrt{X} = \sqrt{I - Z} \simeq a_n(Z) b_n(Z)^{-1},$$

where $Z := I - X$.

Denman–Beavers square root

If X has no negative eigenvalues, the iteration

$$M_0 := Y_0 := X,$$
$$M_{k+1} := \frac{M_k + M_k^{-1}}{4} + \frac{I}{2},$$
$$Y_{k+1} := Y_k \, \frac{I + M_k^{-1}}{2}$$

has $Y_k \to \sqrt{X}$ and $M_k \to I$ as $k \to \infty$.

This iteration is **numerically stable**.

(Denman, Beavers 1976; Cheng, Higham, Kenney, Laub 1999)

Cheng–Higham–Kenney–Laub logarithm

$$\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k} \quad (|z| \leq 1, z \neq 1).$$

Assume X has no negative eigenvalues.
Since $\log X = 2 \log \sqrt{X}$,

1. iterate square roots until $\|I - X\|$ is "small",
2. use a Padé approximant to $\log(I - Z)$, where $Z := I - X$,
3. rescale.

C-H-K-L's "incomplete square root cascade":

- Stop Denman–Beavers iterations early, estimate error in log.

(Cheng, Higham, Kenney, Laub 1999)

What about negative eigenvalues?

Use an algebra with $\mathbf{i}$: $\mathbf{i}^2 = -1$, $\mathbf{i}\boldsymbol{X} = \boldsymbol{X}\mathbf{i}$ for all $\boldsymbol{X}$:
Full $\mathbb{C}$ matrix algebra.

Embeddings:

$$\mathbb{R} \equiv \mathbb{R}_{(0,0)} \subset \mathbb{R}_{(0,1)} \equiv \mathbb{C}.$$

$$^2\mathbb{R} \equiv \mathbb{R}_{(1,0)} \subset \mathbb{R}_{(1,2)} \equiv \mathbb{C}(2).$$

$$\mathbb{R}(2) \equiv \mathbb{R}_{(1,1)} \subset \mathbb{R}_{(1,2)} \equiv \mathbb{C}(2).$$

$$\mathbb{H} \equiv \mathbb{R}_{(0,2)} \subset \mathbb{R}_{(1,2)} \equiv \mathbb{C}(2).$$

Complex chessboard

$q \rightarrow$ / $p \downarrow$	0	1	2	3	4	5	6	7	8
0	1	1	2	4	4	4	8	16	16
1	2	2	2	4	8	8	8	16	32
2	2	4	4	4	8	16	16	16	32
3	2	4	8	8	8	16	32	32	32
4	4	4	8	16	16	16	32	64	64
5	8	8	8	16	32	32	32	64	128
6	8	16	16	16	32	64	64	64	128
7	8	16	32	32	32	64	128	128	128
8	16	16	32	64	64	64	128	256	256

(Cartan and Study 1908; Porteous 1969; Lounesto 1997)

Real–complex chessboard

q → / p ↓	0	1	2	3	4	5	6	7	8
0	1	2	4	8	8	8	8	16	16
1	2	2	4	8	16	16	16	16	32
2	2	4	4	8	16	32	32	32	32
3	4	4	8	8	16	32	64	64	64
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(Cartan and Study 1908; Porteous 1969; Lounesto 1997)

Example: ${}^2\mathbb{R} \equiv \mathbb{R}_{(1,0)}$

$$ {}^2\mathbb{R} \equiv \mathbb{R}_{(1,0)} \subset \mathbb{R}_{(1,2)} \subset \mathbb{R}_{(2,2)} \equiv \mathbb{R}(4). $$

$$ \rho(x + y\{1\}) = \begin{bmatrix} x & y & & \\ y & x & & \\ & & x & y \\ & & y & x \end{bmatrix} $$

$$ \mathbf{i} = \begin{bmatrix} & & 1 & 0 \\ & & 0 & -1 \\ -1 & 0 & & \\ 0 & 1 & & \end{bmatrix} $$

(Cartan and Study 1908; Porteous 1969; Lounesto 1997)

Special case — square is positive

If $\boldsymbol{X}^2 = \boldsymbol{\lambda}^2 \boldsymbol{I}$ we have $(\boldsymbol{X} + \boldsymbol{\lambda I})(\boldsymbol{X} - \boldsymbol{\lambda I}) = \boldsymbol{0}$.

With an algebra containing $\mathbf{i}$, we can use:

$$\sqrt{\boldsymbol{X}} = \sqrt{-\mathbf{i}\boldsymbol{X}}\,\sqrt{\mathbf{i}} = \sqrt{-\mathbf{i}\boldsymbol{X}}\,\frac{\mathbf{i} + \boldsymbol{I}}{\sqrt{2}},$$

$$\log \boldsymbol{X} = \log(-\mathbf{i}\boldsymbol{X}) + \log(\mathbf{i}) = \log(-\mathbf{i}\boldsymbol{X}) + \frac{\pi}{2}\mathbf{i}.$$

Clifford **basis elements** have eigenvalues ± 1 and square $\pm \boldsymbol{I}$.

General case — functions by similarity

For invertible $\boldsymbol{Y}$, $\quad \boldsymbol{f(YXY^{-1}) = Yf(X)Y^{-1}}$.

Find $\boldsymbol{Y}$ to make $\boldsymbol{f(YXY^{-1})}$ easy. Ideally $\boldsymbol{YXY^{-1} = D}$,

$$\boldsymbol{D} = \begin{bmatrix} \lambda_1 & & \\ & \ldots & \\ & & \lambda_N \end{bmatrix}$$

so that

$$\boldsymbol{f(D)} = \begin{bmatrix} f(\lambda_1) & & \\ & \ldots & \\ & & f(\lambda_N) \end{bmatrix}$$

Diagonalization is **not always possible**. Closest is Jordan form.

(Golub and van Loan 1983, 1996; Davies and Higham 2002)

Schur form and QR algorithm

Schur form:

Block triangular, eigenvalues on diagonal. Eg.

$$T = \begin{bmatrix} -2 & 7 & 19 & 3i \\ & -2 & -5 & 0 \\ & & i & 1 \\ & & & 9 \end{bmatrix}$$

QR algorithm:

Iterative algorithm for Schur decomposition

$X = QTQ^*$: originally iterated QR decomposition.

Schur form is more numerically stable than Jordan form.

(Golub and van Loan 1983, 1996; Davies and Higham 2002)

Parlett recurrence

For $T = (t_{i,j})$, $f(T) = (f_{i,j})$,

$$f_{i,j} = t_{i,j}\frac{f_{i,i} - f_{j,j}}{t_{i,i} - t_{j,j}} + \sum_{k=i+1}^{j-1} \frac{f_{i,k}t_{k,j} - t_{i,k}f_{k,j}}{t_{i,i} - t_{j,j}}.$$

Unstable for close eigenvalues.

(Golub and van Loan 1983, 1996; Davies and Higham 2002)

Schur–Parlett–Davies–Higham approximation

- Schur decomposition, reordering, blocking, then block form of Parlett recurrence.
- Puts close clusters of eigenvalues into separate diagonal blocks.

Can be numerically **unstable** depending on blocking parameter.

(Golub and van Loan 1983, 1996; Davies and Higham 2002)

Hale–Higham–Trefethen quadrature

- Conformal map using Jacobi elliptic functions, then approximate contour integral using trapezoidal rule.
- Contour must surround spectrum, so must estimate eigenvalues.

Quadrature is equivalent to rational approximation.

$$f(X) := \frac{1}{2\pi i}\int_{\partial\Omega} f(z)\,(zI - X)^{-1}\,dz$$
$$\simeq \frac{1}{N}\sum_{k=1}^{N} w_k f(z_k)\,(z_k I - X)^{-1}.$$

(Hale, Higham and Trefethen 2008)

GluCat — Clifford algebra library

- Generic library of universal Clifford algebra templates.
- C++ template library for use with other libraries.
- Implements algorithms for matrix functions.

For details, see **http://glucat.sf.net**

(Lounesto et al. 1987; Lounesto 1992; Raja 1996; Leopardi 2001-2008)