Discrepancy, separation and Riesz energy of finite point sets on compact connected Riemannian manifolds

Paul Leopardi

Mathematical Sciences Institute, Australian National University. For presentation at Constructive Functions 2014, Vanderbilt University

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Discrepancy, separation and energy

Topics

- Discrepancy, separation and energy on the unit sphere
- Generalization to compact connected Riemannian manifolds
- The main result
- A sketch of the proof
- Further questions
In 2004, here at Vanderbilt University, Ed Saff asked me a question about, separation, discrepancy and discrete energy on the unit sphere $S^d$. The answer to this question is:

**Theorem 1**

For a well separated admissible sequence $\mathcal{X}$ of $S^d$ spherical codes, with discrepancy function $\delta$, the normalized Riesz $s$ energy for $0 < s < d$ satisfies the inequality

$$E_{X_\ell} U_s = E_M U_s + O(\delta(|X_\ell|)^{1-s/d}).$$

This talk describes a generalization of this result.

(L 2007, L 2013)
Compact connected Riemannian manifolds

Let $M$ be a smooth, connected $d$-dimensional Riemannian manifold, without boundary, with metric $g$ and geodesic distance $\text{dist}$, such that $M$ is compact in the metric topology of $\text{dist}$.

(Sinclair and Tanaka, 2007, Figure 1)
Metric and measure, sequences of $M$-codes

Let $\lambda_M$ be the volume measure on $M$ given by the volume element corresponding to $g$ and therefore to $\text{dist}$. Since $M$ is compact, it has finite volume.

Let $\sigma_M := \lambda_M/\lambda_M(M)$, so $\sigma_M(M) = 1$.

Consider an infinite sequence $\mathcal{X} := (X_1, X_2, \ldots)$ of $M$-codes, each a finite subset of $M$.

A sequence $(X_1, X_2, \ldots)$ whose cardinalities $(|X_1|, |X_2|, \ldots)$ diverge to $+\infty$ is called pre-admissible.
Normalized ball discrepancy

For any probability measure $\mu$ on $M$, the normalized ball discrepancy is

$$D(\mu) := \sup_{x \in M, \ 0 < r \leq \text{diam}(M)} \left| \mu(B_x(r)) - \sigma_M(B_x(r)) \right|,$$

where $\text{diam}(M)$ is the diameter of $M$ and $B_x(r)$ is the geodesic ball of radius $r$ about the point $x$.

An $M$-code $X$ with cardinality $|X|$ has probability measure

$$\sigma_X(S) := |S \cap X| / |X|,$$

and therefore normalized ball discrepancy

$$D(X) := \sup_{y \in M, \ r > 0} \left| |B_y(r) \cap X| / |X| - \sigma_M(B_y(r)) \right|.$$

(Blümlinger 1990, Damelin and Grabner 2003)
Asymptotic equidistribution

A sequence $\mathcal{X} := (X_1, X_2, \ldots)$, of $M$-codes is asymptotically equidistributed if $D(X_\ell) < \delta(|X_\ell|)$, where $\delta$ is a positive decreasing function $\delta : \mathbb{N} \to (0, \infty)$ with $\delta(N) \to 0$ as $N \to \infty$.

It is easy to see that $\delta(|X|) > 1/|X|$.

Consider each $B_x(r)$ with $x \in X$, and the limit as $r \to 0$.

(Blümlinger 1990, Damelin and Grabner 2003)
Separation of points, admissible sequences

An admissible sequence of $\mathcal{M}$-codes is an asymptotically equidistributed pre-admissible sequence with discrepancy function $\delta$ that also has a lower bound on the minimum separation:

$$\text{dist}(x, y) > \Delta(N_\ell) \quad \text{for all } x, y \in X_\ell,$$

where $\Delta : \mathbb{N} \to (0, \infty)$ is a positive decreasing function with $\Delta(N) \to 0$ as $N \to \infty$.

(Tammes 1930, Rankin 1955, Flatto and Newman 1977)
Well separated sequences of codes

The order of the lower bound $\Delta(N)$ for the separation of the sequence with the largest separation for each $N$ is $\Omega(N^{-1/d})$.

Therefore, for all sequences of $M$-codes,
$\Delta(|X_\ell|) = O(|X_\ell|^{-1/d})$.

A sequence of $M$-codes is called well separated if there exists a separation constant $\gamma > 0$ such that we can set $\Delta(N) = \gamma N^{-1/d}$.

(Tammes 1930, Rankin 1955, Flatto and Newman 1977)
Normalized Riesz $s$ energy

The normalized normalized Riesz $s$ energy of an $M$ code is $E_X U_s$, where $U_s(r) := r^{-s}$ and $E_X$ is the normalized discrete energy functional

$$E_X u := \frac{1}{|X|^2} \sum_{x \in X} \sum_{y \in X, y \neq x} u (\text{dist}(x, y)).$$

for $u : (0, \infty) \to \mathbb{R}$.

The corresponding normalized continuous energy functional is

$$E_M u := \int_M \int_M u (\text{dist}(x, y)) \, d\sigma_M(y) \, d\sigma_M(x).$$

Convergence of the energy of $M$ codes

The generalization of the result on the unit sphere $S^d$ is:

**Theorem 2**

Let $M$ be a compact connected $d$-dimensional Riemannian manifold. If $0 < s < d$ then, for a well separated admissible sequence $X$ of $M$-codes,

$$\left| \left( E_X - E_M \right) U \right| = O \left( \delta(|X_\ell|)^{(1-s/d)/(d+2-s/d)} \right),$$

where $\delta(|X_\ell|)$ is the upper bound on the geodesic ball discrepancy of $X_\ell$ used to satisfy the admissibility condition.
Discrepancy, separation and energy

A sketch of the proof

Proof (sketch)

The proof proceeds along the lines of the proof for the sphere, except for two issues.

1. The volume of a geodesic ball does not behave in exactly the same way as the volume of a spherical cap.

2. The normalized mean potential function

\[ \Phi_M(x) := \int_M U_s(\text{dist}(x,y)) \, d\sigma_M(y) \]

varies with \( x \), unlike the case of the sphere.

Both issues are overcome using estimates from Blümlinger (1990).
Blümlinger’s first estimate

Blümlinger (1990) gives us the estimate:

**Lemma 3**

Let $M$ be a compact connected $d$-dimensional Riemannian manifold without boundary. Then

$$\left| \frac{\lambda_M(B_x(r))}{V_d(r)} - 1 \right| = O(r^2)$$

uniformly in $M$, where $V_d(r)$ is the volume of the Euclidean ball of radius $r$ in $\mathbb{R}^d$.

That is, the unnormalized volume of a small enough geodesic ball in $M$ is similar to the volume of a ball of the same radius in $\mathbb{R}^d$.

(Blümlinger 1990)
Blümlinger’s second estimate

Blümlinger (1990) also yields the following estimate.

**Theorem 4**

For $f \in C(M)$, and a measure $\nu$ on $M$ where $\nu(M) = \lambda_M(M)$,

$$|\nu(f) - \lambda_M(f)| \leq T_1(r) + T_2(r) + T_3(r),$$

where

$$T_1(r) := \|f - f_r\|_{\infty} \lambda_M(M),$$

$$T_2(r) := 2 \|f\|_{\infty} \lambda_M(M) \sup_{x \in M} \left| \frac{\lambda_M(B(x, r))}{\mathcal{V}_d(r)} - 1 \right|,$$

$$T_3(r) := \frac{\|f\|_{\infty}}{\mathcal{V}_d(r)} \int_M \left| \nu(B(x, r)) - \lambda_M(B(x, r)) \right| \, d\lambda_M(x).$$
Some notation

For integrable $f : M \to \mathbb{R}$, the mean of $f$ on $M$ is

$$\mathcal{I}_M f := \int_M f(y) \, d\sigma_M(y).$$

For a function $f : M \to \mathbb{R}$ that is finite on the $M$-code $X$, the mean of $f$ on $X$ is

$$\mathcal{I}_X f := \int_M f(y) \, d\sigma_X(y) = \frac{1}{|X|} \sum_{y \in X} f(y).$$
Some notation

For an $M$-code $X$, a point $x \in M$ and a measurable subset $S \subset M$, the punctured normalized counting measure of $S$ with respect to $X$, excluding $x$ is

$$\sigma^{[x]}_X(S) := |S \cap X \setminus \{x\}| / |X|,$$

and for a function $f : M \to \mathbb{R}$ that is finite on $X \setminus \{x\}$, the corresponding punctured mean is

$$\mathcal{I}^{[x]}_X f := \int_M f(y) d\sigma^{[x]}_X(y) = \frac{1}{|X|} \sum_{y \in X \setminus \{x\}} f(y).$$
Some notation

For a point \( x \in M \), define the function \( U_x : M \setminus \{x\} \rightarrow \mathbb{R} \) as

\[
U_x(y) := \text{dist}(x, y)^{-s}.
\]

The mean Riesz \( s \)-potential at \( x \) with respect to \( M \) is then

\[
\Phi_M(x) = \mathcal{I}_M U_x,
\]

and the normalized energy of the Riesz \( s \)-potential on \( M \) is

\[
E_M U = \mathcal{I}_M \Phi_M = \int_M \int_M \text{dist}(x, y)^{-s} \, d\sigma_M(y) \, d\sigma_M(x).
\]
Some notation

For an $\mathcal{M}$-code $X$, the mean Riesz $s$-potential at $x$ with respect to $X$ but excluding $x$ is

$$\Phi_X(x) := \mathcal{I}_X^{[x]} U_x,$$

the normalized energy of the Riesz $s$-potential on $X$ is

$$E_X U = \mathcal{I}_X \Phi_X = \frac{1}{|X|^2} \sum_{x \in X} \sum_{y \in X \backslash \{x\}} \text{dist}(x, y)^{-s},$$

and the mean on $X$ of the mean Riesz $s$-potential is

$$\mathcal{I}_X \Phi_M = \frac{1}{|X|} \sum_{x \in X} \int_M \text{dist}(x, y)^{-s} \, d\sigma_M(y).$$
First, split the energy difference \((E_X - E_M)U\) into two parts:

\[
(E_X - E_M)U = \mathcal{I}_X \Phi_X - \mathcal{I}_M \Phi_M
= (\mathcal{I}_X \Phi_X - \mathcal{I}_X \Phi_M) + (\mathcal{I}_X \Phi_M - \mathcal{I}_M \Phi_M)
= \mathcal{I}_X (\Phi_X - \Phi_M) + (\mathcal{I}_X - \mathcal{I}_M) \Phi_M.
\]

Next, estimate each part.

Lemma 3 yields the estimate

\[
|\mathcal{I}_X (\Phi_X - \Phi_M)| = O(\delta^{1-s/d}).
\]
Proof (sketch, continued)

We apply Theorem 4 with \( f := \Phi_M \) and \( \nu := \lambda(M)\sigma_X \).

It turns out that for \( r \) sufficiently small,

\[
T_1(r) = O\left(r^{(d-s)/(d+1)}\right).
\]

Lemma 3 yields \( T_2(r) = O(r^2) \).

The bound \(|\nu(B(x,r)) - \lambda_M(B(x,r))| \leq \delta \lambda(M)\) yields

\[
T_3(r) = O(\delta r^{-d}).
\]

Setting \( r = \delta^{(d+1)/(d^2+2d-s)} \) then results in the estimate

\[
|(\mathcal{I}_X - \mathcal{I}_M)\Phi_M| = O\left(\delta^{(d-s)/(d^2+2d-s)}\right).
\]
Questions

1. Is the convergence rate given in Theorem 2 best possible?
2. For a compact connected Riemannian manifold $M$, for what function spaces $F_M$ does a Koksma-Hlawka type inequality

$$|(I_X - I_M)f| \leq D(X) V(f)$$

hold for all $f \in F_M$, where $D(X)$ is the geodesic ball discrepancy? What is the appropriate functional $V$?

3. For which compact connected Riemannian manifolds $M$ does the space $F_M$ contain the mean potential function $\Phi_M$?

4. For which compact connected Riemannian manifolds $M$ is there an efficient construction for a well-separated admissible sequence $X$?