Constructions for Hadamard matrices, Clifford algebras, and their relation to amicability - anti-amicability graphs

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Topics

- Kronecker product constructions for Hadamard matrices
- Signed groups, 2-cocycles and Clifford algebras
- Graphs of amicability and anti-amicability

Kronecker product constructions (1)

We aim to find

$$A_k \in \{-1, 0, 1\}^{n \times n}, \quad B_k \in \{-1, 1\}^{p \times p}, \quad k \in \{1, \ldots, n\},$$

such that

$$G = \sum_{k=1}^{n} B_k \otimes A_k, \quad GG^T = npI_{(np)}, \tag{G1}$$

$$H = \sum_{k=1}^{n} A_k \otimes B_k, \quad HH^T = npI_{(np)}. \tag{H1}$$

(Gastineau-Hills 1980, 1982)

Kronecker product constructions (2)

Since

$$HH^T = \sum_{j=1}^{n} A_j \otimes B_j \sum_{k=1}^{n} A_k^T \otimes B_k^T,$$

we impose the stronger conditions

$$\sum_{j=1}^{n} A_j A_j^T \otimes B_j B_j^T = np I_{(np)},$$

$$\sum_{j=1}^{n} \sum_{k=j+1}^{n} \left(A_j A_k^T \otimes B_j B_k^T + A_k A_j^T \otimes B_k B_j^T \right) = 0. \qquad \text{(H2)}$$

Similarly, (G2) with Kronecker product reversed.

(Gastineau-Hills 1980, 1982)

Kronecker product constructions (3)

Stronger conditions:

$$\sum_{k=1}^{n} A_k A_k^T \otimes B_k B_k^T = npI_{(np)},$$
$$A_j A_k^T \otimes B_j B_k^T + A_k A_j^T \otimes B_k B_j^T = 0 \quad (j \neq k). \qquad \text{(H3)}$$

Similarly, (G3) with Kronecker product reversed.

(Gastineau-Hills 1980, 1982)

Kronecker product constructions (4)

Still stronger conditions (• is Hadamard product):

$$\begin{gathered}
A_j \bullet A_k = 0 \quad (j \neq k), \quad \sum_{k=1}^{n} A_k \in \{-1, 1\}^{n \times n}, \\
A_k A_k^T = I_{(n)}, \\
\sum_{k=1}^{n} B_k B_k^T = npI_{(p)}, \\
A_j A_k^T + \lambda_{jk} A_k A_j^T = 0 \quad (j \neq k), \\
B_j B_k^T - \lambda_{jk} B_k B_j^T = 0 \quad (j \neq k), \\
\lambda_{jk} \in \{-1, 1\}.
\end{gathered} \tag{4}$$

(Gastineau-Hills 1980, 1982)

Example: Sylvester-type construction

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & - \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \lambda_{12} = 1$$

$$\Rightarrow \text{ We need } \quad B_1 B_1^T + B_2 B_2^T = 2pI_{(p)}, \quad B_1 B_2^T - B_2 B_1^T = 0,$$

e.g.

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & - \\ 1 & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 1 & 1 & - \\ 1 & - & - & - \\ 1 & 1 & - & 1 \\ 1 & - & 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 & 1 & - \\ 1 & - & 1 & 1 \\ 1 & - & - & - \\ 1 & 1 & - & 1 \end{bmatrix}.$$

Example: Anti-amicable construction

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \lambda_{12} = -1$$

$$\Rightarrow \text{ We need } \quad B_1 B_1^T + B_2 B_2^T = 2pI_{(p)}, \quad B_1 B_2^T + B_2 B_1^T = 0,$$

e.g.

$$B_1 = \begin{bmatrix} - & 1 \\ 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} - & - \\ - & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} - & - & 1 & - \\ - & - & - & 1 \\ 1 & - & 1 & 1 \\ - & 1 & 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} - & 1 & - & - \\ 1 & 1 & - & 1 \\ - & - & - & 1 \\ - & 1 & 1 & 1 \end{bmatrix}.$$

(Gastineau-Hills 1980, 1982)

More examples

Williamson-like construction (uses 4 amicable B matrices):

$$A_1 = I_{(4)}, \quad A_k^T = -A_k \quad (k > 1), \quad \lambda_{jk} = 1 \quad (j \neq k).$$

Octonion-like construction (uses 8 amicable B matrices):

$$A_1 = I_{(8)}, \quad A_k^T = -A_k \quad (k > 1), \quad \lambda_{jk} = 1 \quad (j \neq k).$$

(Gastineau-Hills 1980, 1982)

Hurwitz-Radon limit

A theorem of Hurwitz and Radon puts an upper limit of 8 on the order n such that

$$A_j \bullet A_k = 0 \quad (j \neq k), \quad \sum_{k=1}^{n} A_k \in \{-1, 1\}^{n \times n},$$

$$A_k A_k^T = I_{(n)},$$

$$A_j A_k^T + A_k A_j^T = 0 \quad (j \neq k).$$

(Geramita and Pullman 1974)

Recap: ingredients

We need n-tuples $(A_1, \ldots, A_n)$, $(B_1, \ldots, B_n)$, with

$$A_k \in \{-1, 0, 1\}^{n \times n},$$

$$B_k \in \{-1, 1\}^{p \times p},$$

satisfying the conditions (4).

For the A matrices, we look at signed groups, 2-cocycles and Clifford algebras.

For the B matrices, we look at graphs of amicability and anti-amicability.

Signed groups and 2-cocycles

Signed group is an extension E of $\mathbb{Z}_2 \equiv \{-1, 1\}$ by G,

$$\begin{aligned}
&\psi : G \times G \to \mathbb{Z}_2,\ E = (s, \mathbf{g}),\ s \in \mathbb{Z}_2,\ \mathbf{g} \in G, \\
&(s, \mathbf{g})(t, \mathbf{h}) = \big(st\ \psi(\mathbf{g}, \mathbf{h}), \mathbf{gh}\big), \\
&(r, \mathbf{f})\big((s, \mathbf{g})(t, \mathbf{h})\big) = (rst\ \psi(\mathbf{f}, \mathbf{gh})\psi(\mathbf{g}, \mathbf{h}), \mathbf{fgh}) \\
=&\big((r, \mathbf{f})(s, \mathbf{g})\big)(t, \mathbf{h}) = (rst\ \psi(\mathbf{f}, \mathbf{g})\psi(\mathbf{fg}, \mathbf{h}), \mathbf{fgh}).
\end{aligned}$$

So ψ is a 2-cocycle.

(Craigen 1995; Horadam and de Launey 1993)

Clifford algebras via signed groups (1)

$\mathbb{G}_{p,q}$ is extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2^{p+q}$, defined by the signed group presentation

$$\begin{aligned}\mathbb{G}_{p,q} := \Big\langle -1,\ \mathbf{e}_{\{k\}}\ (k \in S_{p,q})\ \ \big|\ & \\ & \mathbf{e}_{\{k\}}^2 = -1\ (k < 0), \quad \mathbf{e}_{\{k\}}^2 = 1\ (k > 0), \\ & \mathbf{e}_{\{j\}}\mathbf{e}_{\{k\}} = -\mathbf{e}_{\{k\}}\mathbf{e}_{\{j\}}\ (j \neq k)\Big\rangle,\end{aligned}$$

where $S_{p,q} := \{-q, \ldots, -1, 1, \ldots, p\}$. $\qquad |\mathbb{G}_{p,q}| = 2^{1+p+q}$.

(Porteous 1969, 1995; Lam 1973; Gastineau-Hills 1980, 1982; Lounesto 1997, L 2005)

Clifford algebras via signed groups (2)

Multiplication in $\mathbb{Z}_2^{p+q}$ is isomorphic to XOR of bit vectors, or symmetric set difference of subsets of $S_{p,q}$,
so elements of $\mathbb{G}_{p,q}$ can be written as $\pm\mathbf{e}_T,\ T \subset S_{p,q}$.

$\mathbb{G}_{p,q}$ extends to the real Clifford algebra $\mathbb{R}_{p,q}$, of dimension 2^{p+q}.
For $\mathbf{x} \in \mathbb{R}_{p,q}$,

$$\mathbf{x} = \sum_{T \subset S_{p,q}} x_T \mathbf{e}_T.$$

2^{p+q} basis elements $\mathbf{e}_T$; $-1\mathbf{e}_\emptyset$ in $\mathbb{G}_{p,q}$ is identified with -1 in $\mathbb{R}$.

(Porteous 1969, 1995; Lam 1973; Gastineau-Hills 1980, 1982; Lounesto 1997, L 2005)

Remreps for $\mathbb{G}_{m,m}$ and $\mathbb{R}_{m,m}$ (1)

Real monomial representations for $\mathbb{G}_{m,m}$ and $\mathbb{R}_{m,m}$ are generated by Kronecker products of the 2×2 matrices

$$I_{(2)}, \quad J := \begin{bmatrix} 0 & - \\ 1 & 0 \end{bmatrix}, \quad K := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

These representations are *faithful*: $\mathbb{R}_{m,m}$ is isomorphic to $\mathbb{R}^{2^m \times 2^m}$. Thus $\mathbb{R}^{2^m \times 2^m}$ has a basis consisting of 4^m real monomial matrices.

(Porteous 1969, 1995; Lam 1973; Gastineau-Hills 1980, 1982; Lounesto 1997, L 2005)

Remreps for $\mathbb{G}_{m,m}$ and $\mathbb{R}_{m,m}$ (2)

Pairs of basis elements of $\mathbb{R}_{m,m}$ either commute or anticommute. Remreps of basis elements of $\mathbb{R}_{m,m}$ are either symmetric or skew, and so remreps A_j, A_k satisfy

$$A_k A_k^T = I_{(2^m)}, \quad A_j A_k^T + \lambda_{jk} A_k A_j^T = 0 \quad (j \neq k), \quad \lambda_{jk} \in \{-1, 1\}.$$

We can choose $n := 2^m$ of these such that

$$A_j \bullet A_k = 0 \quad (j \neq k), \quad \sum_{k=1}^{n} A_k \in \{-1, 1\}^{n \times n}.$$

(Porteous 1969, 1995; Lam 1973; Gastineau-Hills 1980, 1982; Lounesto 1997, L 2005)

Anti-amicable pairs of $\{-1,1\}$ matrices

Given the A_k, this fixes λ_{jk}.

We now must find an n-tuple of $\{-1,1\}$ matrices with a complementary graph of amicability and anti-amicability.

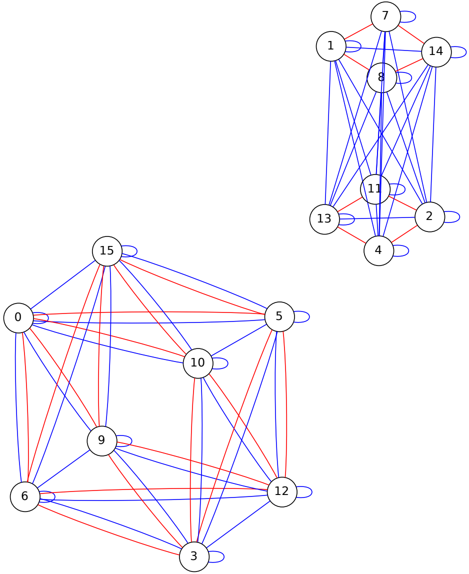
For anti-amicable pairs of matrices in $\{-1,1\}^{p \times p}$,

$$B_1 B_2^T + B_2 B_1^T = 0,$$

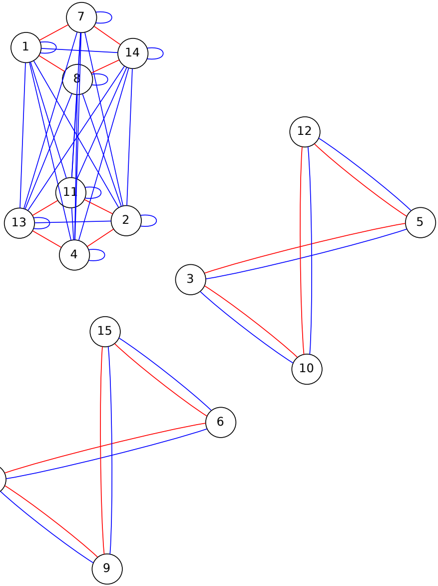
therefore $B_1 B_2^T$ is skew, so p must be even.

(Gastineau-Hills 1980, 1982)

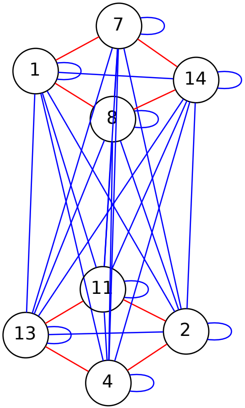
$\{-1, 1\}^{2\times 2}$, Amicable, Anti-amicable



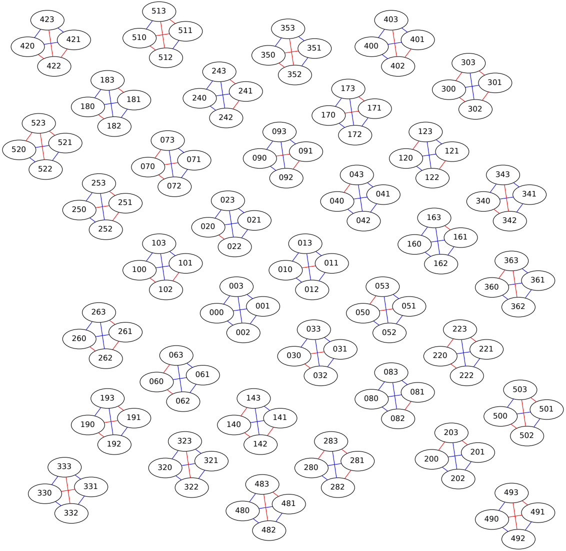
$\{-1,1\}^{2\times 2}$, $B_1B_1^T + B_2B_2^T = 4I_{(2)}$



$\{-1,1\}^{2\times 2}$, $B_1B_1^T = B_2B_2^T = 2I_{(2)}$



$$\{-1,1\}^{2\times 2},\ B_1B_1^T + B_2B_2^T + B_3B_3^T + B_4B_4^T = 8I_{(2)}$$



$$\{-1,1\}^{2\times 2},\ B_1B_1^T + B_2B_2^T + B_3B_3^T + B_4B_4^T = 8I_{(2)}$$