An abstract Hodge–Dirac operator and its stable discretization

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Subjects with parallel 40+ year histories

Finite Element Method

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Clifford analysis

D. Hestenes, *Multivector calculus.* J. Math. Anal. and Appl., 24:2, 1968, pp. 313-325.

More recent developments

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M. Holst and A. Stern, *Geometric variational crimes: Hilbert complexes, Finite element exterior calculus, and problems on hypersurfaces.* Found. Comput. Math. 12, 2012, pp. 263-293.

Topics

- Finite Element Exterior Calculus
- Example: Discretization of the Hodge Laplacian
- Discretization of the Hodge–Dirac operator

Finite Element Method

The Finite Element Method solves boundary value problems based on partial differential equations.

The original problem in a Hilbert space of functions is put into variational form, and is mapped into a problem defined on a finite dimensional function space, whose basis consists of functions supported in small regions, such as simplices.

(Iserles 1996; Braess 2001).

Finite element exterior calculus (FEEC)

FEEC is based on the Finite Element Method over Hilbert complexes. These are cochain complexes where the relevant vector spaces are Hilbert spaces.

For the de Rham complex, FEEC uses Hodge decomposition, the exterior derivative and differential forms.

The numerical stability of the FEEC discretization depends on the existence of a bounded cochain projection from a Hilbert complex to a subcomplex. FEEC uses smoothed projections to obtain this numerical stability.

(Arnold, Falk and Winther 2006, 2010; Christiansen and Winther 2008)

Hodge decomposition of differential forms

If (M, g) is an oriented, compact Riemannian manifold, then each space of smooth k-forms has an L^2-inner product,

$$\langle u, v \rangle_{L^2\Omega^k(M)} = \int_M \langle u, v \rangle_g \operatorname{vol}_g = \int_M u \wedge \star_g v.$$

This gives an adjoint operator $\mathrm{d}_k^* : \Omega^k(M) \to \Omega^{k-1}(M)$ for each k.

$$0 \rightleftarrows \Omega^0(U) \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^1(U) \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^2(U) \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^3(U) \rightleftarrows 0$$

$$0 \rightleftarrows C^\infty(U) \underset{\mathrm{div}}{\overset{\mathrm{grad}}{\rightleftarrows}} \mathfrak{X}(U) \underset{\mathrm{curl}}{\overset{\mathrm{curl}}{\rightleftarrows}} \mathfrak{X}(U) \underset{\mathrm{grad}}{\overset{\mathrm{div}}{\rightleftarrows}} C^\infty(M) \rightleftarrows 0$$

The Hodge decomposition says that each $f \in L^2\Omega^k(M)$ can be orthogonally decomposed as $f = \mathrm{d}\alpha + \mathrm{d}^*\beta + \gamma$, where $\mathrm{d}\gamma = 0$, $\mathrm{d}^*\gamma = 0$.

(Arnold, Falk and Winther 2006, 2010)

Hilbert complexes

Definition

A Hilbert complex (W, d) consists of:

- a sequence of Hilbert spaces W^k, and
- closed, densely-defined linear maps $\mathrm{d}^k : V^k \subset W^k \to V^{k+1} \subset W^{k+1}$, (possibly unbounded) such that $\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$ for each k.

The complex is *closed* if each d^k has closed image $\mathrm{d}^k V^k \subset W^{k+1}$, and *bounded* if $\mathrm{d}^k \in \mathcal{L}(W^k, W^{k+1})$.

Definition

Given a Hilbert complex (W, d), the *domain complex* (V, d) consists of the domains $V^k \subset W^k$, endowed with the graph inner product

$$\langle u, v \rangle_{V^k} = \langle u, v \rangle_{W^k} + \langle \mathrm{d}^k u, \mathrm{d}^k v \rangle_{W^{k+1}}$$

(Arnold, Falk and Winther 2006, 2010)

Hodge theory for closed Hilbert complexes

Definition

Given a Hilbert complex (W, d),

- $\mathfrak{Z}^k = \ker \mathrm{d}^k$ (closed k-forms), $\mathfrak{Z}^*_k = \ker \mathrm{d}^*_k$ (coclosed k-forms),
- $\mathfrak{B}^k = \mathrm{d}^{k-1} V^{k-1}$ (exact k-forms), $\mathfrak{B}^*_k = \mathrm{d}^*_{k+1} V^*_{k+1}$ (coexact k-forms),
- $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp} = \mathfrak{Z}^k \cap \mathfrak{Z}^*_k$ (harmonic k-forms) $\cong \mathfrak{Z}^k / \mathfrak{B}^k$ (kth cohomology)

Theorem (abstract Hodge decomposition)

If (W, d) is a closed Hilbert complex with domain complex (V, d), then

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}^*_k,$$
$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp}.$$

(Arnold, Falk and Winther 2006, 2010)

The abstract Poincaré inequality

Theorem

If (V, d) is a bounded, closed Hilbert complex, then there exists a Poincaré constant c_P such that

$$\|v\|_V \leqslant c_P \left\|\mathrm{d}^k v\right\|_V, \forall v \in \mathfrak{Z}^{k\perp}.$$

Proof

The map d_k is a V-bounded bijection from $\mathfrak{Z}^{k\perp}$ to $\mathfrak{B}^{k+1}$, which are both closed subspaces. The result follows by Banach's bounded inverse theorem.

Example

Let $U \subset \mathbb{R}^n$ be bounded and connected with Lipschitz boundary, and take $V^0 = H^1(U)$. Since $\mathfrak{Z}^0 \subset H^1(U)$ consists of the constant functions, this recovers the classical Poincaré inequality,

$$\|v\|_{H^1(U)} \leqslant c_P \left\|\mathrm{d}^k v\right\|_{L^2(U)}, \forall v \in H^1(U) : \int_U v(x)\mathrm{d}x = 0.$$

(Arnold, Falk and Winther 2006, 2010)

The abstract Hodge Laplacian

The *abstract Hodge Laplacian* is the operator $L = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$, which is an unbounded operator $W^k \to W^k$ with domain

$$D_L = \left\{ u \in V^k \cap V_k^* \mid \mathrm{d}u \in V_{k+1}^*, \mathrm{d}^* u \in V^{k-1} \right\}.$$

We wish to solve the problem $Lu = f$.

Example

For the de Rham complex on $U \subset \mathbb{R}^3$, we have the vector proxies:

$$\begin{aligned} L^0 &= 0 - \operatorname{div} \operatorname{grad}, \\ L^1 &= -\operatorname{grad} \operatorname{div} + \operatorname{curl} \operatorname{curl}, \\ L^2 &= \operatorname{curl} \operatorname{curl} - \operatorname{grad} \operatorname{div}, \\ L^3 &= -\operatorname{div} \operatorname{grad} + 0. \end{aligned}$$

(Arnold, Falk and Winther 2006, 2010)

Variational problem for the abstract Hodge Laplacian

If $u \in D_L$ solves $Lu = f$, then it satisfies the variational principle

$$\langle \mathrm{d}u, \mathrm{d}v \rangle + \langle \mathrm{d}^* u, \mathrm{d}^* v \rangle = \langle f, v \rangle, \quad \forall v \in V^k \cap V_k^*.$$

It can be hard to construct finite elements for the intersection space $V^k \cap V_k^*$, and there is also an existence and uniqueness problem.
If $v \in \mathfrak{H}^k$, then

$$\langle \mathrm{d}u, \mathrm{d}v \rangle + \langle \mathrm{d}^* u, \mathrm{d}^* v \rangle = 0,$$

so a solution exists only if $f \perp \mathfrak{H}^k$. Also, for any $q \in \mathfrak{H}^k$,

$$\langle \mathrm{d}(u+q), dv \rangle + \langle \mathrm{d}^*(u+q), \mathrm{d}^* v \rangle = \langle \mathrm{d}u, \mathrm{d}v \rangle + \langle \mathrm{d}^* u, \mathrm{d}^* v \rangle,$$

so the solution is not unique when $\mathfrak{H}^k \neq \{0\}$.
Therefore, we solve the problem:

$$\sigma = \mathrm{d}^* u, \quad \mathrm{d}\sigma + \mathrm{d}^* \mathrm{d}u + p = f, \quad p \in \mathfrak{H}^k, \quad u \in \mathfrak{H}^{k\perp}.$$

(Arnold, Falk and Winther 2006, 2010)

Mixed variational problem for the abstract Hodge Laplacian

Find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ satisfying

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, \mathrm{d}\tau \rangle &= 0, && \forall v \in V^{k-1}, \\ \langle \mathrm{d}\sigma, v \rangle + \langle \mathrm{d}u, \mathrm{d}v \rangle + \langle p, v \rangle &= \langle f, v \rangle, && \forall v \in V^k, \\ \langle u, q \rangle &= 0, && \forall q \in \mathfrak{H}^k. \end{aligned}$$

The solution computes the Hodge decomposition,

$$f = \mathrm{d}\sigma + \mathrm{d}^* \mathrm{d}u + p.$$

Theorem (Arnold, Falk, Winther, 2010, Th. 3.1)

Let (W, d) be a closed Hilbert complex with domain complex (V, d). The mixed formulation of the abstract Hodge Laplacian is well-posed, i.e., the unique solution (σ, u, p) satisfies

$$\|\sigma\|_V + \|u\|_V + \|p\|_W \leqslant c \, \|f\|_W \, .$$

(Arnold, Falk and Winther 2006, 2010)

The problem on a finite-dimensional subcomplex

A Hilbert subcomplex $V_h \subset V$ is a sequence of Hilbert subspaces $V_h^h \subset V^k$, such that the inclusions $i_h^k : V_h^k \to V^k$ are unitary and commute with the differentials. We also assume that these are equipped with bounded projections, $\pi_h^k : V^k \to V_h^k$, which also commute with the differentials.

Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ satisfying

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle u_h, \mathrm{d}\tau \rangle &= 0, && \forall v \in V_h^{k-1}, \\ \langle \mathrm{d}\sigma_h, v \rangle + \langle \mathrm{d}u_h, \mathrm{d}v \rangle + \langle p_h, v \rangle &= \langle f, v \rangle, && \forall v \in V_h^k, \\ \langle u_h, q \rangle &= 0, && \forall q \in \mathfrak{H}_h^k. \end{aligned}$$

(Arnold, Falk and Winther 2006, 2010)

Error estimate for the Hodge–Laplace problem

Theorem (Arnold, Falk, Winther, 2006, Th. 7.4)

If (σ, u, p) is the solution of the continuous problem, and (σ_h, u_h, p_h) is the solution of the discrete problem, and if the projections π_h are V-bounded uniformly, independently of h, then the error can be estimated by

$$\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\|_W$$
$$\leqslant C \left(\inf_{\tau \in V_h} \|\sigma - \tau\| + \inf_{v \in V_h} \|u - v\| + \inf_{q \in V_h} \|p - q\|_W + \mu \inf_{v \in V_h} \|P_{\mathfrak{B}} u - v\| \right),$$

where $\mu := \sup_{r \in \mathfrak{H}, \|r\|=1} \|(1 - \pi_h) r\|$, and where C depends only on the Poincaré constant c_P.

(Arnold, Falk and Winther 2006, 2010)

An abstract Hodge–Dirac problem: setting

Let (W, d) be a closed Hilbert complex with domain complex (V, d). In V, we use the inner product

$$\langle u, v \rangle_V := \langle u, v \rangle + \langle \mathrm{d}u, \mathrm{d}v \rangle.$$

We use the orthogonal Hodge decomposition

$$\begin{aligned} W &= \mathfrak{B} \oplus \mathfrak{H} \oplus \mathfrak{B}^*, \\ u &= u_{\mathfrak{B}} \oplus u_{\mathfrak{H}} \oplus u_{\mathfrak{B}^*}, \quad \forall u \in W. \end{aligned}$$

where $V \in W$ is the domain of d, $\mathfrak{B}$ is the range of d, and $\mathfrak{Z}$ is the null space of d.

An abstract Hodge–Dirac problem

We want to solve the problem $(\mathrm{d}+\mathrm{d}^*)u = f - f_{\mathfrak{H}}$, where $\mathrm{d}+\mathrm{d}^*$ is the *abstract Hodge–Dirac operator*.
Consider the following mixed variational problem:
Find $(u,p) \in V \times \mathfrak{H}$ satisfying

$$\begin{aligned} \langle \mathrm{d}u, v\rangle + \langle u, \mathrm{d}v\rangle + \langle p, v\rangle &= \langle f, v\rangle, && \forall v \in V, \\ \langle u, q\rangle &= 0, && \forall q \in \mathfrak{H}. \end{aligned} \tag{1}$$

To show that this problem is well-posed, it suffices to prove the inf-sup condition for the symmetric bilinear form

$$B(u,p;v,q) := \langle \mathrm{d}u, v\rangle + \langle u, \mathrm{d}v\rangle + \langle p, v\rangle + \langle u, q\rangle$$

on $V \times \mathfrak{H}$.

The problem is well-posed

Theorem

There exists a constant $\gamma > 0$, depending only on the Poincaré constant c_P, such that for all non-zero $(u, p) \in V \times \mathfrak{H}$, there exists non-zero $(v, q) \in V \times \mathfrak{H}$ such that

$$B(u, p; v, q) \geq \gamma(\|u\|_V + \|p\|)(\|v\|_V + \|q\|).$$

Proof (hint). Consider the test functions

$$v := \rho + p + du, q := u_{\mathfrak{H}},$$

where $\rho \in \mathfrak{Z}^{\perp}$ is the unique element such that $\mathrm{d}\rho = u_{\mathfrak{B}}$.

A corresponding discrete problem

Suppose $V_h \subset V$ is a Hilbert subcomplex, with a bounded cochain projection $\pi_h : V \to V_h$.

Consider the discrete problem:

Find $(u_h, p_h) \in V_h \times \mathfrak{H}_h$ satisfying

$$\begin{aligned} \langle \mathrm{d}u_h, v_h \rangle + \langle u_h, \mathrm{d}v_h \rangle + \langle p_h, v_h \rangle &= \langle f, v_h \rangle, & \forall v_h \in V_h, \\ \langle u_h, q_h \rangle &= 0, & \forall q_h \in \mathfrak{H}_h. \end{aligned} \tag{2}$$

This problem is well-posed, with a discrete inf-sup condition, where the constant γ_h depends only on c_P and the norm $\|\pi_h\|$.

An error estimate

Theorem

Let (u, p) be the solution to (1) and (u_h, p_h) be the solution to (2). If the projections π_h are V-bounded uniformly, independently of h, then the error can be estimated by

$$\|u - u_h\|_V + \|p - p_h\| \\ \leqslant C \left(\inf_{v \in V_h} \|u - v\| + \inf_{q \in V_h} \|p - q\| + \mu \inf_{v \in V_h} \|P_{\mathfrak{B}} u - v\| \right),$$

where

$$\mu := \sup_{r \in \mathfrak{H}, \|r\| = 1} \|(1 - \pi_h) r\|,$$

and where C depends only on the Poincaré constant c_P.

Relationship to the discrete Hodge Laplacian

We can solve the discrete Hodge–Laplace equation by first obtaining the solution $(u_h, p_h) \in V_h \times \mathfrak{H}_h$ of the Hodge–Dirac equation for f, then obtaining the solution $(w_h, 0) \in V_h \times \mathfrak{H}_h$ of the Hodge–Dirac equation for u_h.

Corollary

Convergence of solution of discrete Hodge–Laplace equation.

Further considerations

Arnold, Falk and Winther define finite dimensional k-form spaces based on polynomials defined piecewise on simplices of the appropriate dimension. For Clifford-valued functions, we would use the direct sum of the appropriate spaces selected from those of Arnold, Falk and Winther.

Finite element exterior calculus uses *bounded cochain projections* to map from the continuous problem to the discrete problem. We would use the corresponding direct sum of these same projections.

On an embedded Riemannian manifold, subdivision into Euclidean simplices may introduce geometric errors. Holst and Stern (2012) have addressed this issue with their work on *geometric variational crimes*. This idea should also apply to various Dirac operators on Riemannian manifolds, but is yet to be tried.

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