Approximate Fekete points and discrete Leja points based on equal area partitions of the unit sphere

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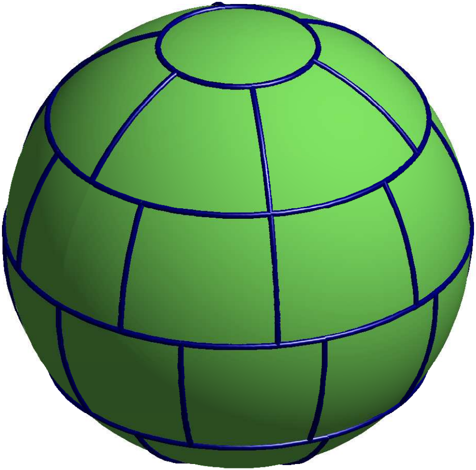
Outline of talk

- The EQ spherical codes
- Approximately optimal interpolating sets
- Results for the EQ spherical codes

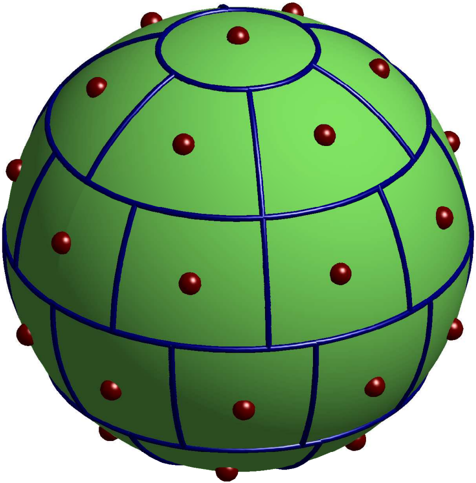
The partition EQ(2,33) on $\mathbb{S}^2 \subset \mathbb{R}^3$

EQ partitions: Recursive Zonal Equal Area partitions of the sphere,
$\bigcup \mathbf{EQ(d,\mathcal{N})} = \mathbb{S}^{\mathbf{d}}$, with $|\mathbf{EQ(d,\mathcal{N})}| = \mathcal{N}$.



The spherical code EQP(2,33) on $\mathbb{S}^2$

EQ codes: The Recursive Zonal Equal Area spherical codes, $\mathbf{EQP(d,\mathcal{N})} \subset \mathbb{S}^{\mathbf{d}}$, with $|\mathbf{EQP(d,\mathcal{N})}| = \mathcal{N}$.



Equal-area partitions of $\mathbb{S}^d \subset \mathbb{R}^d$

An *equal area partition* of $\mathbb{S}^d \subset \mathbb{R}^d$ is a finite set $\mathcal{P}$ of Lebesgue measurable subsets of $\mathbb{S}^d$, such that

$$\bigcup_{R \in \mathcal{P}} R = \mathbb{S}^d,$$

and for each $R \in \mathcal{P}$,

$$\lambda_d(R) = \frac{\lambda_d(\mathbb{S}^d)}{|\mathcal{P}|},$$

where λ_d is the Lebesgue area measure on $\mathbb{S}^d$.

Diameter bounded sets of partitions

The *diameter* of a region $\mathbf{R} \subset \mathbb{R}^{\mathbf{d+1}}$ is defined by

$$\mathbf{diam\,R := \sup\{\|x - y\| \mid x, y \in R\}}.$$

A set $\boldsymbol{\Xi}$ of partitions of $\mathbb{S}^{\mathbf{d}} \subset \mathbb{R}^{\mathbf{d+1}}$ is *diameter-bounded* with *diameter bound* $\mathbf{K} \in \mathbb{R}_+$ if for all $\mathcal{P} \in \boldsymbol{\Xi}$, for each $\mathbf{R} \in \mathcal{P}$,

$$\mathbf{diam\,R} \leqslant \mathbf{K}\,|\mathcal{P}|^{-\mathbf{1/d}}.$$

Key properties of the **EQ** partition of $\mathbb{S}^{\mathbf{d}}$

$\mathbf{EQ(d,\mathcal{N})}$ is the *recursive zonal equal area* partition of $\mathbb{S}^{\mathbf{d}}$ into $\mathcal{N}$ regions.

The set of partitions $\mathbf{EQ(d)} := \{\mathbf{EQ(d,\mathcal{N})} \mid \mathcal{N} \in \mathbb{N}_+\}$.

The **EQ** partition satisfies:

Theorem 1

For $\mathbf{d} \geqslant \mathbf{1}$, $\mathcal{N} \geqslant \mathbf{1}$, $\mathbf{EQ(d,\mathcal{N})}$ *is an equal-area partition.*

Theorem 2

For $\mathbf{d} \geqslant \mathbf{1}$, $\mathbf{EQ(d)}$ *is diameter-bounded.*

Geometric properties

For $\mathbf{EQP}(\mathbf{d}, \mathcal{N})$

Good:

- Centre points of regions of diameter $= \mathbf{O}(\mathcal{N}^{-1/\mathbf{d}})$,
- Mesh norm (covering radius) $= \mathbf{O}(\mathcal{N}^{-1/\mathbf{d}})$,
- Minimum distance and packing radius $= \boldsymbol{\Omega}(\mathcal{N}^{-1/\mathbf{d}})$.

Bad:

- Mesh ratio $= \boldsymbol{\Omega}(\sqrt{\mathbf{d}})$,
- Packing density $\leqslant \frac{\pi^{\mathbf{d}/2}}{2^{\mathbf{d}}\ \Gamma(\mathbf{d}/2+1)}$ as $\mathcal{N} \to \infty$.

Approximation properties

Not so bad?

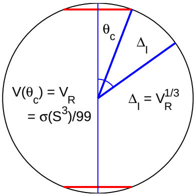
- Normalized spherical cap discrepancy $= O(\mathcal{N}^{-1/d})$,
- Normalized s-energy

$$E_s = \begin{cases} I_s \pm O(\mathcal{N}^{-1/d}) & 0 < s < d-1 \\ I_s \pm O(\mathcal{N}^{-1/d} \log \mathcal{N}) & s = d-1 \\ I_s \pm O(\mathcal{N}^{s/d-1}) & d-1 < s < d \\ O(\log \mathcal{N}) & s = d \\ O(\mathcal{N}^{s/d-1}) & s > d. \end{cases}$$

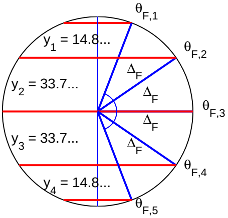
Ugly:

- *Cannot be used for polynomial interpolation:* proven for large enough $\mathcal{N}$, conjectured for small $\mathcal{N}$.

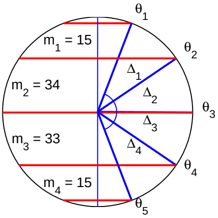
EQ(3,99) Steps 1 to 2

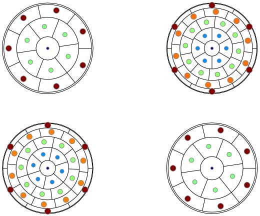


EQ(3,99) Steps 3 to 5



EQ(3,99) Steps 6 to 7





Admissible meshes

Definition 3

For compact $\mathbf{D} \subset \mathbb{R}^{\mathbf{d}}$, and $\mathbf{C(D)}$ the space of continuous functions on $\mathbf{D}$, given a sequence of finite dimensional subspaces $\mathbf{P_t(D)} \subset \mathbf{C(D)}$, a $\mathbf{P_t}$ *-norming mesh* is a sequence $(\mathbf{Z_t})$ of finite subsets of $\mathbf{D}$ such that

$$\|\mathbf{p}\|_\infty \leq \mathbf{c} \sup_{\mathbf{z} \in \mathbf{Z_t}} |\mathbf{p(z)}| \quad \text{for all } \mathbf{p} \in \mathbf{P_t}.$$

For a $\mathbb{P}_\mathbf{t}$ *-admissible mesh*,
$\mathbb{P}_\mathbf{t}(\mathbf{D})$ is the space of polynomials of maximum degree $\mathbf{t}$ on $\mathbf{D}$,
and
the cardinality $|\mathbf{Z_t}| = \mathbf{O(t^s)}$ for some $\mathbf{s} \geqslant \mathbf{1}.$

(Calvi and Levenberg 2008, Vianello 2013)

Approximate Fekete points

Given a $\mathbb{P}_t$-admissible mesh with

$$\mathbf{n_t := |Z_t| \geq d_t := dim(P_t(D))},$$

points $\mathbf{z_1, \ldots, z_{n_t}} \in \mathbf{Z_t}$, and a basis $\{\mathbf{p_1, \ldots, p_{d_t}}\}$ of $\mathbf{P_t(D)}$, the *approximate Fekete points* of order $\mathbf{t}$ are a subset of $\mathbf{Z_t}$ with cardinality $\mathbf{d_t}$, obtained from the Vandermonde matrix $\mathbf{A_t := [p_i(z_j)]}$ via QR decomposition with column pivoting.

They approximate the maximal determinant *Fekete points* by having a large Vandermonde determinant and a small Lebesgue constant of interpolation.

(Sommariva and Vianello 2009)

Approximate Fekete points

The approximate Fekete points `z_af` are obtained from the points `z` and corresponding Vandermonde matrix `A` as

```
dim = rows(A);
y = zeros(dim, 1);
y(1) = 1;

[Q, R, P] = qr(A);
w = P(:, 1:dim) * (R(:, 1:dim) \ (Q' * y));

z_af = z(:, abs(w) > tol);
```

where `w` is the optimal quadrature weight vector and `tol` is a small tolerance. Here all elements of `w` are non-zero, but `z_af` may have dimension less than `dim`.

(Sommariva and Vianello 2009)

Discrete Leja points

For the *Discrete Leja* points, LU decomposition with partial row pivoting is used instead.

```
dim = rows(A);
y = zeros(dim, 1);
y(1) = 1;

[L, U, p] = lu(A', 'vector');
w = zeros(n, 1);
w(p(1:dim)) = L(1:dim, :)' \ (U' \ y);

z_dl = z(:, p(1:dim));
```

Here `z_dl` has dimension `dim`, but some elements of `w` may be zero.

(Bos, De Marchi, Sommariva and Vianello 2010)

Non-negative least squares points

We can also use *non-negative least squares* instead of either QR or LU decomposition.

```
dim = rows(A);
y = zeros(dim, 1);
y(1) = 1;

w = lsqnonneg(A, y);

z_nn = z(:, abs(w) > tol);
```

Here all elements of `w` are positive, but `z_nn` may have dimension less than `dim`.

(Sommariva and Vianello 2014)

The EQ codes form an admissible mesh

Theorem 4

The EQ codes form a $\mathbb{P}_\mathbf{t}$*-admissible mesh .*

Proof.

Any finite point set on the unit sphere $\mathbb{S}^\mathbf{d}$ with mesh norm at most $\mathbf{(1-c)/t}$ generates a norming set with constant $\mathbf{c}$ for $\mathbb{P}_\mathbf{t}$.
The EQ spherical codes have mesh norm at most $\mathbf{C_d}\mathcal{N}^{-1/\mathbf{d}}$.
Thus if $\mathcal{N} \geqslant \mathbf{(C_d/(1-c))^d\ t^d}$, then $\mathbf{EQP(d},\mathcal{N}\mathbf{)}$ is a norming set with constant $\mathbf{c}$ for $\mathbb{P}_\mathbf{t}$. □

(Jetter, Stöckler and Ward, 1998; L and Vianello 2014)

The Fekete points on the sphere $\mathbb{S}^2$

The *Fekete* (maximal determinant) points on the sphere $\mathbb{S}^2$ are the points that maximize the determinant of the Vandermonde-type matrix $\mathbf{A_t} := [\mathbf{p_{t,i}}(\mathbf{x_j})]$, where $\mathbf{i}, \mathbf{j} \in \{1, \ldots, (\mathbf{t}+1)^2\}$, the $\mathbf{p_{t,i}}$ form an orthonormal basis of the spherical polynomials of degree at most $\mathbf{t}$, and $\mathbf{x_j} \in \mathbb{S}^2$.

Rob Womersley has (approximately) computed these points up to degree $\mathbf{t} = \mathbf{165}$, and their corresponding optimal quadrature weights, as well as the log of the determinant of the *Gram matrix* $\mathbf{G_t} := \mathbf{A_t^T A_t}$.

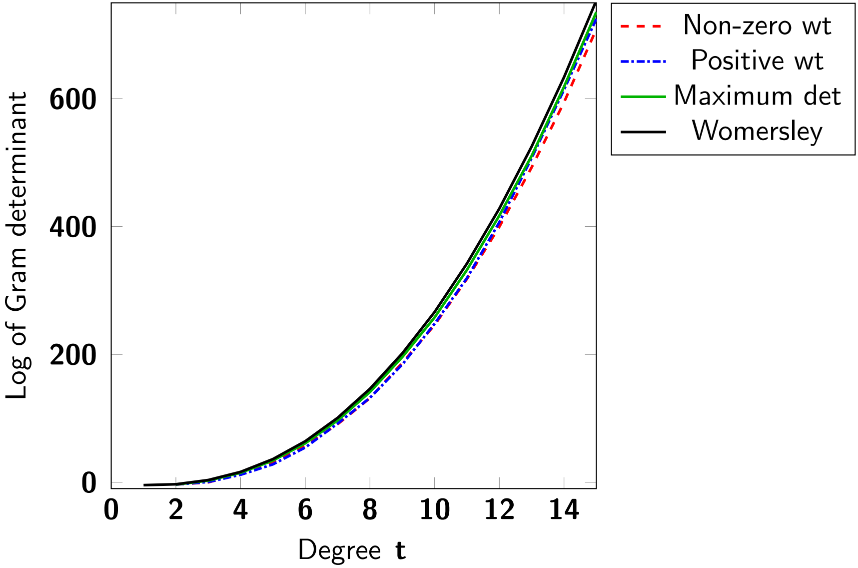
(Sloan and Womersley 2004; Womersley 2007)

The search algorithm

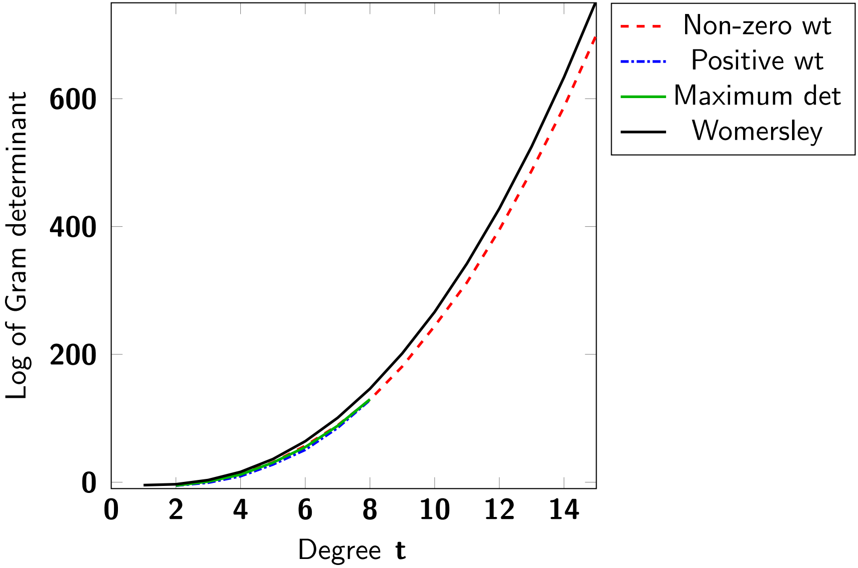
For each type of point set (approximate Fekete points, discrete Leja points, non-negative least squares points), for $\mathbf{t}$ from 1 to 15, I used Octave with the EQ codes $\mathbf{EQP(2,\mathcal{N})}$ for $\mathcal{N}$ from $\mathbf{(t+1)^2}$ to $\mathbf{(t+1)^3}$ to find:

- The smallest $\mathcal{N}$ such that the matrix $\mathtt{A}$ has full rank, and the point set has all corresponding *weights non-zero*.
- The smallest $\mathcal{N}$ such that the matrix $\mathtt{A}$ has full rank, and the point set has all corresponding *weights positive*.
- The value of $\mathcal{N}$ such that the matrix $\mathtt{A}$ has full rank, the point set has all corresponding weights positive, and the Gram *determinant is maximal*.

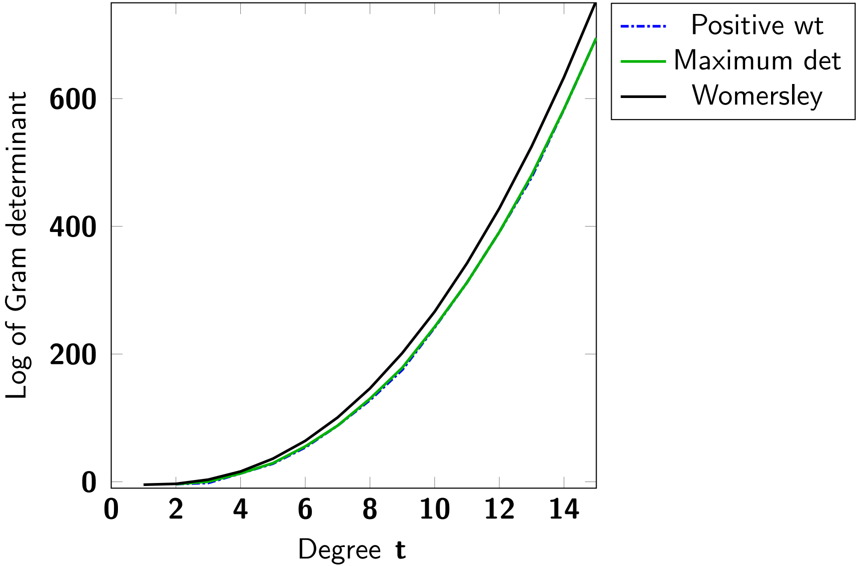
Approximate Fekete points



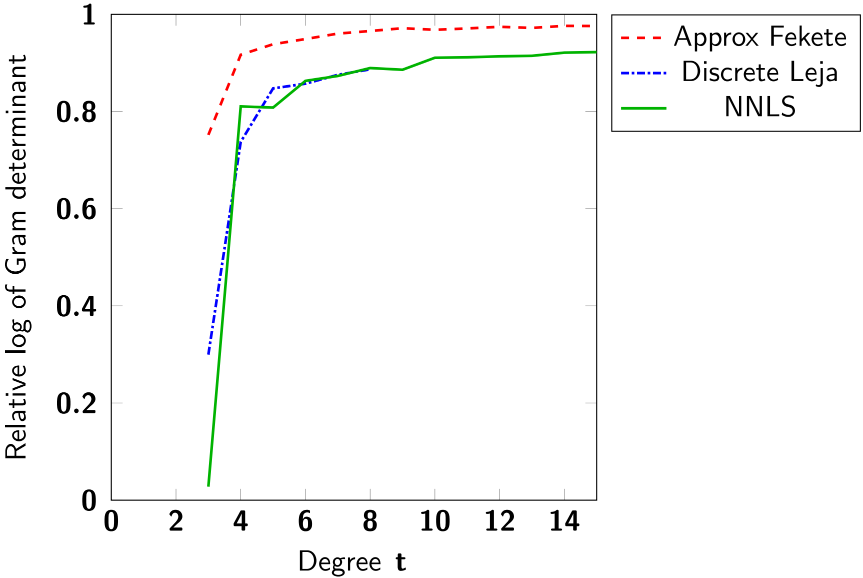
Discrete Leja points



Non-negative least squares points



Maximum determinant positive weight points



Left to do

- Numerical examples for larger degree $\mathbf{t}$.

  Searching can be done in parallel.

- Prove that for some $\mathbf{T} > \mathbf{0}$, for all $\mathbf{t} > \mathbf{T}$, for sufficiently large $\mathcal{N}$, the EQ codes $\mathbf{EQP(2, \mathcal{N})}$ yield positive weights for each of the 3 types of points. Estimate the $\mathcal{N}$ required.

- Investigate properties (mesh norm, discrepancy, energy) of the resulting point sets.