Spherical codes with good separation, discrepancy and energy

Paul Leopardi

Mathematical Sciences Institute, Australian National University.

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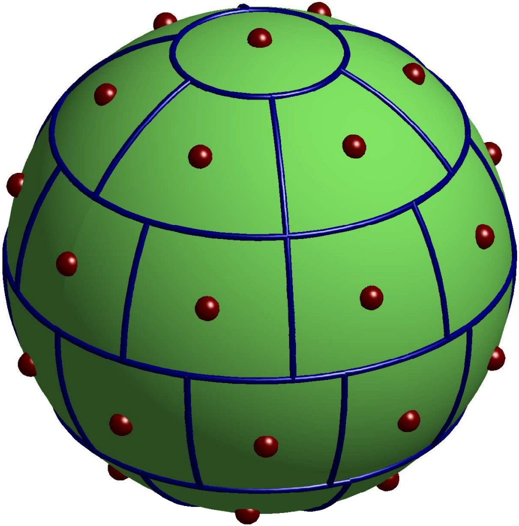
Outline of talk

EQ codes: The Recursive Zonal Equal Area spherical codes,

$$\mathbf{EQP}(\mathbf{d}, \mathcal{N}) \subset \mathbb{S}^{\mathbf{d}}, \text{ with } |\mathbf{EQP}(\mathbf{d}, \mathcal{N})| = \mathcal{N}.$$

- Overview of properties of the EQ codes
- Construction of the EQ codes
  - Some precedents
  - Definitions: coordinates, partitions, diameter bounds
  - The Recursive Zonal Equal Area (EQ) partition
- Details of properties of the EQ codes
  - Separation and discrepancy bounds imply energy bounds
  - Separation and diameter bounds imply energy bounds
  - More details of properties (if time permits)

The spherical code EQP(2,33) on $\mathbb{S}^2 \subset \mathbb{R}^3$



Geometric properties

For $\mathbf{EQP(d, \mathcal{N})}$

Good:

- Centre points of regions of diameter $= \mathbf{O(\mathcal{N}^{-1/d})}$,
- Mesh norm (covering radius) $= \mathbf{O(\mathcal{N}^{-1/d})}$,
- Minimum distance and packing radius $= \mathbf{\Omega(\mathcal{N}^{-1/d})}$.

Bad:

- Mesh ratio $= \mathbf{\Omega(\sqrt{d})}$,
- Packing density $\leqslant \frac{\pi^{d/2}}{2^d \; \Gamma(d/2+1)}$ as $\mathcal{N} \to \infty$.

Approximation properties

Not so bad?

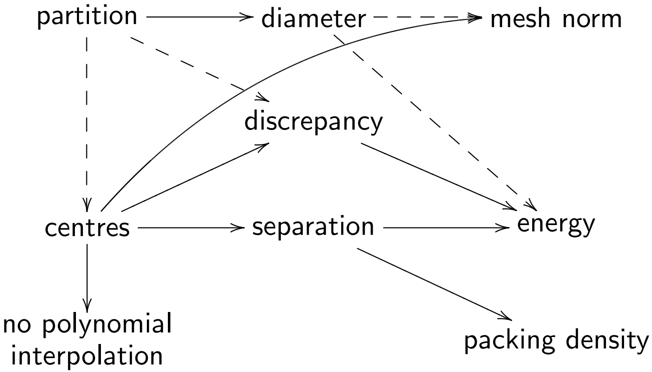
- Normalized spherical cap discrepancy $= \mathrm{O}(\mathcal{N}^{-1/d})$,
- Normalized s-energy

$$E_s = \begin{cases} I_s \pm \mathrm{O}(\mathcal{N}^{-1/d}) & 0 < s < d-1 \\ I_s \pm \mathrm{O}(\mathcal{N}^{-1/d} \log \mathcal{N}) & s = d-1 \\ I_s \pm \mathrm{O}(\mathcal{N}^{s/d-1}) & d-1 < s < d \\ \mathrm{O}(\log \mathcal{N}) & s = d \\ \mathrm{O}(\mathcal{N}^{s/d-1}) & s > d. \end{cases}$$

Ugly:

- Cannot be used for polynomial interpolation:
  proven for large enough $\mathcal{N}$, conjectured for small $\mathcal{N}$.

Relationships between properties



Some precedents

The **EQ** partition is based on Zhou's (1995) construction for $\mathbb{S}^2$ as modified by Saff, and on Sloan's sketch of a partition of $\mathbb{S}^3$ (2003).

Separation without equidistribution: Hamkins (1996) and Hamkins and Zeger (1997) constructed $\mathbb{S}^d$ codes with asymptotically optimal packing density.

Equidistibution without separation: Many constructions for $\mathbb{S}^2$, eg. mapped Hammersley, Halton, (t, s) etc. sequences.
Feige and Schechtman (2002) constructed a diameter bounded equal area partition of $\mathbb{S}^d$. Put one point in each region.

Equal-area partitions of $\mathbb{S}^{\mathbf{d}} \subset \mathbb{R}^{\mathbf{d}}$

An *equal area partition* of $\mathbb{S}^{\mathbf{d}} \subset \mathbb{R}^{\mathbf{d}}$ is a finite set $\mathcal{P}$ of Lebesgue measurable subsets of $\mathbb{S}^{\mathbf{d}}$, such that

$$\bigcup_{\mathbf{R} \in \mathcal{P}} \mathbf{R} = \mathbb{S}^{\mathbf{d}},$$

and for each $\mathbf{R} \in \mathcal{P}$,

$$\sigma(\mathbf{R}) = \frac{\sigma(\mathbb{S}^{\mathbf{d}})}{|\mathcal{P}|},$$

where σ is the Lebesgue area measure on $\mathbb{S}^{\mathbf{d}}$.

Diameter bounded sets of partitions

The *diameter* of a region $\mathbf{R} \subset \mathbb{R}^{\mathbf{d+1}}$ is defined by

$$\mathbf{diam\, R := sup\{\|x - y\| \mid x, y \in R\}.}$$

A set $\boldsymbol{\Xi}$ of partitions of $\mathbb{S}^{\mathbf{d}} \subset \mathbb{R}^{\mathbf{d+1}}$ is *diameter-bounded* with *diameter bound* $\mathbf{K} \in \mathbb{R}_+$ if for all $\mathcal{P} \in \boldsymbol{\Xi}$, for each $\mathbf{R} \in \mathcal{P}$,

$$\mathbf{diam\, R \leqslant K\, |\mathcal{P}|^{-1/d}}.$$

Key properties of the **EQ** partition of $\mathbb{S}^d$

$\mathbf{EQ(d,\mathcal{N})}$ is the *recursive zonal equal area* partition of $\mathbb{S}^d$ into $\mathcal{N}$ regions.

The set of partitions $\mathbf{EQ(d) := \{EQ(d,\mathcal{N}) \mid \mathcal{N} \in \mathbb{N}_+\}}$.

The **EQ** partition satisfies:

Theorem 1

For $\mathbf{d \geqslant 1}$, $\mathcal{N} \geqslant \mathbf{1}$, $\mathbf{EQ(d,\mathcal{N})}$ *is an equal-area partition.*

Theorem 2

For $\mathbf{d \geqslant 1}$, $\mathbf{EQ(d)}$ *is diameter-bounded.*

Spherical polar coordinates on $\mathbb{S}^d$

Spherical polar coordinates describe $\mathbf{x} \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$ by one longitude, $\xi_1 \in \mathbb{R}$ (modulo 2π), and $d-1$ colatitudes, $\xi_j \in [0, \pi]$, for $j \in \{2, \dots, d\}$.

The spherical polar to Cartesian coordinate map $\odot : \mathbb{R} \times [0, \pi]^{d-1} \to \mathbb{S}^d \subset \mathbb{R}^{d+1}$ is

$$\odot(\xi_1, \xi_2, \dots, \xi_d) = (x_1, x_2, \dots, x_{d+1}),$$

$$\text{where} \quad x_1 := \cos \xi_1 \prod_{j=2}^{d} \sin \xi_j, \quad x_2 := \prod_{j=1}^{d} \sin \xi_j,$$

$$x_k := \cos \xi_{k-1} \prod_{j=k}^{d} \sin \xi_j, \quad k \in \{3, \dots, d+1\}.$$

Spherical caps, zones, and collars

The *spherical cap* $\mathbf{S}(\mathbf{p}, \theta) \subset \mathbb{S}^{\mathbf{d}}$ is

$$\mathbf{S}(\mathbf{p}, \theta) := \left\{ \mathbf{q} \in \mathbb{S}^{\mathbf{d}} \mid \mathbf{p} \cdot \mathbf{q} \geqslant \cos(\theta) \right\}.$$

For $\mathbf{d} > 1$, a *zone* can be described by

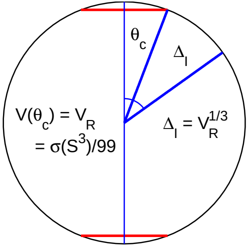
$$\mathbf{Z}(\tau, \beta) := \left\{ \odot(\xi_1, \ldots, \xi_{\mathbf{d}}) \in \mathbb{S}^{\mathbf{d}} \mid \xi_{\mathbf{d}} \in [\tau, \beta] \right\},$$

where $0 \leqslant \tau < \beta \leqslant \pi$.

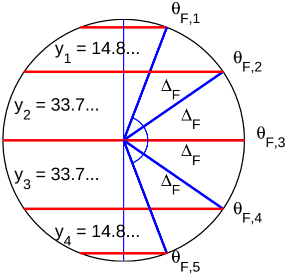
$\mathbf{Z}(0, \beta)$ is a North polar cap and $\mathbf{Z}(\tau, \pi)$ is a South polar cap.

If $0 < \tau < \beta < \pi$, $\mathbf{Z}(\tau, \beta)$ is a *collar*.

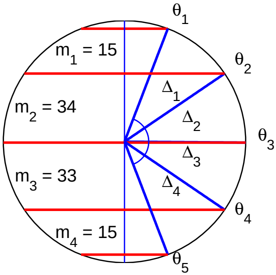
EQ(3,99) Steps 1 to 2

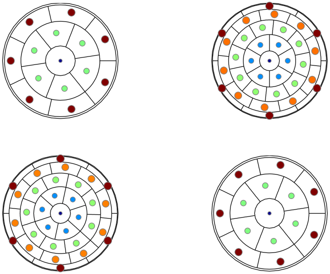


EQ(3,99) Steps 3 to 5



EQ(3,99) Steps 6 to 7





Centre points of regions of $\mathbf{EQ(d,\mathcal{N})}$

The placement of the centre point $\mathbf{a} = \odot(\alpha)$ of a region

$$\mathbf{R} = \odot\left([\tau_1, \beta_1] \times \ldots \times [\tau_d, \beta_d]\right) \text{ is}$$

$$\alpha_1 := \begin{cases} 0 & \beta_1 = \tau_1 \pmod{2\pi} \\ (\tau_1 + \beta_1)/2 \pmod{2\pi} & \text{otherwise,} \end{cases}$$

and for $j > 1$,

$$\alpha_j := \begin{cases} 0 & \tau_j = 0 \\ \pi & \beta_j = \pi \\ (\tau_j + \beta_j)/2 & \text{otherwise.} \end{cases}$$

Minimum distance and packing radius

The *minimum distance* of $\mathbf{X} := \{\mathbf{x}_1, \ldots, \mathbf{x}_\mathcal{N}\} \subset \mathbb{S}^\mathbf{d}$ is

$$\text{min dist } \mathbf{X} := \min_{\mathbf{x} \neq \mathbf{y} \in \mathbf{X}} \|\mathbf{x} - \mathbf{y}\|,$$

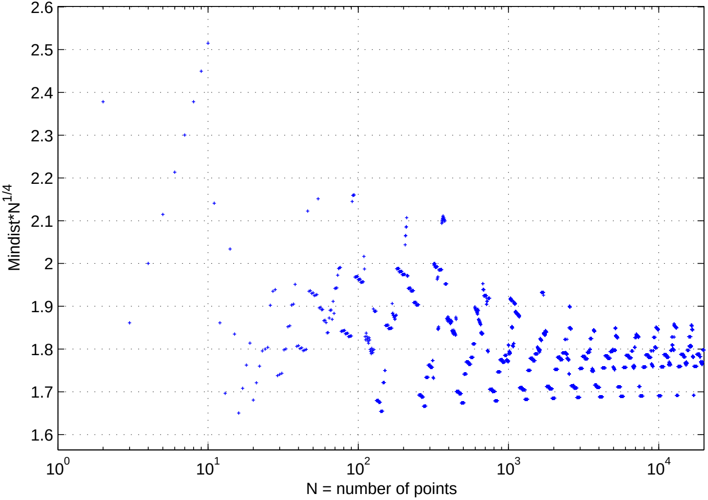
and the *packing radius* of $\mathbf{X}$ is

$$\text{prad } \mathbf{X} := \min_{\mathbf{x} \neq \mathbf{y} \in \mathbf{X}} \cos^{-1}(\mathbf{x} \cdot \mathbf{y})/2.$$

It can be shown that $\text{min dist } \mathbf{EQP}(\mathbf{d}, \mathcal{N}) = \Omega(\mathcal{N}^{-1/\mathbf{d}})$,

and therefore $\text{prad } \mathbf{EQP}(\mathbf{d}, \mathcal{N}) = \Omega(\mathcal{N}^{-1/\mathbf{d}})$.

Minimum distance of EQP(4) codes



Normalized spherical cap discrepancy

We use the probability measure $\mathring{\sigma} := \sigma / \sigma(\mathbb{S}^d)$.

For $X := \{x_1, \ldots, x_{\mathcal{N}}\} \subset \mathbb{S}^d$ the *normalized spherical cap discrepancy* is

$$\operatorname{disc} X := \sup_{y \in \mathbb{S}^d} \sup_{\theta \in [0,\pi]} \left| \frac{|X \cap S(y,\theta)|}{\mathcal{N}} - \mathring{\sigma}\big(S(y,\theta)\big) \right|.$$

It can be shown that

$$\operatorname{disc}\ \mathbf{EQP}(d, \mathcal{N}) = O(\mathcal{N}^{-1/d}).$$

Normalized $\mathbf{s}$-energy

For $\mathbf{X} := \{\mathbf{x}_1, \ldots, \mathbf{x}_\mathcal{N}\} \subset \mathbb{S}^\mathbf{d}$, $\mathbf{s} \in \mathbb{R}$,
the *normalized* $\mathbf{s}$*-energy* is

$$\mathbf{E_s}(\mathbf{X}) := \mathcal{N}^{-2} \sum_{\mathbf{i}=1}^{\mathcal{N}} \sum_{\mathbf{x_i} \neq \mathbf{x_j} \in \mathbf{X}} \|\mathbf{x_i} - \mathbf{x_j}\|^{-\mathbf{s}},$$

and the *normalized energy double integral* for $\mathbf{0} < \mathbf{s} < \mathbf{d}$ is

$$\mathbf{I_s} := \int_{\mathbb{S}^\mathbf{d}} \int_{\mathbb{S}^\mathbf{d}} \|\mathbf{x} - \mathbf{y}\|^{-\mathbf{s}} \, \mathbf{d}\overset{*}{\sigma}(\mathbf{x}) \mathbf{d}\overset{*}{\sigma}(\mathbf{y}).$$

Separation and discrepancy imply energy

Theorem 3

Let $(\mathbf{X}_1, \mathbf{X}_2, \ldots)$ *be a sequence of* $\mathbb{S}^d$ *codes for which there exist* $c_1, c_2 > 0$ *and* $0 < q < 1$ *such that each* $\mathbf{X}_\mathcal{N} = \{x_{\mathcal{N},1}, \ldots, x_{\mathcal{N},\mathcal{N}}\}$ *satisfies*

$$\|x_{\mathcal{N},i} - x_{\mathcal{N},j}\| > c_1\, \mathcal{N}^{-1/d}, \quad (i \neq j)$$
$$\textit{disc}\, \mathbf{X}_\mathcal{N} \leqslant c_2\, \mathcal{N}^{-q}.$$

Then for the normalized s *energy for* $0 < s < d$, *we have for some* $c_3 \geqslant 0$,

$$E_s(\mathbf{X}_\mathcal{N}) \leqslant I_s + c_3\, \mathcal{N}^{(s/d-1)q}.$$

Separation and diameter imply energy

Theorem 4

Let $((\mathbf{X}_1, \mathcal{P}_1), (\mathbf{X}_2, \mathcal{P}_2), \ldots)$ *be a sequence of pairs of* $\mathbb{S}^d$ *codes and equal area partitions such that* $|\mathbf{X}_\mathcal{N}| = |\mathcal{P}_\mathcal{N}| = \mathcal{N}$*, with* $(\mathbf{X}_1, \mathbf{X}_2, \ldots)$ *well separated and* $(\mathcal{P}_1, \mathcal{P}_2, \ldots)$ *diameter bounded, where each* $\mathbf{x}_{\mathcal{N},i} \in \mathbf{X}_\mathcal{N}$ *lies in* $\mathbf{R}_{\mathcal{N},i} \in \mathcal{P}_\mathcal{N}$*. Then*

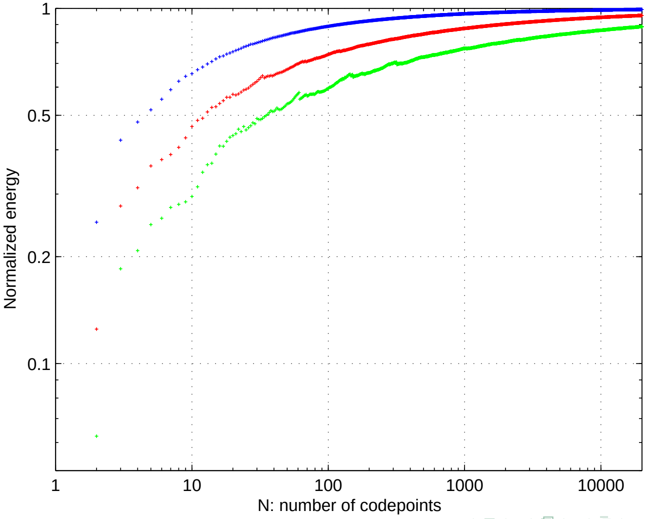
$$\mathbf{E}_s(\mathbf{X}_\mathcal{N}) = \begin{cases} \mathbf{I}_s \pm O(\mathcal{N}^{-1/d}) & 0 < s < d-1 \\ \mathbf{I}_s \pm O(\mathcal{N}^{-1/d} \log \mathcal{N}) & s = d-1 \\ \mathbf{I}_s \pm O(\mathcal{N}^{s/d-1}) & d-1 < s < d \\ O(\log \mathcal{N}) & s = d \\ O(\mathcal{N}^{s/d-1}) & s > d. \end{cases}$$

Comparison to minimum energy

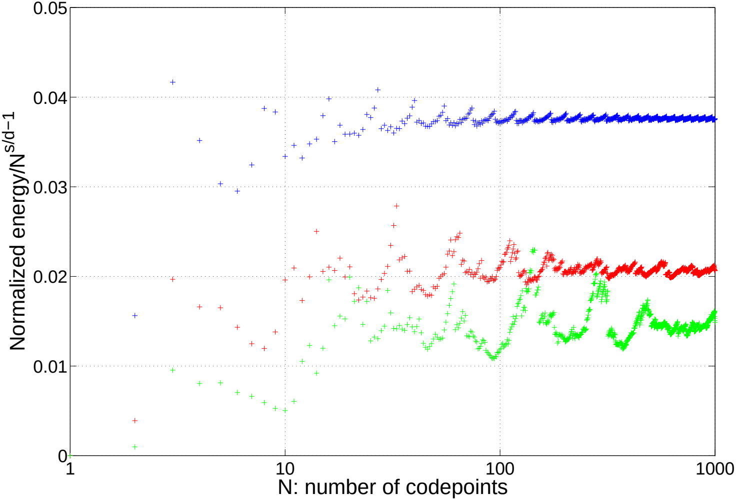
For $\mathbf{s} > \mathbf{d} - 1$, Theorem 4 gives energy bounds of the same order as $\mathcal{E}_{\mathbf{s}}(\mathcal{N})$, the minimum normalized $\mathbf{s}$ energy for $\mathcal{N}$ points on $\mathbb{S}^{\mathbf{d}}$.

$$\mathcal{E}_{\mathbf{s}}(\mathcal{N}) = \begin{cases} \mathbf{I}_{\mathbf{s}} - \Theta(\mathcal{N}^{\mathbf{s}/\mathbf{d}-1}) & 0 < \mathbf{s} < \mathbf{d} \\ & \text{(Wagner;} \\ & \text{Rakhmanov, Saff \& Zhou;} \\ & \text{Brauchart)} \\ \mathbf{O}(\log \mathcal{N}) & \mathbf{s} = \mathbf{d} \quad \text{(Kuijlaars \& Saff)} \\ \mathbf{O}(\mathcal{N}^{\mathbf{s}/\mathbf{d}-1}) & \mathbf{s} > \mathbf{d} \quad \text{(Hardin \& Saff).} \end{cases}$$

$\mathbf{d-1}$ energy of EQP(2), EQP(3), EQP(4)



2d energy of EQP(2), EQP(3), EQP(4)



Mesh norm (covering radius)

The *mesh norm* of $\mathbf{X} := \{\mathbf{x}_1, \ldots, \mathbf{x}_\mathcal{N}\} \subset \mathbb{S}^\mathbf{d}$ is

$$\text{mesh norm } \mathbf{X} := \sup_{\mathbf{y} \in \mathbb{S}^\mathbf{d}} \min_{\mathbf{x} \in \mathbf{X}} \cos^{-1}(\mathbf{x} \cdot \mathbf{y}).$$

Since $\mathbf{EQ(d)}$ is diameter bounded,

$$\text{mesh norm } \mathbf{EQP(d, \mathcal{N}) = O(\mathcal{N}^{-1/d})}.$$

Mesh ratio and packing density

The *mesh ratio* of $\mathbf{X} := \{\mathbf{x_1}, \ldots, \mathbf{x_\mathcal{N}}\} \subset \mathbb{S}^\mathbf{d}$ is

$$\text{mesh ratio } \mathbf{X} := \text{mesh norm } \mathbf{X} \ / \ \text{prad } \mathbf{X}.$$

The *packing density* of $\mathbf{X}$ is

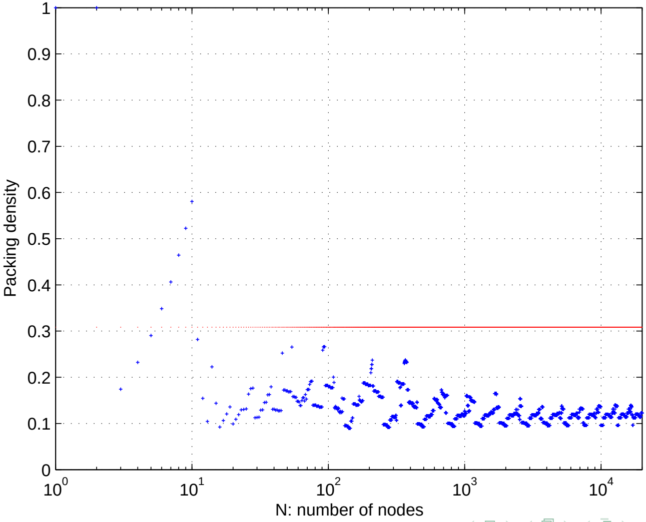
$$\text{pdens } \mathbf{X} := \mathcal{N} \mathring{\sigma}^*(\mathbf{S}(\mathbf{x}, \text{prad } \mathbf{X})).$$

Regions of $\mathbf{EQ(d, \mathcal{N})}$ near equators $\longrightarrow$ cubic as $\mathcal{N} \rightarrow \infty$, so

$$\text{mesh ratio } \mathbf{EQP(d, \mathcal{N}) = \Omega(\sqrt{d})}, \quad \text{and}$$

$$\text{pdens } \mathbf{EQP(d, \mathcal{N})} \leqslant \frac{\pi^{\mathbf{d/2}}}{\mathbf{2^d \ \Gamma(d/2+1)}} \quad \text{as } \mathcal{N} \rightarrow \infty.$$

Packing density of EQP(4) codes



For EQSP Matlab code

See SourceForge web page for EQSP:

Recursive Zonal Equal Area Sphere Partitioning Toolbox:

`http://eqsp.sourceforge.net`