

# An abstract Hodge–Dirac operator and its stable discretization

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# Subjects with parallel 40+ year histories

## Finite Element Method

M. Zlámal, *On the finite element method*. Numer. Math., 12, 1968, pp. 394-409.

## Clifford analysis

D. Hestenes, *Multivector calculus*. J. Math. Anal. and Appl., 24:2, 1968, pp. 313-325.

# Topics

- ▶ Finite Element Exterior Calculus
- ▶ Discretization of the Hodge–Dirac operator
- ▶ Further considerations

# Finite Element Method

The Finite Element Method solves boundary value problems based on partial differential equations.

The original problem in a Hilbert space of functions is put into variational form, and is mapped into a problem defined on a finite dimensional function space, whose basis consists of functions supported in small regions, such as simplices.

(Iserles 1996; Braess 2001).

# Finite element exterior calculus (FEEC)

FEEC is based on the Finite Element Method over Hilbert complexes. These are cochain complexes where the relevant vector spaces are Hilbert spaces.

For the de Rham complex, FEEC uses Hodge decomposition, the exterior derivative and differential forms.

The numerical stability of the FEEC discretization depends on the existence of a bounded cochain projection from a Hilbert complex to a subcomplex. FEEC uses smoothed projections to obtain this numerical stability.

(Arnold, Falk and Winther 2006, 2010; Christiansen and Winther 2008)

# Hodge decomposition of differential forms

If  $(M, g)$  is an oriented, compact Riemannian manifold, then each space of smooth  $k$ -forms has an  $L^2$ -inner product,

$$\langle u, v \rangle_{L^2 \Omega^k(M)} = \int_M \langle u, v \rangle_g \operatorname{vol}_g = \int_M u \wedge \star_g v.$$

This gives an adjoint operator  $d_k^* : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$  for each  $k$ .

$$\begin{array}{ccccccc} 0 & \xrightleftharpoons{\quad} & \Omega^0(U) & \xrightleftharpoons[\text{d}^*]{\text{d}} & \Omega^1(U) & \xrightleftharpoons[\text{d}^*]{\text{d}} & \Omega^2(U) & \xrightleftharpoons[\text{d}^*]{\text{d}} & \Omega^3(U) & \xrightleftharpoons{\quad} & 0 \\ 0 & \xrightleftharpoons{\quad} & C^\infty(U) & \xrightleftharpoons[\text{div}]{\text{grad}} & \mathfrak{X}(U) & \xrightleftharpoons[\text{curl}]{\text{curl}} & \mathfrak{X}(U) & \xrightleftharpoons[\text{grad}]{\text{div}} & C^\infty(M) & \xrightleftharpoons{\quad} & 0 \end{array}$$

The Hodge decomposition says that each  $f \in L^2 \Omega^k(M)$  can be orthogonally decomposed as  $f = d\alpha + d^*\beta + \gamma$ , where  $d\gamma = 0$ ,  $d^*\gamma = 0$ .

# Hilbert complexes

## Definition

A Hilbert complex  $(W, d)$  consists of:

- ▶ a sequence of Hilbert spaces  $W^k$ , and
- ▶ closed, densely-defined linear maps  $d^k : V^k \subset W^k \rightarrow V^{k+1} \subset W^{k+1}$ , (possibly unbounded) such that  $d^k \circ d^{k-1} = 0$  for each  $k$ .

The complex is *closed* if each  $d^k$  has closed image  $d^k V^k \subset W^{k+1}$ , and *bounded* if  $d^k \in \mathcal{L}(W^k, W^{k+1})$ .

## Definition

Given a Hilbert complex  $(W, d)$ , the *domain complex*  $(V, d)$  consists of the domains  $V^k \subset W^k$ , endowed with the graph inner product

$$\langle u, v \rangle_{V^k} = \langle u, v \rangle_{W^k} + \langle d^k u, d^k v \rangle_{W^{k+1}}$$



# Hodge theory for closed Hilbert complexes

## Definition

Given a Hilbert complex  $(W, d)$ ,

- ▶  $\mathfrak{Z}^k = \ker d^k$  (closed  $k$ -forms),  $\mathfrak{Z}_k^* = \ker d_k^*$  (coclosed  $k$ -forms),
- ▶  $\mathfrak{B}^k = d^{k-1}V^{k-1}$  (exact  $k$ -forms),  $\mathfrak{B}_k^* = d_{k+1}^*V_{k+1}^*$  (coexact  $k$ -forms),
- ▶  $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp} = \mathfrak{Z}^k \cap \mathfrak{Z}_k^*$  (harmonic  $k$ -forms)  $\cong \mathfrak{Z}^k / \mathfrak{B}^k$  ( $k$ th cohomology)

## Theorem (abstract Hodge decomposition)

If  $(W, d)$  is a closed Hilbert complex with domain complex  $(V, d)$ , then

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}_k^*,$$

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp}.$$

# The abstract Poincaré inequality

## Theorem

If  $(V, d)$  is a bounded, closed Hilbert complex, then there exists a Poincaré constant  $c_P$  such that

$$\|v\|_V \leq c_P \left\| d^k v \right\|_V, \forall v \in \mathfrak{Z}^{k\perp}.$$

## Proof

The map  $d_k$  is a  $V$ -bounded bijection from  $\mathfrak{Z}^{k\perp}$  to  $\mathfrak{B}^{k+1}$ , which are both closed subspaces. The result follows by Banach's bounded inverse theorem.

## Example

Let  $U \subset \mathbb{R}^n$  be bounded and connected with Lipschitz boundary, and take  $V^0 = H^1(U)$ . Since  $\mathfrak{Z}^0 \subset H^1(U)$  consists of the constant functions, this recovers the classical Poincaré inequality,

$$\|v\|_{H^1(U)} \leq c_P \left\| d^k v \right\|_{L^2(U)}, \forall v \in H^1(U) : \int_U v(x) dx = 0.$$

# An abstract Hodge–Dirac problem: setting

Let  $(W, d)$  be a closed Hilbert complex with domain complex  $(V, d)$ . In  $V$ , we use the inner product

$$\langle u, v \rangle_V := \langle u, v \rangle + \langle du, dv \rangle.$$

We use the orthogonal Hodge decomposition

$$\begin{aligned} W &= \mathfrak{B} \oplus \mathfrak{H} \oplus \mathfrak{B}^*, \\ u &= u_{\mathfrak{B}} \oplus u_{\mathfrak{H}} \oplus u_{\mathfrak{B}^*}, \quad \forall u \in W. \end{aligned}$$

where  $V \in W$  is the domain of  $d$ ,  $\mathfrak{B}$  is the range of  $d$ , and  $\mathfrak{H}$  is the null space of  $d$ .

# An abstract Hodge–Dirac problem

We want to solve the problem  $(d + d^*)u = f - f_{\mathfrak{H}}$ , where  $d + d^*$  is the *abstract Hodge–Dirac operator*.

Consider the following mixed variational problem:

Find  $(u, p) \in V \times \mathfrak{H}$  satisfying

$$\begin{aligned} \langle du, v \rangle + \langle u, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, & \forall v \in V, \\ \langle u, q \rangle &= 0, & \forall q \in \mathfrak{H}. \end{aligned} \quad (1)$$

To show that this problem is well-posed, it suffices to prove the inf-sup condition for the symmetric bilinear form

$$B(u, p; v, q) := \langle du, v \rangle + \langle u, dv \rangle + \langle p, v \rangle + \langle u, q \rangle$$

on  $V \times \mathfrak{H}$ .

# The problem is well-posed

## Theorem

There exists a constant  $\gamma > 0$ , depending only on the Poincaré constant  $c_P$ , such that for all non-zero  $(u, p) \in V \times \mathfrak{H}$ , there exists non-zero  $(v, q) \in V \times \mathfrak{H}$  such that

$$B(u, p; v, q) \geq \gamma(\|u\|_V + \|p\|)(\|v\|_V + \|q\|).$$

Proof (hint). Consider the test functions

$$v := \rho + p + du, q := u_{\mathfrak{H}},$$

where  $\rho \in \mathfrak{Z}^\perp$  is the unique element such that  $d\rho = u_{\mathfrak{B}}$ .

# A corresponding discrete problem

Suppose  $V_h \subset V$  is a Hilbert subcomplex, with a bounded cochain projection  $\pi_h : V \rightarrow V_h$ .

Consider the discrete problem:

Find  $(u_h, p_h) \in V_h \times \mathfrak{H}_h$  satisfying

$$\begin{aligned} \langle du_h, v_h \rangle + \langle u_h, dv_h \rangle + \langle p_h, v_h \rangle &= \langle f, v_h \rangle, & \forall v_h \in V_h, & \quad (2) \\ \langle u_h, q_h \rangle &= 0, & \forall q_h \in \mathfrak{H}_h. & \end{aligned}$$

This problem is well-posed, with a discrete inf-sup condition, where the constant  $\gamma_h$  depends only on  $c_P$  and the norm  $\|\pi_h\|$ .

# An error estimate

## Theorem

Let  $(u, p)$  be the solution to (1) and  $(u_h, p_h)$  be the solution to (2). If the projections  $\pi_h$  are  $V$ -bounded uniformly, independently of  $h$ , then the error can be estimated by

$$\|u - u_h\|_V + \|p - p_h\| \leq C \left( \inf_{v \in V_h} \|u - v\| + \inf_{q \in V_h} \|p - q\| + \mu \inf_{v \in V_h} \|P_{\mathfrak{B}} u - v\| \right),$$

where

$$\mu := \sup_{r \in \mathfrak{H}, \|r\|=1} \|(1 - \pi_h)r\|,$$

and where  $C$  depends only on the Poincaré constant  $c_P$ .

# Relationship to the discrete Hodge Laplacian

We can solve the discrete Hodge–Laplace equation by first obtaining the solution  $(u_h, p_h) \in V_h \times \mathfrak{H}_h$  of the Hodge–Dirac equation for  $f$ , then obtaining the solution  $(w_h, 0) \in V_h \times \mathfrak{H}_h$  of the Hodge–Dirac equation for  $u_h$ .

## Corollary

Convergence of solution of discrete Hodge–Laplace equation.



## Further considerations

Arnold, Falk and Winther define finite dimensional  $k$ -form spaces based on polynomials defined piecewise on simplices of the appropriate dimension. For Clifford-valued functions, we would use the direct sum of the appropriate spaces selected from those of Arnold, Falk and Winther.

Finite element exterior calculus uses *bounded cochain projections* to map from the continuous problem to the discrete problem. We would use the corresponding direct sum of these same projections.

On an embedded Riemannian manifold, subdivision into Euclidean simplices introduces geometric errors. Holst and Stern (2012) have addressed this with their work on *geometric variational crimes*. This idea should also apply to various Dirac operators on Riemannian manifolds, but is yet to be tried.

## Further considerations – implementation

The *Exterior* package of the *FEniCS* project is an experimental implementation of the concepts of Finite Element Exterior Calculus (FEEC). It is not yet integrated with the rest of *FEniCS*, and so it is difficult to use it to formulate and solve problems involving the Hodge-Laplacian. The *UFL* package of the *FEniCS* project implements some of the FEEC finite element spaces, but not direct sums of spaces.

A group under Doug Arnold has a current project, “Finite Element Exterior Calculus With FEniCS” which may address these issues.

(Logg et al. 2010, 2012, Arnold et al. 2012)

## Marie Rognes (via DOLFIN answers):

You can actually write (for instance)

```
from dolfin import exterior_derivative as d
```

... However, this functionality is exceptionally rudimentary: the 'd' just defaults to grad/curl/div/ based on element families when used.

## Marie Rognes (via DOLFIN answers):

Here is a complete example:

```
from dolfin import *
from dolfin import exterior_derivative as d
mesh = UnitSquare(16, 16)
V = FunctionSpace(mesh, "P Lambda", 2, 0)
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("1.0")
a = inner(d(u), d(v))*dx
L = f*v*dx
bc = DirichletBC(V, 0.0, "on_boundary")
u = Function(V)
solve(a == L, u, bc)
```

## Some references

D. N. Arnold, R. S. Falk and R. Winther, *Finite element exterior calculus, homological techniques, and applications*. Acta Numerica, 15, 2006, pp. 1-155.

D. N. Arnold, R. S. Falk and R. Winther, *Finite element exterior calculus: from Hodge theory to numerical stability*. Bull. Amer. Math. Soc. 47, 2010, pp. 281-354.

D. N. Arnold, *Spaces of finite element differential forms* (to appear in U. Gianazza et al., (eds.) *Analysis and Numerics of Partial Differential equations*, Springer, 2013.)

M. Holst and A. Stern, *Geometric variational crimes: Hilbert complexes, Finite element exterior calculus, and problems on hypersurfaces*. Found. Comput. Math. 12, 2012, pp. 263-293.