Skew, bent and fractious: a confession

Paul Leopardi

Mathematical Sciences Institute, Australian National University.
Presented on 3 October 2013 at AustMS 2013, Sydney.

Corrected, 4 October 2013

Acknowledgements

Richard Brent, Padraig Ó Catháin, Judy-anne Osborn.

National Computational Infrastructure.

Australian Mathematical Sciences Institute.

Australian National University.

Result 1: anti-amicability

The graph of *anti-amicability* of the canonical basis matrices of the neutral Clifford algebra $\mathbb{R}_{m,m}$ is *strongly regular* with parameters

$$(\nu, k, \lambda = \mu) = (4^m, 2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1}).$$

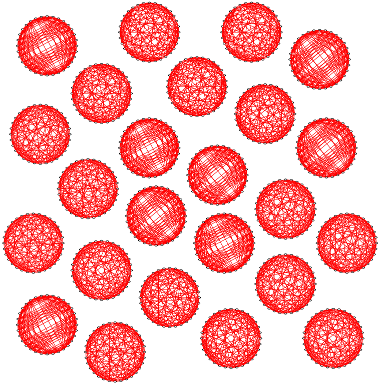
Result 2: anti-amicability

The graph of anti-amicability of the canonical basis matrices of the neutral Clifford algebra $\mathbb{R}_{2,2}$ is *strongly regular* with parameters $(\nu, k, \lambda = \mu) = (16, 6, 2)$ is the 4×4 *lattice graph* and not the Shrikande graph.

Overview

- What led to this investigation?
- Key concepts.
- Specific construction.
- Why is Result 1 true?

Motivation



Anti-amicability of 4×4 Hadamard matrices: 24 components.

A long history and a deep literature

- Difference sets.
  Bruck (1955), Hall (1956), Menon (1960, 1962),
  Mann (1965), Turyn (1965), Baumert (1969),
  Dembowski (1969), McFarlane (1973), Dillon (1974),
  Kantor (1975, 1985), Ma (1994), ...

- Bent functions.
  Dillon (1974), Rothaus (1976), Dempwolff (2006), ...

- Strongly regular graphs.
  Brouwer, Cohen and Neumaier (1989), Ma (1994),
  Bernasconi and Codenotti (1999),
  Bernasconi, Codenotti and VanderKam (2001) ...

Difference sets

The k-element set D is a (v, k, λ, n) *difference set* in an abelian group G of order v if for every non-zero element g in G, the equation $g = d_i - d_j$ has exactly λ solutions (d_i, d_j) with d_i, d_j in D.

The parameter $n := k - \lambda$.

(Dillon 1974).

Hadamard difference sets

A (v, k, λ, n) difference set with $v = 4n$ is called a *Hadamard difference set.*

Theorem 1

(Menon 1962)

A Hadamard difference set has parameters of the form

$$\begin{aligned}(v, k, \lambda, n) &= (4N^2, 2N^2 - N, N^2 - N, N^2) \\ \textit{or} \quad &(4N^2, 2N^2 + N, N^2 + N, N^2).\end{aligned}$$

(Menon 1962, Dillon 1974).

Bent functions

H_m, the Sylvester Hadamard matrix of order 2^m, is defined by

$$H_1 := \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix},$$
$$H_m := H_{m-1} \otimes H_1, \quad \text{for} m > 1.$$

For a boolean function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2$, define the vector $[f]$ by

$$[f] = [(-1)^{f(0)}, (-1)^{f(1)}, \ldots, (-1)^{f(2^m-1)}]^T,$$

where $f(i)$ uses the binary expansion of i.

Bent functions

The Boolean function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2$ is *bent* if its Hadamard transform has constant magnitude.

In other words,

$$|H_m[f]| = C[1, \ldots, 1]^T.$$

for some constant C.

(Dillon 1974)

Bent functions and Hadamard difference sets

Theorem 2

(Dillon 1974, Theorem 6.2.2)

The Boolean function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2$ is bent if and only if $D := f^{-1}(1)$ is a Hadamard difference set.

Theorem 3

(Dillon 1974, Remark 6.2.4)

Bent functions exist on $\mathbb{Z}_2^m$ only when m is even.

(Dillon 1974)

Strongly regular graphs

A simple graph Γ of order v is *strongly regular* with parameters (v, k, λ, μ) if

- each vertex has degree k,
- each adjacent pair of vertices has λ common neighbours, and
- each nonadjacent pair of vertices has μ common neighbours.

(Brouwer, Cohen and Neumaier 1989)

Bent functions and strongly regular graphs

The *Cayley graph* of a binary function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2$ is the undirected graph with adjacency matrix F given by $F_{i,j} = f(g_i - g_j)$, for some ordering $(g_1, g_2, \ldots)$ of $\mathbb{Z}_2^m$.

Theorem 4

(Bernasconi and Codenotti 1999, Lemma 12)
The Cayley graph of a bent function on Z_2^m is a strongly regular graph with $\lambda = \mu$.

Theorem 5

(Bernasconi, Codenotti and VanderKam 2001, Theorem 3)
Bent functions are the only binary functions on Z_2^m whose Cayley graph is a strongly regular graph with $\lambda = \mu$.

The groups $\mathbb{G}_{1,1}$ and $\mathbb{Z}_2^2$

The 2×2 orthogonal matrices

$$\mathrm{e}_1 := \begin{bmatrix} . & - \\ 1 & . \end{bmatrix}, \quad \mathrm{e}_2 := \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix}$$

generate the group $\mathbb{G}_{1,1}$ of order 8, an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2^2$, with $\mathbb{Z}_2 \simeq \{I, -I\}$, and cosets

$$\begin{aligned} 0 &\leftrightarrow 00 \leftrightarrow \{\pm I\}, \\ 1 &\leftrightarrow 00 \leftrightarrow \{\pm \mathrm{e}_1\}, \\ 2 &\leftrightarrow 10 \leftrightarrow \{\pm \mathrm{e}_2\}, \\ 3 &\leftrightarrow 11 \leftrightarrow \{\pm \mathrm{e}_1 \mathrm{e}_2\}. \end{aligned}$$

The groups $\mathbb{G}_{m,m}$ and $\mathbb{Z}_2^{2m}$

For $m > 1$, the group $\mathbb{G}_{m,m}$ of order 2^{2m+1} consists of matrices of the form $g_1 \otimes g_{m-1}$ with g_1 in $\mathbb{G}_{1,1}$ and g_{m-1} in $\mathbb{G}_{m-1,m-1}$.

This group is an extension of $\mathbb{Z}_2 \simeq \{\pm I\}$ by $\mathbb{Z}_2^{2m}$:

$$\begin{aligned}
0 &\leftrightarrow 00\ldots00 \leftrightarrow \{\pm I\}, \\
1 &\leftrightarrow 00\ldots01 \leftrightarrow \{\pm I_{(2)}^{\otimes(m-1)} \otimes \mathrm{e}_1\}, \\
2 &\leftrightarrow 00\ldots10 \leftrightarrow \{\pm I_{(2)}^{\otimes(m-1)} \otimes \mathrm{e}_2\}, \\
&\ldots \\
2^{2m}-1 &\leftrightarrow 11\ldots11 \leftrightarrow \{\pm(\mathrm{e}_1\mathrm{e}_2)^{\otimes m}\}.
\end{aligned}$$

Canonical basis matrices of $\mathbb{R}_{m,m}$

A canonical ordered basis of the matrix representation of the Clifford algebra $\mathbb{R}_{m,m}$ is given by an ordered transversal of $\mathbb{Z}_2 \simeq \{\pm I\}$ in $\mathbb{Z}_2^{2m}$.

For example, $(I, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_1\mathrm{e}_2)$ is one such ordered basis.

We define a function $\gamma_m : \mathbb{Z}_{2^{2m}} \to \mathbb{G}_{m,m}$ to choose the corresponding canonical basis matrix for $\mathbb{R}_{m,m}$ for some transversal, and use binary expansion to get a function on $\mathbb{Z}_2^{2m}$.

For example, $\gamma_1(1) = \gamma_1(01) := \mathrm{e}_1$.

The sign function s_1 on $\mathbb{Z}_4$ and $\mathbb{Z}_2^2$

We use the function γ_1 to define the sign function s_1 :

$$s_1(i) := \begin{cases} 1 \leftrightarrow \gamma_1(i)^2 = -I \\ 0 \leftrightarrow \gamma_1(i)^2 = I, \end{cases}$$

for all i in $\mathbb{Z}_2^2$.

Using our notation, we see that $[s_1] = [1, -1, 1, 1]^T$.

The sign function s_m on $\mathbb{Z}_{2^{2m}}$ and $\mathbb{Z}_2^{2m}$

We use the function γ_m to define the sign function s_m :

$$s_m(i) := \begin{cases} 1 \leftrightarrow \gamma_m(i)^2 = -I \\ 0 \leftrightarrow \gamma_m(i)^2 = I, \end{cases}$$

for all i in $\mathbb{Z}_2^{2m}$.

Properties of the sign function s_m

If we define $\odot : \mathbb{Z}_2 \times \mathbb{Z}_2^{2m-2} \to \mathbb{Z}_2^{2m}$ as concatenation of bit vectors, e.g.. $01 \odot 1111 := 011111$, it becomes easy to verify that

$$s_m(i_1 \odot i_{m-1}) = s_1(i_1) + s_{m-1}(i_{m-1})$$

for all i_1 in $\mathbb{Z}_2$ and i_{m-1} in $\mathbb{Z}_2^{2m-2}$, and therefore

$$[s_m] = [s_1] \otimes [s_{m-1}].$$

Also, since each $\gamma_m(i)$ is orthogonal,
$s_m(i) = 1$ if and only if $\gamma_m(i)$ is skew.

Proof of Result 1: s_m is bent

Recall that $[s_1] = [1, -1, 1, 1]^T$.

We show that s_1 is bent by forming

$$H_2[s_1] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix} \begin{bmatrix} 1 \\ - \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix}.$$

Proof of Result 1: s_m is bent

Recall that for $m > 1$, $H_{2m} = H_2 \otimes H_{2m-2}$ and $[s_m] = [s_1] \otimes [s_{m-1}]$.

Therefore

$$H_{2m}[s_m] = H_2[s_1] \otimes H_{2m-2}[s_{m-1}] = (H_2[s_1])^{(\otimes m)},$$

which has constant absolute value.

The 4×4 lattice graph

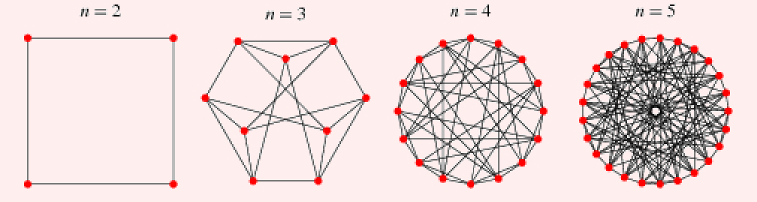


Image from
http://mathworld.wolfram.com/LatticeGraph.html