HOLOMORPHIC CURVES IN LAGRANGIAN TORUS FIBRATIONS

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Abstract

This thesis deals with the problem of finding holomorphic curves in symplectic manifolds which are Lagrangian torus fibrations with a well defined action of the torus. A family of complex structures are defined which can be viewed as collapsing the torus fibers. Under this degeneration, it is seen that holomorphic curves converge to objects called holomorphic graphs, similar to what are called tropical curves in the algebraic setting of tropical geometry.

A moduli space of objects called J^ϵ holomorphic graphs is defined, and proved to be cobordant to the moduli space of holomorphic curves. Thus the moduli space of J^ϵ holomorphic graphs can be used to calculate invariants of the moduli space of holomorphic curves.

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Chapter 1

Introduction

This thesis provides methods for studying the moduli space of (pseudo)holomorphic curves in a class of symplectic manifolds which are fibered by Lagrangian tori, subject to assumptions listed in appendix B. Examples of such Lagrangian torus fibrations are given by the symplectization of the unit cotangent bundle of $\mathbb{T}^n$ or the region in a toric manifold where the torus action is free. Another space where these techniques apply is given by the structure on $\mathbb{R} \times (S^1 \times S^2)$ considered by Taubes in [8].

A family of complex structures J^ϵ is considered which can be viewed as collapsing the torus fibers. Under this degeneration, holomorphic curves become solutions of a finite dimensional problem. A perturbation of this finite dimensional problem gives a moduli space of objects called J^ϵ holomorphic graphs. Invariants associated to the moduli space of holomorphic curves can then be computed using J^ϵ holomorphic graphs.

The qualitative properties of these J^ϵ holomorphic graphs (and hence the moduli space of holomorphic curves) are determined by studying the integral curves of a lattice of vector fields determined by the original complex structure and the torus fibration. Sending ϵ to zero creates objects called holomorphic graphs. Loosely speaking, an edge of a holomorphic graph consists of a closed geodesic in a torus fiber translated along by a vector field determined by the homology class of the geodesic. The fact that homology classes of edge geodesics sum to zero at vertices gives a kind of conservation of momentum condition at vertices. Apart from this, vertices contain the information of a model holomorphic map of a punctured Riemann surface to $(\mathbb{C}P^1)^n$ and determine the relative positioning of edges.

The class of manifolds under consideration are Lagrangian torus fibrations of the form

$$\begin{array}{ccc} \mathbb{T}^n & \longrightarrow & \mathbb{T}^n \rtimes B^n \\ & & \downarrow \pi \\ & & B^n \end{array}$$

with a structure group consisting of rotations of the torus fibers $\mathbb{T}^n$. This means that there is a well defined action of torus rotations on the fibers.

$$\mathbb{T}^n \times (\mathbb{T}^n \rtimes B^n) \xrightarrow{m} \mathbb{T}^n \rtimes B^n$$

Note that it would also be possible to work with torus fiber bundles that have a structure group consisting of affine transformations of the torus, however the analysis is complicated in this case by not having nice metrics. For a study of Lagrangian torus fibrations, see [7]. In this article, it is pointed out by Mishachev that any Lagrangian fibration admits a canonical affine structure on the fibers, identified with the affine structure on T^*B^n by lifting a covector in the the base manifold to the total space and then taking its symplectic dual, which consists of a vectorfield tangent to the fiber. Locally, we always have a symplectic action of $\mathbb{T}^n$ on the fibers given by the flow defined by these vectorfields.

We need an (almost) complex structure

$$J^2 = -Id : T_p(\mathbb{T}^n \rtimes B^n) \longrightarrow T_p(\mathbb{T}^n \rtimes B^n)$$

which is symmetric with respect to this structure in the sense that it is preserved by the torus rotations, and a symplectic form

$$\omega \in \Omega^2(\mathbb{T}^n \rtimes B^n), \quad d\omega = 0, \quad \omega^n \neq 0$$

which is also symmetric with respect to the torus rotations. ω needs to tame J holomorphic curves in a sense described in section 3.1. The torus fibers should be Lagrangian with respect to ω.

$$\omega(v_1, v_2) = 0 \text{ for } v_1, v_2 \in \ker(d\pi)$$

Note that our torus rotations and J provide a canonical trivialization of our tangent space $T(\mathbb{T}^n \rtimes B^n)$. Giving $\mathbb{T}^n$ coordinates $x \in \mathbb{R}^n/\mathbb{Z}^n$, the torus multiplication on fibers provides vertical vector fields we'll denote as ∂_{x_i} and an identification of each fiber with $\mathbb{R}^n/\mathbb{Z}^n$ up to translation. We then have the following important basis for $T(\mathbb{T}^n \rtimes B^n)$.

$$\{\partial_{x_i}, J\partial_{x_i}\}$$

We shall see that the dynamics of the vector fields generated by those above will determine the moduli space of holomorphic curves. Note that as described in [7], the existence of a torus fibration over a manifold B^n is equivalent to T^*B^n carrying an integrable affine structure. This affine structure is the one given by choosing a one form λ over a ball in the base so that $d\lambda = \omega$. λ then restricts to a closed one form on each fiber, which represents a class in $H^1(\mathbb{T}^n)$. The Lagrangian neighborhood theorem implies that this gives coordinates for the base. A different choice of primitive λ simply shifts these coordinates by a constant, so we have an identification of H^1 of the fiber with the tangent space of the base. The projection of $\{J\partial_{x_i}\}$ to the base gives an identification of the tangent space of the base with H_1 of the fibers. The lattice defined by the projection of $\{J\partial_{x_i}\}$ is not necessarily the same as the lattice defined by $H^1(\mathbb{T}^n, \mathbb{Z})$. These two lattices do however obey a positivity condition due to the constraint that ω is positive on holomorphic planes.

This basis $\{\partial_{x_i}, J\partial_{x_i}\}$ gives an identification of $(T_p(\mathbb{T}^n \rtimes B^n), J)$ with $\mathbb{C}^n$ by sending ∂_{x_k} to x_k and $J\partial_{x_k}$ to y_k using the standard $z = x + iy$ coordinates for $\mathbb{C}$. We can put a metric g on $T(\mathbb{T}^n \rtimes B^n)$ in which $\{\partial_{x_i}, J\partial_{x_i}\}$ gives an orthonormal frame. There is a canonical flat connection, ∇ which preserves this trivialization.

We are interested in the moduli space of J holomorphic curves. A J holomorphic curve is a map $u : (S, j) \longrightarrow (\mathbb{T}^n \rtimes B^n, J)$ so that $u^*J = j$, where (S, j) denotes a Riemann surface S with its complex structure j. This is equivalent to

$$\bar{\partial}u := \frac{1}{2}(du - J \circ du \circ j) = 0$$

We consider J holomorphic curves tamed by ω so that

$$E_\omega(u) := \int_S u^*(\omega) < \infty$$

Consider the degenerating family of complex structures J^ϵ for $\epsilon \in (0, 1]$ characterized by

$$J^\epsilon \partial_x = \epsilon J \partial_x \text{ for } \partial_x \in \ker(d\pi)$$

The conditions on our taming form ω have been chosen to ensure that ω tames J^ϵ holomorphic curves for all $\epsilon \neq 0$. In chapter 2, for ϵ small we will construct the moduli space of bounded energy solutions of a slightly weakened $\bar{\partial}$ equation, using as models objects called J^ϵ quasi holomorphic graphs described in sections 1.1, 2.5, and 1.1.4. We shall show in chapter 3 that any bounded energy J^ϵ holomorphic curve can be constructed in this way. This is put together in chapter 4 by showing that the moduli space of holomorphic curves is cobordant to the a moduli space of objects called J^ϵ holomorphic graphs.

It is shown in appendix B that the following spaces obey our technical assumptions.

Example 1.0.1. $\mathbb{C}^n/\mathbb{Z}^n$

We can consider $\mathbb{C}^n/\mathbb{Z}^n$ as $(\mathbb{C}P^1 - \{0, \infty\})^n$. Pulling back a rotationally symmetric symplectic form ω from $(\mathbb{C}P^1)^n$ to $\mathbb{C}^n/\mathbb{Z}^n$, we can consider holomorphic maps of punctured Riemann surfaces to $\mathbb{C}^n/\mathbb{Z}^n$ which have finite ω energy. These extend by the removable singularity theorem to holomorphic maps to $(\mathbb{C}P^1)^n$. Prescribing the homology class in $\mathbb{C}^n/\mathbb{Z}^n$ represented by a puncture corresponds to prescribing the order of poles and zeros at that puncture in $(\mathbb{C}P^1)^n$. This integrable case will be an important local model for the constructions that follow.

We could also consider different compactifications of $\mathbb{C}^n/\mathbb{Z}^n$ to toric manifolds. It is interesting to note that the moduli space of holomorphic curves in $\mathbb{C}^n/\mathbb{Z}^n$ tamed by any symplectic form pulled back from a compact toric manifold in this way is always the same, however we need to consider different compactifications of our moduli space depending on the compactification of $\mathbb{C}^n/\mathbb{Z}^n$.

The degeneration of complex structures has been studied in this algebraic setting by Grigory Mikhalkin and other tropical geometers, see for instance [6], [5]. This thesis can be considered as a smooth version of this tropical scheme for counting holomorphic curves, with an idea of what a perturbation theory of tropical curves would be. The smooth notion of a holomorphic graph however is sufficiently different from the more degenerate idea of a tropical curve that we are justified in using a different name.

Example 1.0.2.

We can give $\mathbb{T}^n$ the flat metric from $\mathbb{R}^n/\mathbb{Z}^n$. The unit cotangent bundle then is a contact manifold with a $\mathbb{T}^n$ symmetry. We can give the symplectization $\mathbb{T}^n \times (\mathbb{R}^n - \{0\})$ a cylindrical complex structure J given in coordinates $(x, y) \in (\mathbb{R}^n/\mathbb{Z}^n) \times (\mathbb{R}^n - \{0\})$ by

$$J\partial_{x_i} = |y|\,\partial_{y_i}$$

Example 1.0.3.

The symplectization of any compact three dimensional contact manifold with a $\mathbb{T}^2$ symmetry, where locally, either the $\mathbb{T}^2$ action is free or there is a neighborhood which has a contact form modeled on

$$d\theta_2 + r^2 d\theta_1$$

$$(r, \theta_1, \theta_2) \in D^2 \times S^1$$

and the $\mathbb{T}^2$ action is given by rotating θ_1 and θ_2.

1.1 The moduli space of holomorphic graphs

A holomorphic graph is a kind of decorated graph. The underlying graph has vertices which have associated to them model maps of a stable punctured Riemann surface into $\mathbb{C}^n/\mathbb{Z}^n$, and edges associated to trivial holomorphic cylinders, the ends of which may be attached to a vertex or free.

Recall that we have a torus fibration

$$\begin{array}{ccc} \mathbb{T}^n & \longrightarrow & \mathbb{T}^n \rtimes B^n \\ & & \downarrow \pi \\ & & B^n \end{array}$$

a degenerating family of complex structures

$$J^\epsilon \partial_x = \epsilon J \partial_x \text{ for } \partial_x \in \ker(d\pi)$$

and a symplectic form, ω which tames all J^ϵ holomorphic curves. Labeling the vector fields induced by the $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ action ∂_{x_i}, we also have a metric g^ϵ in which $\{\partial_{x_i}, J^\epsilon \partial_{x_i}\}$ gives an orthonormal frame, and a flat connection, ∇ which preserves this frame.

1.1.1 Trivial holomorphic cylinders

Given a point $p \in \mathbb{T}^n \rtimes B^n$ and a lattice direction $\alpha \in \mathbb{Z}^n \subset T\mathbb{T}^n$, consider the map $C_{p,\alpha}(\theta, t) : S^1 \times \mathbb{R} \longrightarrow \mathbb{T}^n \rtimes B^n$ given by

$$C_{p,\alpha}(\theta, t) = \exp_p(\theta\alpha + tJ\alpha)$$

Here exp denotes the exponentiation given by our flat connection ∇. $\alpha \in \mathbb{Z}^n$ is identified with a vertical tangent vector by our identification of $T_p(\mathbb{T}^n \rtimes B^n)$ with $\mathbb{C}^n$

given by the basis $\{\partial_{x_i}, J\partial_{x_i}\}$.

Note that $C_{p,\alpha}$ is J^ϵ holomorphic if $S^1 \times \mathbb{R}$ is given the complex structure $j\partial_\theta = \epsilon\partial_t$. Maps of this type and their images will be called *trivial holomorphic cylinders*. In chapter 3, we will see that for ϵ small enough, J^ϵ holomorphic maps of long cylinders with bounded energy must converge to some trivial holomorphic cylinder at a uniform rate which is exponential in the distance to the ends of the cylinder, thus we see that parts of J^ϵ holomorphic curves with bounded energy which are conformal to long cylinders can be approximated by trivial holomorphic cylinders. Actually, if we add in an averaging condition, each part of a holomorphic map conformal to a long cylinder will be approximated by a unique trivial holomorphic cylinder.

An edge of a holomorphic graph consists of a subset of a trivial holomorphic cylinder parametrized by the cylinder $\mathbb{R}/\mathbb{Z} \times (a, b) \subset \mathbb{R}/\mathbb{Z} \times \mathbb{R}$. An end of an edge can either go off to the edge of our manifold (which will require an edge of infinite length) or be attached to a vertex.

1.1.2 Vertex model curves

A vertex of a holomorphic graph will correspond to an equivalence class of pairs $[p, f]$ where p is a point in our manifold $\mathbb{T}^n \rtimes B^n$ and f is a model holomorphic map

$$f : S \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$$

of the type considered in example 1.0.1. In particular, f is a holomorphic map of a punctured Riemann surface which extends to a finite energy holomorphic map to $(\mathbb{C}P^1)^n$. f should be thought of as a map to the torus fiber containing p. Identifying the torus fiber containing p with $\mathbb{R}^n/\mathbb{Z}^n \subset \mathbb{C}^n/\mathbb{Z}^n$, we consider pairs $[p, f]$ up to the following equivalence relation:

$$[p, f] = [p + x_0, f - x_0 + Jx_1]$$

$$x_0, x_1 \in \mathbb{R}^n$$

Equivalently, $[p, f] = [\tilde{p}, \tilde{f}]$ if $\exp_p\left(\pi_{\mathbb{R}^n/\mathbb{Z}^n} f\right) = \exp_{\tilde{p}}\left(\pi_{\mathbb{R}^n/\mathbb{Z}^n} \tilde{f}\right)$

f is a holomorphic map of a stable Riemann surface with labeled punctures which extends to a holomorphic map to $\left(\mathbb{CP}^1\right)^n$. Each edge attached to a vertex corresponds to one of these punctures. At each puncture, f converges in the torus fiber containing p to some closed geodesic α with an orientation induced from S. The image of the end of the edge attached to this puncture must be α. In particular, as α also gives a class in $H_1(\mathbb{T}^n)$ and a corresponding lattice direction $[\alpha]$, we can choose a point $p_{i,\alpha} \in \alpha$, and then the edge attached to this puncture can be parametrized as

$$C_{p_{i,\alpha},[\alpha]}(\theta, t) = \exp_{p_{i,\alpha}}(\theta[\alpha] + tJ[\alpha])$$

Lemma 1.1.1. *Any holomorphic curve $f : S \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ from a punctured Riemann surface S which extends to a holomorphic map of the entire Riemann surface to $\left(\mathbb{CP}^1\right)^n$ is determined up to translation by $f_* : H_1(S) \longrightarrow H_1(\mathbb{C}^n/\mathbb{Z}^n)$ and the complex structure of S.*

Proof:

Consider one factor of the extension of f to $\left(\mathbb{CP}^1\right)^n$. All poles and zeroes of this map are determined by f_*, and hence this map is determined up to multiplication by a constant. Multiplication in $\mathbb{CP}^1$ corresponds to translation in $\mathbb{C}/\mathbb{Z}$.

□

Note that this tells us that the edges attached to a vertex and the complex structure of its model curve determine a model curve up to translation. Translations in the real or torus fiber directions will give us distinct model curves, however we want to quotient out by any translation in the imaginary direction. To this end we add a normalizing condition to our model curves. One that will come in useful later on is the following.

Given a Riemann surface with punctures, we have a way of partitioning it into subsets called 'vertices' and subsets called 'edges' which is similar to the partitioning of a Riemann surface into 'thick' and 'thin' given by the uniformisation theorem. This decomposition is discussed in appendix C. We use the notation $\mathfrak{E}$ to refer to a way of partitioning Riemann surfaces in this way. The important fact that we use now about this decomposition is that each puncture is surrounded by an 'edge' region which is conformal to $\mathbb{R}/\mathbb{Z} \times (0, \infty)$.

It is proved in section 3.3 that if f has finite energy, it must converge in these coordinates to some map

$$\lim_{t\to\infty} f(\theta, t) = \zeta + \theta\alpha + tJ\alpha \in \mathbb{C}^n/\mathbb{Z}^n$$

$$\zeta \in \mathbb{C}^n/\mathbb{Z}^n, \alpha \in \mathbb{Z}^n$$

Note that rotating our coordinates changes ζ by some multiple of α, but the imaginary part of ζ is well defined. A normalizing condition on our holomorphic model curves can them be

$$\text{Imaginary part of} \left(\sum_{\text{punctures}} \zeta \right) = 0$$

Example 1.1.2. *Trivalent Graphs*

An easy to deal with subset of holomorphic graphs consist of the trivalent graphs which have model curves at vertices consisting of three-punctured spheres. These graphs are simple because there is only one possible complex structure on a three-punctured sphere, and thus up to torus rotations only one possible model curve for a given set of homological data. For example, suppose we take as coordinates for our three punctured sphere $\mathbb{C} - \{0, 1\}$ with the third puncture at ∞. If the image of a loop around 0 is $\alpha \in \mathbb{Z}^n$ and 1 is $\beta \in \mathbb{Z}^n$, then we know that the image of a loop around ∞ is $-\alpha - \beta$. We can choose a normalization so that our model curve $f : \mathbb{C} - \{0, 1\} \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ is given by

$$f(z) = \alpha\frac{\log(z)}{2\pi i} + \beta\frac{\log(z-1)}{2\pi i}$$

The asymptotics of the real (torus) part of f are given by

$$\lim_{r\to 0} \pi_{\mathbb{R}^n/\mathbb{Z}^n} f_i(re^{2\pi i\theta}) = \alpha\theta + \frac{\beta}{2} \text{ at } 0$$

$$\lim_{r\to 0} \pi_{\mathbb{R}^n/\mathbb{Z}^n} f(1 + re^{2\pi i\theta}) = \beta\theta \text{ at } 1$$

$$\lim_{r\to \infty} \pi_{\mathbb{R}^n/\mathbb{Z}^n} f(re^{-2\pi i\theta}) = (-\alpha - \beta)\theta \text{ at } \infty$$

Note that these three geodesics in $\mathbb{T}^n$ will not necessarily have a point in common, so we can't make all the edges leaving a vertex leave from the same point p. We can parametrize the trivial holomorphic cylinders attached to each puncture as follows:

$$\text{attach to 0: } \exp_p(\frac{\beta}{2} + \theta\alpha + tJ\alpha)$$

$$\text{attach to 1: } \exp_p(\theta\beta + tJ\beta)$$

$$\text{attach to } \infty\text{: } \exp_p(\theta(-\alpha - \beta) + tJ(-\alpha - \beta))$$

Suppose we want to attach the other end of our edge attached to 1 to another model curve located at $\tilde{p}$ at its 1 or ∞ punctures. The image of a positively oriented loop around this puncture will need to be $-\beta$. The location of $\tilde{p}$ must be on this edge, so

$$\tilde{p} = \exp_p(\theta\beta + lJ\beta)$$

The only constraints on constructing these trivalent holomorphic graphs come from the correct placement of the ends of edges, arising from equations such as the one above. θ is the twist of this edge, and l its length. This can be thought of as changing the domain of the holomorphic graph. Note that we needed to make choices such as labeling the 0, 1 and ∞ punctures and giving a parametrization of their blow ups to define the twist θ of an edge, so an edge's twist is far from being a canonical coordinate.

Often, we can calculate invariants of the moduli space of holomorphic curves using the space of trivalent holomorphic graphs.

1.1.3 J^ϵ holomorphic $\mathfrak{E}$ graphs: first attempt, $\mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}}$

A J^ϵ holomorphic $\mathfrak{E}$ graph is an object that can be thought of as mimicking the behavior of a holomorphic map. In this section we will see a first attempt at saying what a J^ϵ holomorphic graph is. We will call the space of such objects $\mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}}$ and refer to the individual objects as 'graphs' $u \in \mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}}$. The reasons for choosing this notation will be clear in section 1.1.6, when we define the correct space $M^{\epsilon,[\bar\partial_0],\mathfrak{E}}$. Our description in this section will not be perfect, as $\mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}}$ will not be continuous when the combinatorics of edges and vertices changes. Nevertheless, $\mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}}$ can be used to calculate invariants of the moduli space of holomorphic curves given some transversality conditions and when it is possible to restrict the moduli space to regions where the combinatorics of edges and vertices is constant. The shortcomings of the description given in this section are remedied in section 1.1.6.

The $\mathfrak{E}$ refers to a way of partitioning Riemann surfaces into subsets which are called 'edges', conformal to $\mathbb{R}/\mathbb{Z} \times (a,b)$ and 'vertices' which are the connected components of the compliments of the edges. This choice of edges is similar to the thin parts of a Riemann surface when it is given the complete hyperbolic metric provided by the uniformisation theorem. $\mathfrak{E}$ must obey the axioms listed in appendix C. The important difference from the usual thick-thin decomposition given by the uniformisation theorem is that the decomposition must be preserved by surgeries on edges which change the length of an edge. In particular, if we take a vertex region V and replace each $\mathfrak{E}$ edge region surrounding it, $\mathbb{R}/\mathbb{Z} \times (0,R)$ with $\mathbb{R}/\mathbb{Z} \times (0,\infty)$, we get a Riemann surface with punctures S_V. The $\mathfrak{E}$ edge regions of S_V surrounding these punctures will consist of these cylinders $\mathbb{R}/\mathbb{Z} \times (0,\infty)$. Graphs in $\mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}}$ obey the restriction that all model curves have domains S_V of this type.

We can associate a domain Riemann surface S to our graph $u \in \mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}}$, so that every $\mathfrak{E}$ vertex region $V \subset S$ corresponds to a vertex of u with domain S_V, and every $\mathfrak{E}$ edge region of S corresponds to an edge of u. An edge of a graph $u \in \mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}}$ is then part of a trivial holomorphic cylinder parametrized by a $\mathfrak{E}$ edge region of S.

The connections between vertex model curves and edges are reflected by the connections between the vertex and edge parts of the domain. Parameterizing the edge part of a holomorphic model curve $[p, f]$ surrounding a puncture by $\mathbb{R}/\mathbb{Z} \times (0, \infty)$ the domain of the edge attached to this puncture is considered as a subset of this, $\mathbb{R}/\mathbb{Z} \times (0, R)$. Then the trivial holomorphic cylinder this edge parametrizes is given by

$$C(\theta, t) = \exp_{\exp_p \zeta}(\theta\alpha + tJ^\epsilon \alpha)$$

$$\text{where } \lim_{t\to\infty} f(\theta, t) = \zeta + \theta\alpha + tJ\alpha$$

The map exp should be understood by the identification we have of $T(\mathbb{T}^n \rtimes B^n)$, J^ϵ with $\mathbb{C}^n$.

Note that apart from the stupid holomorphic curves that map entirely to a point, every vertex model curve $[p, f]$ has at least one puncture. f is then determined by the complex structure of the model curve and the location of one edge relative to p. This means that the equivalence class of $[p, f]$ is determined by any edge attached to it and the complex structure of the domain S.

The above observation motivates putting a topology on $\mathcal{M}^{\epsilon,\bar{\partial}_0,\mathfrak{E}}$ that keeps track of the complex structure of the domain surface and the trivial holomorphic cylinders at its edges. A sequence of graphs $\{u_i\} \subset \mathcal{M}^{\epsilon,\bar{\partial}_0,\mathfrak{E}}$ is said to converge to a given graph u_∞ if the domain surfaces converge in Delinge-Mumford space, and for each edge of u_∞, the corresponding sequence of trivial holomorphic cylinders from u_i converge. (The exact choice of what we mean by 'converge' for a trivial holomorphic cylinder depends on what compactification we wish to put on a space of holomorphic maps.)

The problem with $\mathcal{M}^{\epsilon,\bar{\partial}_0,\mathfrak{E}}$ is that it will not be continuous when the combinatorics of the $\mathfrak{E}$ edge-vertex decomposition jumps. This is taken care of by a smoothing procedure in section 1.1.6. For now the space of J^ϵ holomorphic graphs $\mathcal{M}^{\epsilon,[\bar{\partial}_0],\mathfrak{E}}$ should be thought of as being like $\mathcal{M}^{\epsilon,\bar{\partial}_0,\mathfrak{E}}$ except with some sort of interpolation to make the moduli space smooth when the combinatorics of $\mathfrak{E}$ edge markings changes.

1.1.4 J^ϵ quasi holomorphic graphs

The space of J^ϵ holomorphic $\mathfrak{E}$ graphs, $\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}}$ can be used to calculate invariants of the moduli space of holomorphic curves when some transversality conditions are met. To get a good perturbation theory for these objects, we define the space of J^ϵ quasi holomorphic $\mathfrak{E}$ graphs, $\mathcal{Q}^{\epsilon,\mathfrak{E}}$. Each connected component of $\mathcal{Q}^{\epsilon,\mathfrak{E}}$ will be finite dimensional, and we can view $\mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}}$ and $\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}}$, the space of J^ϵ holomorphic $\mathfrak{E}$ graphs, as embedded in this space. For a given genus and energy, we can choose an ϵ small enough and a system of vertex and edge decompositions, $\mathfrak{E}$ so that the space of J^ϵ holomorphic curves can also be considered as embedded in $\mathcal{Q}^{\epsilon,\mathfrak{E}}$ and cobordant to $\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}}$.

As with the graphs in $\mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}}$ considered in the previous section, the data for a J^ϵ quasi holomorphic $\mathfrak{E}$ graph $u \in \mathcal{Q}^{\epsilon,\mathfrak{E}}$ will consist of a Riemann surface S with trivial holomorphic maps on the edges, and equivalence classes of vertex model curves $[p, f]$ associated to vertex regions, joined together in the manner suggested by S.

Model maps corresponding to a vertex region V now will be continuous maps $f : S_V \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$. S_V denotes the Riemann surface obtained by replacing the edges $\mathbb{R}/\mathbb{Z} \times (0, R)$ surrounding V with semi infinite cylinders $\mathbb{R}/\mathbb{Z} \times (0, \infty)$. f must have the property that in these cylindrical coordinates

$$\lim_{t\to\infty} |f(\theta, t) - \zeta - \theta\alpha - tJ\alpha|\, e^t = 0$$

$$\alpha \in \mathbb{Z}^n, \zeta \in \mathbb{C}^n/\mathbb{Z}^n$$

$$\text{Imaginary part of} \left(\sum_{punctures} \zeta_i \right) = 0$$

Considering the edge part attached to this boundary as the subset $\mathbb{R}/\mathbb{Z} \times (0, R) \subset \mathbb{R}/\mathbb{Z} \times (0, \infty)$, the trivial holomorphic edge attached to this boundary is given by

$$C(\theta, t) = \exp_{\exp_p \zeta}(\theta\alpha + tJ^\epsilon\alpha)$$

Here $\exp_p \zeta$ is to be understood by the identification of $(T_p(\mathbb{T}^n \rtimes B^n), J^\epsilon)$ with $\mathbb{C}^n$

We will consider these model maps up to the following equivalence relation:

$$[p, f] = [p + x, f - x + g]$$

$$g : S_V \longrightarrow \mathbb{C}^n, x \in \mathbb{R}^n/\mathbb{Z}^n$$

$$g = 0 \text{ at punctures}$$

Said in words, the equivalence relation equates isotopy classes of maps with a fixed set of asymptotics modulo an equivariant torus action. Coordinates on $\mathcal{Q}^{\epsilon,\mathfrak{E}}$ are locally given by keeping track of the trivial holomorphic cylinders from edges and the complex structure of the domain S. The extra information carried by the vertices is discrete. Note that the dimension of $\mathcal{Q}^{\epsilon,\mathfrak{E}}$ can change when we change the complex structure of the domain S so that edge regions are created or disappear. We will discuss this in more detail in section 1.1.5.

We now give a notion of what it means for a family of quasi holomorphic graphs to be continuous or smooth. To do this, we associate to each quasi holomorphic graph in a family a map from its domain S to $\mathbb{T}^n \rtimes B^n$ called a gluing. A family of quasi holomorphic graphs is said to be continuous (or smooth) if we can choose a continuous (or smooth) family of gluings. (For an example of a definition of continuous for a family of maps with changing domains, see Definition 2.11.1).

A gluing of a holomorphic graph u with domain S is a smooth map

$$f : S \longrightarrow \mathbb{T}^n \rtimes B^n$$

so that f restricted to the edge regions of S is given by the trivial holomorphic cylinders associated to edges of u, and f restricted to each vertex region is given by $\exp_{p_i} f_i$, where $[p_i, f_i]$ is a choice of model curve for the corresponding vertex of u which is a trivial holomorphic cylinder on all edge regions. The energy of a holomorphic graph, $E_\omega(u)$ is defined to be the energy of a gluing $E_\omega(f)$. Note that this doesn't depend on our choice of gluing. If we want to consider invariants that come from placing restrictions on the moduli space of holomorphic curves such as passing through a particular point, or being tangent to a plane, we place the corresponding conditions on a choice of gluings of quasi holomorphic graphs. We will show in chapter 4 that a choice of gluing of $\mathcal{M}^{\epsilon,[\bar{\partial}_0],\mathfrak{E}}_{g,k,E}$ gives the moduli space of solutions to a perturbed $\bar{\partial}$ equation.

We can define a bundle $\mathcal{E}$ over $\mathcal{Q}^{\epsilon}$.

$$\begin{array}{ccc} \mathcal{H}_{\mathfrak{E}}(S,\mathbb{C}^n) & \longrightarrow & \mathcal{E} \\ & & \downarrow \\ & & \mathcal{Q}^{\epsilon,\mathfrak{E}} \end{array}$$

The fiber $\mathcal{H}_{\mathfrak{E}}(S,\mathbb{C}^n)$ is a vector space depending on the vertex regions $\{V_i\}$ of S defined as follows: Denote by S_{V_i} the Riemann surface with punctures created by attaching semi infinite cylinders to the vertex region V_i. Now, let the space of holomorphic one forms on S_{V_i} with poles of order at most 1 at punctures be $\Lambda^{1,0}(S_{V_i})$. Then

$$\mathcal{H}_{\mathfrak{E}}(S,\mathbb{C}^n) := \oplus_i \hom_{\mathbb{C}}(\Lambda^{1,0}(S_{V_i}),\mathbb{C}^n)$$

Note that if we change the complex structure of the domain S so that one edge region disappears, consolidating the vertex regions that it joins, the dimensions of $\mathcal{H}_{\mathfrak{E}}(S,\mathbb{C}^n)$ and $\mathcal{Q}^{\epsilon,\mathfrak{E}}$ drop by the same amount.

We now define a section $\bar{\partial}_0$ of this bundle, which should be thought of as an approximate measure of how close to being holomorphic a quasi holomorphic graph is.

First, note that given a smooth map $v : S_V \longrightarrow \mathbb{C}^n$ which vanishes at punctures of S_V, the following identity holds:

$$\int_{S_V} \bar{\partial} v \wedge \theta = \int_{S_V} d(v\theta) = 0 \text{ for } \theta \in \Lambda^{1,0}(S_V)$$

Choosing a of model curve $[p,f]$ for a vertex of a quasi holomorphic graph defines a map $\Lambda^{1,0}(S_V) \longrightarrow \mathbb{C}^n$ by

$$\theta \mapsto \int_{S_V} \bar{\partial} f \wedge \theta$$

This doesn't depend on the choice of representatives $[p, f]$ for our model curve and therefore doing the same for all vertices defines a section

$$\bar{\partial}_0 : \mathcal{Q}^{\epsilon,\mathfrak{E}} \longrightarrow \mathcal{E}$$

We will define a gluing procedure in section 2 which will produce from a quasi holomorphic graph u with domain S a map $\mathcal{G}(u) : S \longrightarrow \mathbb{T}^n \rtimes B^n$, which will satisfy

$$\left\|\bar{\partial}\mathcal{G}(u) - \bar{\partial}_0 u\right\| \approx 0$$

Here the above must be understood after an identification of $\mathcal{H}_{\mathfrak{E}}(S, \mathbb{C}^n)$ with a subspace of the Banach space $\mathcal{B}$ which is the target of the $\bar{\partial}$ operator.

There is a family of edge-vertex decompositions $\mathfrak{E}_R$, the edge regions of which consist of the cylinders a distance R into the interior of $\mathfrak{E}$ edge regions. We will show that for R large enough and ϵ small enough (dependent on a choice of energy bound E and genus), the moduli space of bounded energy J^ϵ holomorphic curves with genus g can be viewed as embedded in a subset of $Q^{\epsilon,\mathfrak{E}_R}$ where $\bar{\partial}_0$ is within a neighborhood of 0. Putting a metric on $\mathcal{H}_{\mathfrak{E}_R}(S, \mathbb{C}^n)$, we can choose this neighborhood to be

$$\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} := \{u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R} : \left|\bar{\partial}_0 u\right| < 1, E_\omega(u) \leq E, \text{ genus} = g, k \text{ punctures}\}$$

We will show that on this space of bounded energy approximately holomorphic J^ϵ quasi holomorphic graphs, there exists a section

$$\bar{\partial}_1 : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{E}$$

so that the moduli space of J^ϵ holomorphic curves corresponds to the intersection of

$\bar{\partial}_1$ with the zero section, and

$$\left|\bar{\partial}_1 u - \bar{\partial}_0 u\right| \leq \frac{1}{2}$$

This section is defined by using an iteration procedure to obtain a map

$$\mathcal{G}_\infty : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \text{ maps to } \mathbb{T}^n \rtimes B^n$$

satisfying

$$\bar{\partial}\mathcal{G}_\infty(u) \in \mathcal{H}_{\mathfrak{E}_R}(S, \mathbb{C}^n) \subset \mathcal{B}$$

$\bar{\partial}_1$ is then defined by

$$\bar{\partial}_1(u) = \bar{\partial}\mathcal{G}_\infty(u) \in \mathcal{H}_{\mathfrak{E}_R}(S, \mathbb{C}^n)$$

If $\mathcal{E} \longrightarrow \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ was a vector bundle over a manifold, and we knew that $\bar{\partial}_1$ was a continuous section, we would then have enough information to identify invariants of the moduli space of holomorphic curves. This is often the case in moduli spaces that we are interested in, but at other times $\mathcal{E} \longrightarrow \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ can jump dimensions, so we have to examine the behavior near this dimension jumping.

1.1.5 Local stabilization

We want to have some idea of what it would mean to have a continuous section of our bundle $\mathcal{E} \longrightarrow \mathcal{Q}^{\epsilon,\mathfrak{E}}$. The dimensions of the fiber and base can jump when we change the complex structure of the domain S so that an edge region appears or disappears.

Given a decomposition into edge and vertex regions $\mathfrak{E}$ satisfying the assumptions listed in appendix C, we can define another decomposition $\mathfrak{E}_R$ as follows: The edge regions in $\mathfrak{E}_R$ are in one to one correspondence with the edges in $\mathfrak{E}$ which are longer that $2R$. In cylindrical coordinates on the edges of $\mathfrak{E}$, they consist of the subsets

$$\mathbb{R}/\mathbb{Z} \times (a + R, b - R) \subset \mathbb{R}/\mathbb{Z} \times (a, b)$$

This new decomposition will obey the axioms listed in appendix C for all $R \geq 0$. We can make choices so that our gluing procedure will define inclusions

$$\mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n) \subset \mathcal{H}_{\mathfrak{E}}(S,\mathbb{C}^n) \subset \mathcal{B}$$

where $\mathcal{B}$ indicates the Banach space which is the target of the linearized $\bar\partial$ operator. This inclusion is an isomorphism when the edges of $\mathfrak{E}_R$ are in one to one correspondence with the edges of $\mathfrak{E}$, and otherwise has complex codimension equal to n times the difference in the number of edges. Note that in the case that $\dim(\mathcal{H}_{\mathfrak{E}}(S,\mathbb{C}^n)) = \dim(\mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n))$, the quasi holomorphic graphs in $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}$ are in one to one correspondence with the quasi holomorphic graphs in $\mathcal{Q}^{\epsilon,\mathfrak{E}}$.

Definition 1.1.3. *A section $s : U \subset \mathcal{Q}^{\epsilon,\mathfrak{E}} \longrightarrow \mathcal{E}$ is continuous (smooth, transverse to the zero section respectively) if around every graph $u_0 \in U$ with domain S_0, there exists a neighborhood $U_0 \subset \mathcal{Q}^{\epsilon,\mathfrak{E}}$ of u_0 and some $R \geq 0$ so that*

1. $$\mathcal{H}_{\mathfrak{E}_R}(S_0,\mathbb{C}^n) = \mathcal{H}_{\mathfrak{E}}(S_0,\mathbb{C}^n)$$

2. *The dimension of $\mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n)$ doesn't jump in U_0.*

3. *The subset*

$$U_0^{\mathfrak{E}_R} := \{u \in U_0 \text{ so that } s(u) \subset \mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n) \subset \mathcal{H}_{\mathfrak{E}}(S,\mathbb{C}^n)\}$$

is transversely cut out and homeomorphic (or diffeomorphic) to an open neighborhood of u_0 considered as a graph in $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}$. This homeomorphism should preserve the domain Riemann surface. Note that this means that $U_0^{\mathfrak{E}_R}$ will not exhibit dimension jumping behavior.

4. *The restriction of s to $U_0^{\mathfrak{E}_R}$,*

$$s : U_0^{\mathfrak{E}_R} \longrightarrow \mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n)$$

is continuous (smooth, transverse to the zero section respectively).

As there will be no bubbling of domains, we can take a finite cover of each component of Deligne Mumford space to get rid of finite automorphisms, and not worry about orbifolds and multisections.

Note the transverse intersection of a section $s : \mathcal{Q}^{\epsilon,\mathfrak{E}} \longrightarrow \mathcal{E}$ with the zero section will be a manifold, and that a generic smooth section is transverse to the zero section. Also, a generic family of sections will give a cobordism.

Theorem 1.1.4. *For R large enough, and ϵ small enough dependent on R,*

$$\bar{\partial}_1 : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{E}$$

is a continuous section. The intersection of $\bar{\partial}_1$ with the zero set is homeomorphic to the moduli space of holomorphic curves.

Chapters 2 and 3 are devoted to the proof of the above theorem.

1.1.6 Moduli space of J^ϵ holomorphic graphs

What we want to be able to say is that $\bar{\partial}_1$ is a perturbation of $\bar{\partial}_0$ and that their intersections with the zero section are cobordant. The problem with this is that $\bar{\partial}_0$ is not continuous where dimensions jump. To remedy this, we just need to smooth $\bar{\partial}_0$ over where this dimension jumping happens. The following lemma helps us do this explicitly.

Lemma 1.1.5. *For ϵ small enough, dependent on E, g, k and R', there is a well defined projection*

$$\pi_{R,R'} : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{Q}^{\epsilon,\mathfrak{E}_{R'}}$$

for all $0 \leq R \leq R'$.

This preserves the domain Riemann surface, and satisfies

$$\bar{\partial}_0 \circ \pi_{R,R'} = \bar{\partial}_0$$

when $\mathcal{H}_{\mathfrak{E}_R}(S, \mathbb{C}^n) = \mathcal{H}_{\mathfrak{E}_{R'}}(S, \mathbb{C}^n)$.

Moreover,

$$\pi_{R,R'} : \{u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \text{ so that } \bar\partial_0 u \in \mathcal{H}_{\mathfrak{E}_{R'}}(S,\mathbb{C}^n) \subset \mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n)\} \longrightarrow \mathcal{Q}^{\epsilon,\mathfrak{E}_{R'}}$$

is a diffeomorphism onto its image.

This is constructed in appendix C.3.

Now define the section

$$[\bar\partial_0] := \int_0^1 \bar\partial_0 \circ \pi_{R,(R+t)} dt$$

Theorem 1.1.6. *For R large enough and ϵ small enough,*

$$[\bar\partial_0] : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{E}$$

is a C^1 smooth section hopotopic to $\bar\partial_1$. If $[\bar\partial_0]$ and $\bar\partial_1$ are transverse to the zero section, the intersection of the homotopy with the zero section defines a cobordism contained in the interior of $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ between the intersection of $[\bar\partial_0]$ with the zero section and the moduli space of holomorphic curves identified with the intersection of $\bar\partial_1$ with the zero section.

The homotopy is given by the following family

$$[\bar\partial_s] := \int_0^1 ((1-s)\bar\partial_0 + s\bar\partial_1) \circ \pi_{R,(R+t)} dt \text{ for } s \in [0,1]$$

composed with a homotopy between $[\bar\partial_1]$ and $\bar\partial_1$. This theorem is proved in chapter 4.

We call intersection of $[\bar\partial_0]$ with the zero section the moduli space of J^ϵ holomorphic graphs, $\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}_R}_{g,k,E}$. This moduli space can be used instead of the moduli space of holomorphic curves. By taking a smooth gluing of $\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}_R}_{g,k,E}$, it can be regarded as the moduli space of solutions to some perturbed $\bar\partial$ equation. This is proved in chapter 4.

Chapter 2

Gluing

2.1 Introduction

The purpose of this chapter is to use quasi holomorphic graphs to construct J^ϵ holomorphic curves for ϵ small enough. The plan is to glue together the model holomorphic curves at vertices of the graph using the positioning of the graph as a guide. This will result in approximately J^ϵ holomorphic curves. We will see that the linearization of a weakened $\bar{\partial}$ equation at these approximately holomorphic curves is surjective, and use this to prove that there are close by solutions. The structure of the moduli space J^ϵ holomorphic curves within this moduli space can then be found by purely topological means.

Recall that we have a torus fibration

$$\begin{array}{c} \mathbb{T}^n \longrightarrow \mathbb{T}^n \rtimes B^n \\ \downarrow \pi \\ B^n \end{array}$$

and a degenerating family of complex structures

$$J^\epsilon \partial_x = \epsilon J \partial_x \text{ for } \partial_x \in \ker(d\pi)$$

Giving $\mathbb{T}^n$ coordinates $x \in \mathbb{R}^n/\mathbb{Z}^n$, there is a metric g^ϵ in which $\{\partial_{x_i}, J^\epsilon\partial_{x_i}\}$ gives an orthonormal frame, and a flat connection, ∇ which preserves this frame. ∇ is not torsion free, and most of the estimates for the behavior of holomorphic curves in this chapter involve its torsion tensor.

$$\mathbf{T}(v,w) := \nabla_v w - \nabla_w v - [v,w]$$

For this reason the major technical assumption used in this chapter is that $|\mathbf{T}|$ and $|\nabla\mathbf{T}|$ are bounded.

$$\|\mathbf{T}\|_\infty + \|\nabla\mathbf{T}\|_\infty < c$$

This will automatically be true for reasonable choices of J on compact manifolds or manifolds with cylindrical ends. Note that

$$\begin{aligned} \|\mathbf{T}\|^\epsilon_\infty &< \epsilon c \\ \|\nabla\mathbf{T}\|^\epsilon_\infty &< \epsilon^2 c \end{aligned} \tag{2.1}$$

Here the superscript ϵ denotes the norm using g^ϵ. For convenience, the ϵ will also usually be omitted from notation in what follows.

2.2 The linearized $\bar\partial$ operator, $D_{\bar\partial,u}$

We are interested in holomorphic maps

$$u : S \longrightarrow \mathbb{T}^n \rtimes B^n$$

Here S denotes a Riemann surface with complex structure j. This map is holomorphic if

$$\bar\partial(u) := \frac{1}{2}(du + J\circ du\circ j) = 0$$

We need an expression for the linearization of $\bar{\partial}$ at a map u:

$$D_{\bar{\partial},u} : \Omega^0(S, u^*T(\mathbb{T}^n \rtimes B^n)) \longrightarrow \Omega^{0,1}(S, u^*T(\mathbb{T}^n \rtimes B^n))$$

$D_{\bar{\partial},u}$ maps from sections of the bundle over S consisting of the pullback of the tangent bundle of $\mathbb{T}^n \rtimes B^n$ to antiholomorphic one-forms with values in this bundle. As we have a canonical holomorphic frame for $T(\mathbb{T}^n \rtimes B^n)$ we really have

$$D_{\bar{\partial},u} : \Omega^0(S, \mathbb{C}^n) \longrightarrow \Omega^{0,1}(S, \mathbb{C}^n)$$

Note that this is the linearization of $\bar{\partial}$ restricted to curves with a fixed complex structure. The variations from changing complex structures will be included at a later point.

The trivial parallel transport provided by ∇ makes $D_{\bar{\partial},u}$ relatively easy to define.

$$D_{\bar{\partial},u}(\xi) := \left.\frac{d}{dt}\right|_{t=0} \left(\bar{\partial} \exp_u(t\xi)\right)$$

Lemma 2.2.1.

$$\left.\frac{d}{dt}\right|_{t=0} (d \exp_u(t\xi)) = \nabla\xi + \mathbf{T}(\xi, du)$$

Proof:

Define $U : \mathbb{R} \times S \longrightarrow \mathbb{T}^n \rtimes B^n$ by

$$U(t, z) = \exp_{u(z)} t\xi$$

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} (d \exp_u(t\xi)) &= \frac{\partial}{\partial t} \left(\partial_z U\right)(0, z) \\
&= \left(\nabla_\xi \partial_z U\right)(0, z) \\
&= \nabla_{\partial_z U(0,z)}\xi + \mathbf{T}\left(\xi, \partial_z U(0, z)\right) \\
&= \nabla\xi + \mathbf{T}(\xi, du)
\end{aligned}$$

□

Lemma 2.2.1 allows us to compute an expression for $D_{\bar{\partial},u}$

Proposition 2.2.2.

$$D_{\bar{\partial},u}(\xi) = \frac{1}{2}(\nabla\xi + J \circ \nabla\xi \circ j) + \frac{1}{2}\left(\mathbf{T}(\xi, du) + J \circ \mathbf{T}(\xi, du \circ j)\right)$$

Note that in the above expression for $D_{\bar{\partial},u}$, the only terms which change when we change our map u are du and the torsion tensor, $\mathbf{T}$, which depends on position. Thus we have

Proposition 2.2.3. *If u_1 and u_2 are two maps $S \longrightarrow \mathbb{T}^n \rtimes B^n$, then*

$$\left|\left(D_{\bar{\partial},u_1} - D_{\bar{\partial},u_2}\right)(\xi)\right| \leq \left(|\mathbf{T}_{u_1} - \mathbf{T}_{u_2}|\,|du_1| + |\mathbf{T}_{u_2}|\,|du_1 - du_2|\right)|\xi|$$

Note that in the above proposition, $|du|$ indicates the size of du in the g^ϵ metric.

In order to get estimates comparing $D_{\bar{\partial},u}$ and $D_{\bar{\partial},exp_u\phi}$, it is necessary to estimate $|d(\exp_u\phi) - du|$. In the proofs below, $\exp_u\phi$ will often be abbreviated to u_ϕ

Lemma 2.2.4.

$$|d(\exp_u\phi) - du - \nabla\phi| \leq \left(\|\mathbf{T}\|_\infty |\phi|\,|du| + |\nabla\phi|\right)e^{\|\mathbf{T}\|_\infty|\phi|} - |\nabla\phi|$$

In particular, if $\|\mathbf{T}\|_\infty |\phi| < 1$, then

$$|d(\exp_u\phi) - du| \leq 3\,\|\mathbf{T}\|_\infty |\phi|\left(|du| + |\nabla\phi|\right)$$

Proof:

$$|du_\phi - du - \nabla\phi| \leq \int_0^1 \left|\frac{d}{dt}(du_{t\phi}) - \nabla\phi\right| dt \tag{2.2}$$

$$\begin{aligned}\left|\frac{d}{dt}(du_{t\phi}) - \nabla\phi\right| &= |\mathbf{T}(\phi, du_{t\phi})| \\ &\leq \|\mathbf{T}\|_\infty |\phi|\,|du_{t\phi}|\end{aligned} \tag{2.3}$$

$$\frac{d}{dt}|du_{t\phi}| \leq |\nabla\phi| + \|\mathbf{T}\|_\infty |\phi|\,|du_{t\phi}| \tag{2.4}$$

Integrating 2.4 for $\|\mathbf{T}\|_\infty |\phi| > 0$ gives

$$|du_{t\phi}| \leq \left(|du| + \frac{|\nabla\phi|}{\|\mathbf{T}\|_\infty |\phi|}\right) e^{t\|\mathbf{T}\|_\infty |\phi|} - \frac{|\nabla\phi|}{\|\mathbf{T}\|_\infty |\phi|} \tag{2.5}$$

Substituting the estimate of $|du_\phi|$ from 2.5 into 2.3 and 2.2 gives the desired result,

$$|du_\phi - du| \leq (\|\mathbf{T}\|_\infty |\phi| \, |du| + |\nabla\phi|) \, e^{\|\mathbf{T}\|_\infty |\phi|} - |\nabla\phi|$$

□

Lemma 2.2.4 and Proposition 2.2.3 give the following pointwise estimate for the behavior of $D_{\bar{\partial},u}$. This will be used in Proposition 2.3.3 to bound $\left\|D_{\bar{\partial},u}(\xi) - D_{\bar{\partial},u_\phi}(\xi)\right\|$ after appropriate norms for ϕ and ξ (which dominate the L^∞ norm) have been chosen.

Proposition 2.2.5. *If* $\|\mathbf{T}\|_\infty |\phi| < 1$*, then*

$$\left|D_{\bar{\partial},u}(\xi) - D_{\bar{\partial},u_\phi}(\xi)\right| \leq \left(|\phi| \, \|\nabla\mathbf{T}\|_\infty |du| + 3 \, \|\mathbf{T}\|_\infty^2 |\phi| \, (|du| + |\nabla\phi|)\right) |\xi|$$

2.3 Banach norms

It would be nice to have a Banach manifold structure on the space of maps we are dealing with so that $\bar{\partial}$ has a continuous derivative $D_{\bar{\partial},u}$. This section describes a local Banach space structure on a finite codimension subset of maps close to a particular map u. Later, in section 2.7, we will be able to describe a Banach manifold structure on an open set of maps which contains the holomorphic curves. Consider the set of maps

$$u_\phi := \exp_u \phi$$

$$u : S \longrightarrow \mathbb{T}^n \rtimes B^n$$

$$\phi : S \longrightarrow u^*T(\mathbb{T}^n \rtimes B^n) = \mathbb{C}^n$$

Recall that the linearization of $\bar{\partial}$ at u is given by

$$\begin{aligned} D_{\bar{\partial},u}(\xi) &= \frac{1}{2}(\nabla\xi + J \circ \nabla\xi \circ j) + \frac{1}{2}\left(\mathbf{T}(\xi, du) + J \circ \mathbf{T}(\xi, du \circ j)\right) \\ &= \bar{\partial}\xi + \frac{1}{2}\left(\mathbf{T}(\xi, du) + J \circ \mathbf{T}(\xi, du \circ j)\right) \end{aligned}$$

The objective now is to choose a norm for ξ and $D_{\bar{\partial},u}(\xi)$ so that $D_{\bar{\partial},u}$ is continuous. First we choose a metric on the Riemann surface S that satisfies the assumptions listed in appendix C.2. The important properties of this metric are that it is in the conformal class defined by the complex structure on S, and it gives the regions marked as edges the standard metric on $\mathbb{R}/\mathbb{Z} \times (a, b)$. (For a discussion of the partitioning of S into vertex and edge regions, see appendix C)

One important feature of such a metric on S is that for ϵ small enough, bounded energy J^ϵ holomorphic curves will have uniformly bounded derivatives.

The Banach spaces that we will use will be Sobolev spaces with exponential weights $L^{p,\delta}_k$ with the norm

$$(\|\xi\|_{k,p,\delta})^p = \int_S w^\delta \sum_{|\alpha| \le k} |D_\alpha \xi|^p$$

The weight w on each cylindrical part is given by the exponential of the distance to the edge of the cylinder, and 1 elsewhere. For $D_{\bar{\partial},u}$ to be well behaved, we will choose $0 < \delta < \frac{1}{2}$.

We will consider $D_{\bar{\partial},u}$ as a map

$$D_{\bar{\partial},u} : L^{p,\delta}_1(\Omega^0(S, \mathbb{C}^n)) \longrightarrow L^{p,\delta}(\Omega^{0,1}(S, \mathbb{C}^n))$$

The following is a standard Sobolev embedding lemma:

Lemma 2.3.1. *If $p > 2$, then on the unit disk the inclusion of L^p_1 into C^0 is compact.*

This implies a similar fact for S with the metric we have chosen:

Lemma 2.3.2. *For $2 < p < \infty$, there exists a constant c_s so that*

$$c_s \|\xi\|_{1,p,\delta} \ge \left\|w^\delta \xi\right\|_\infty \ge \|\xi\|_\infty$$

This implies the following proposition that tells us that $D_{\bar{\partial},u_\phi}$ is well behaved for $\phi \in L^{p,\delta}_1$ small if $|du|$ is bounded.

Proposition 2.3.3. *If* $\|\mathbf{T}\|_\infty (c_s \|\phi\|_{1,p,\delta} + 1) \leq 1$.

$$\left\| D_{\bar{\partial},u} - D_{\bar{\partial},u_\phi} \right\| \leq c_s \left(\|\nabla \mathbf{T}\|_\infty + 3 \|\mathbf{T}\|_\infty \right) \|du\|_\infty \|\phi\|_{1,p,\delta}$$

Moreover,

$$(D_{\bar{\partial},u} - D_{\bar{\partial},u_\phi}) : L_1^{p,\delta}(\Omega^0(S, \mathbb{C}^n)) \longrightarrow L^{p,\delta}(\Omega^{0,1}(S, \mathbb{C}^n))$$

is compact.

Proof:

Lemma 2.3.2 tells us that $\|\mathbf{T}\|_\infty |\phi| < 1$ so we can apply Proposition 2.2.5 which tells us

$$\left| D_{\bar{\partial},u}(\xi) - D_{\bar{\partial},u_\phi}(\xi) \right| \leq \big(|\phi| \, \|\nabla \mathbf{T}\|_\infty |du| + 3 \|\mathbf{T}\|_\infty^2 |\phi| \left(|du| + |\nabla \phi| \right) \big) |\xi|$$

We integrate this to obtain

$$\begin{aligned} \left\| (D_{\bar{\partial},u} - D_{\bar{\partial},u_\phi})(\xi) \right\|_{p,\delta} &\leq \left\| |\phi| \, \|\nabla \mathbf{T}\|_\infty |du| + 3 \|\mathbf{T}\|_\infty^2 |\phi| \left(|du| + |\nabla \phi| \right) \right\|_{p,\delta} \|\xi\|_\infty \\ &\leq \Big(\left(\|\nabla \mathbf{T}\|_\infty + 3 \|\mathbf{T}\|_\infty^2 \right) \|du\|_\infty \|\phi\|_{p,\delta} \\ &\quad + 3 \|\mathbf{T}\|_\infty^2 \|du\|_\infty \|\phi\|_\infty \|\nabla \phi\|_{p,\delta} \Big) \|\xi\|_\infty \\ &\leq \left(\|\nabla \mathbf{T}\|_\infty + 3 \|\mathbf{T}\|_\infty \right) \|du\|_\infty \|\phi\|_{1,p,\delta} \|\xi\|_\infty \end{aligned}$$

Thus we see that $(D_{\bar{\partial},u} - D_{\bar{\partial},u_\phi}) : L^\infty \longrightarrow L^{p,\delta}$ is bounded. Therefore composing with the compact inclusion $L_1^{p,\delta} \hookrightarrow L^\infty$ from Lemma 2.3.2 gives a compact map.

□

Now we have seen that $D_{\bar{\partial},u} : L_1^{p,\delta} \longrightarrow L^{p,\delta}$ is quite well behaved near maps with bounded derivatives.

2.4 Model left inverse Q

Recall that as we sent ϵ to 0, the torsion tensor becomes smaller and $D_{\bar{\partial},u}$ converges to the linear $\bar{\partial}$ operator. Consider the integrable problem

$$f : S \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$$

where S is a Riemann surface with punctures and f extends to a continuous map to $(\mathbb{C}P^1)^n$. Consider the linearized $\bar{\partial}$ operator

$$D_{\bar{\partial},f} := \bar{\partial} : L_1^{p,\delta}(S, \mathbb{C}^n) \longrightarrow L^{p,\delta}(\Omega^{0,1}(S, \mathbb{C}^n))$$

It is a standard result that $\bar{\partial}$ is Fredholm for the above spaces if $\delta \notin 2\pi\mathbb{Z}$ (see [4]). In particular, we will be interested in the case where $0 < \delta < \frac{1}{2}$.

Assume that our domain S has at least one puncture. Then a section $\phi \in L_1^{p,\delta}(S, \mathbb{C}^n)$ must approach 0 at that puncture. The kernel of $\bar{\partial}$ consists of holomorphic maps $S \longrightarrow \mathbb{C}^n$ that vanish at punctures. Therefore $\bar{\partial}$ restricted to $L_1^{p,\delta}$ is injective. As $\bar{\partial}$ is Fredholm, this means that there must exist a bounded left inverse

$$Q : L^{p,\delta}(\Omega^{0,1}(S, \mathbb{C}^n)) \longrightarrow L_1^{p,\delta}(S, \mathbb{C}^n)$$

$$Q \circ \bar{\partial} = \text{Identity}$$

A choice of left inverse Q is equivalent to a choice of projection $\pi_Q := \bar{\partial} \circ Q$ onto $\bar{\partial}(L_1^{p,\delta})$. This is the same as choosing a kernel for Q. The kernel of Q is dual to the cokernel of $\bar{\partial}$.

An antiholomorphic one form $\nu \in L^{p,\delta}(\Omega(S, \mathbb{C}^n))$ is in $\bar{\partial}(L_1^{p,\delta})$ for $0 < \delta < \frac{1}{2}$ if and only if,

$$\int_S \nu \wedge \theta = 0$$

for all holomorphic one forms θ with at most simple poles at the punctures of S. Calling the space of such holomorphic one forms $\Lambda^{1,0}(S)$, we see that the cokernel of

$\bar{\partial}$ is equal to

$$\ker \pi_Q = \hom_{\mathbb{C}}(\Lambda^{1,0}(S), \mathbb{C}^n)$$

where the above isomorphism is given by

$$\nu \mapsto \int_S \nu \wedge \cdot \in \hom_{\mathbb{C}}(\Lambda^{1,0}(S), \mathbb{C}^n)$$

In section 2.9 we will see some extra compatibility constraints that we want Q to satisfy. To do this, it will be important to choose the kernel of π_Q to consist of smooth one forms supported on some subsets of the Riemann surface S. The characterization above tells us we are able to choose one forms with support in any given open set in S which span $\ker \pi_Q$. The space of such choices is convex, so there is no obstruction to choosing a continuous family such left inverses Q. If we restrict attention to model Riemann surfaces with a compact subset of (non nodal) complex structures, then this family can be chosen to be uniformly bounded.

2.5 Quasi holomorphic model curves

We now define a new type of quasi holomorphic model curve which will exist for complex structures where there are no genuine holomorphic model curves. Given any smooth map $F : S \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ which consists of trivial holomorphic cylinders close to punctures of S, consider the map

$$f = F - Q(\bar{\partial} F)$$

f is characterized by having asymptotics given by the chosen trivial holomorphic cylinders and satisfying the equation

$$\pi_Q \bar{\partial} f := \bar{\partial} \circ Q(\bar{\partial} f) = 0$$

We call solutions of the above weakened $\bar{\partial}$ equation quasi holomorphic. We can measure how far such a quasi holomorphic map is from being holomorphic by taking

$\bar{\partial}$ of it

$$\bar{\partial} f \in \ker \pi_Q \subset L^{p,\delta}$$

We can define the finite dimensional $\bar{\partial}_0$ equation on the space of quasi holomorphic maps by identifying $\ker \pi_Q$ with $\hom_{\mathbb{C}}(\Lambda^{1,0}, \mathbb{C}^n)$

$$\bar{\partial} f = \bar{\partial}_0 f \in \hom_{\mathbb{C}}(\Lambda^{1,0}, \mathbb{C}^n)$$

Recall that the definition of vertices of quasi holomorphic graphs given in section 1.1.4 involved an equivalence class $[p, f]$, where p is a point in $\mathbb{T}^n \rtimes B^n$, and $f : S_V \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ satisfies

$$\lim_{t\to\infty} |f(\theta, t) - \zeta - \theta\alpha - tJ\alpha|\, e^t = 0 \text{ on edge regions surrounding punctures}$$

$$\alpha \in \mathbb{Z}^n, \zeta \in \mathbb{C}^n/\mathbb{Z}^n$$

$$\text{Imaginary part of} \left(\sum_{punctures} \zeta_i \right) = 0$$

Here S_V is created from the vertex region $V \subset S$ by adding semi infinite cylinders with the coordinates $(\theta, t) \in \mathbb{R}/\mathbb{Z} \times (0, R)$. These model curves are considered up to the following equivalence relation:

$$[p, f] = [p + x, f - x + g]$$

$$g : S_V \longrightarrow \mathbb{C}^n, x \in \mathbb{R}^n/\mathbb{Z}^n$$

$$g = 0 \text{ at punctures}$$

Given such an equivalence class $[p, F]$ and a point p, there is a unique quasi holomorphic model curve f so that $[p, F] = [p, f]$. We can consider the alternative definition for quasi holomorphic model curves to be an equivalence class $[p, f]$ where f is a quasi holomorphic map satisfying a normalizing condition as above and $[p, f]$

is defined up to the equivalence relation

$$[p, f] = [p + x, f - x]$$

$$x \in \mathbb{R}^n/\mathbb{Z}^n$$

2.6 Exponentiating out model curves

This section describes how to take a quasi holomorphic model curve

$$f : S \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$$

$$\pi_Q \circ \bar{\partial} f = 0$$

and exponentiate it out to a map $\mathcal{G}(f) : S \longrightarrow (\mathbb{T}^n \rtimes B^n, J^\epsilon)$ which consists of trivial holomorphic cylinders outside a compact set and satisfying

$$\|d\mathcal{G}(f) - df\|_{p,\delta} \approx 0$$

Consider the end of a model curve $f : S \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ near a puncture z_i. After we give a neighborhood of z_i its cylindrical coordinates $(\theta, t) \in \mathbb{R}/\mathbb{Z} \times [0, \infty)$, there exists a trivial holomorphic cylinder C_i so that

$$e^{3t} |f(\theta, t) - C_i(\theta, t)|$$

and

$$e^{3t} |d(f(\theta, t) - C_i(\theta, t))|$$

are uniformly bounded for $t \geq 1$. Note that these cylindrical coordinates come from an edge-vertex decomposition $\mathfrak{E}$ as described in appendix C.

Lemma 2.6.1. *There exists a constant $c(E, g)$ so that all quasi holomorphic model curves f with energy less than E, and genus less than g, satisfying $\pi_Q \bar{\partial} f = 0$ and*

$\|\bar{\partial} f\| \leq 1$, *satisfy the following*

$$e^{3t} \left| f(\theta, t) - C_i(\theta, t) \right| \leq c(E, g)$$

and

$$e^{3t} \left| d(f(\theta, t) - C_i(\theta, t)) \right| \leq c(E, g)$$

for all $t \geq 1$

Proof:

To see this, note that such a cylinder C can be found using the removable singularity theorem on f considered as a map to $(\mathbb{C}P^1)^n$. f is holomorphic on this $\mathfrak{E}$ edge region, and has energy controlled by E and the L^2 norm of $\bar{\partial} f$. Note that the energy of f bounds the number of punctures, and the genus is bounded by g so the $\mathfrak{E}$ vertex regions on which $\bar{\partial} f$ is supported are bounded, and the L^2 norm of $\bar{\partial} f$ is controlled by our $L^{p,\delta}$ norm. The proof of Proposition 3.2.4 can then be used to bound df on the interior of our $\mathfrak{E}$ edge region. The function $d(f - C)$ is then holomorphic, bounded for $t \geq \frac{1}{2}$ and converges to 0 as $t \to \infty$. The Schwartz lemma then implies that $e^{2\pi t} d(f - C)$ must be uniformly bounded for $t \geq 1$. Integrating this gives that $e^{2\pi t}(f - C)$ is also uniformly bounded for $t \geq 1$.

□

In what follows we will first modify f so that it coincides with a trivial holomorphic cylinder close to each of its punctures, and then exponentiate out the resulting modified model curve. Choose a smooth cutoff function

$$\psi : \mathbb{R} \longrightarrow [0, 1]$$

so that $\psi(t) = 0$ for $t \leq 0$, $\psi = 1$ for $t \geq 1$, and $|d\psi| \leq 2$. We can use ψ to obtain a cutoff function in cylindrical coordinates around the ith puncture

$$\psi_{R,i} : \mathbb{R}/\mathbb{Z} \times (0, \infty) \longrightarrow [0, 1]$$

$$\psi_{R,i}(t, \theta) := \psi\left(t - (R + 1)\right)$$

Extend $\psi_{R,i}$ to be 0 elsewhere Now modify f close to punctures

$$\tilde{f} := (1 - \sum_i \psi_{R,i}) f + \sum_i \psi_{R,i} C_i$$

We can modify f similarly close to all other punctures. What we obtain is a map $\tilde{f} : S \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ which consists of trivial holomorphic cylinders close to punctures. In particular, the edges regions surrounding punctures in the decomposition $\mathfrak{E}_R$ described in appendix C are trivial holomorphic cylinders.

Lemma 2.6.2. *There exists a constant $c(E, g) < \infty$ so that all quasi holomorphic model curves f of energy less than E and genus g satisfy*

$$\left\| f - \tilde{f} \right\|_{1,p,\delta} \leq c(E, g) e^{-2R}$$

when we use the cutoff functions ψ_{R_i}.

We want now to take $\tilde{f}$ and exponentiate it out to an approximately J^ϵ holomorphic curve. Taking $\exp_p \tilde{f}$ is not quite good enough as the ends of the resulting curve will not in general be trivial holomorphic cylinders. This is because the trivial holomorphic cylinders around the punctures of $\tilde{f}$ are given by

$$C_i(\theta, t) := \theta\alpha_i + tJ\alpha_i + \zeta_i$$

and ζ_i may not be a multiple of α_i.

Define the function $\psi_\zeta := \sum \zeta_i \psi_{R,i}$, and exponentiate f out from $\exp_{p_i} \psi_\zeta$ instead.

$$\mathcal{G}([p, f]) := \exp_{\exp_p \psi_\zeta} \tilde{f} - \psi_\zeta$$

Note that although each ζ_i depends on the S^1 choice of cylindrical coordinates for the edge region around the puncture at z_i, $\mathcal{G}([p, f])$ is independent of this choice, and well defined depending only on the equivalence class $[p, f]$ so long as quasi holomorphic representatives are used. The exponentiations in the above expression can be understood after identifying $T_p(\mathbb{T}^n \rtimes B^n), J^\epsilon$ with $\mathbb{C}^n/\mathbb{Z}^n$.

For notational convenience, when the point p from which we exponentiate out from is not important, we will often write

$$\mathcal{G}(f) := \mathcal{G}([p, f])$$

Note that $\mathcal{G}([p, f])$ restricted to the $\mathfrak{E}_R$ edge regions surrounding punctures of S coincides exactly with the trivial holomorphic cylinders that attach to the model curve $[p, f]$ in J^ϵ quasi holomorphic $\mathfrak{E}_R$ graphs, as described in section 1.1.4. This means that we can extend $\mathcal{G}$ to $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}$. Explicitly, for a quasi holomorphic graph u, $\mathcal{G}(u)$ restricted to edge regions is given by the associated trivial holomorphic cylinders, and $\mathcal{G}(u)$ restricted to a vertex region V with model curve $[p, f]$ is given by

$$\mathcal{G}(u)|_V := \mathcal{G}([p, f])|_V$$

The next step is to show that $\bar{\partial}\mathcal{G}(f)$ is close to $\bar{\partial} f$. For this to be true when $\mathbf{T} \neq 0$, the parts of $\mathcal{G}(f)$ which are not trivial holomorphic cylinders must not be too far apart. The space of vertex model curves appearing in $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ is contained in a compact set of model curves and independent of ϵ, so there is a bound independent of ϵ on the diameter of model curves restricted to vertex regions. (Recall that $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ is a subset of the space of J^ϵ quasi holomorphic graphs u with edge-vertex decompositions given by $\mathfrak{E}_R$, genus g, k punctures, and energy $\leq E$ which satisfy $\left\|\bar{\partial}_0 u\right\| < 1$. The closure of this set of model curves f will then have bounded energy, genus less than g, and satisfy $\pi_Q \bar{\partial} f = 0$ and $\left\|\bar{\partial}_0 f\right\| \leq 1$. As the number of punctures is bounded by the energy for maps to $\mathbb{C}^n/\mathbb{Z}^n$, this makes this set of curves compact.)

Lemma 2.6.3. *For quasi holomorphic model curves f appearing in $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$, there exists a constant $c < \infty$ so that for ϵ small enough dependent on R,*

$$\|d(\mathcal{G}(f)) - df\|_{p,\delta} \leq ce^{-R}$$

Proof:

This uses the fact, proved in section 3 that the energy of a quasi holomorphic graph bounds the energy of its model curves. $d\mathcal{G}(f) = d\tilde{f}$ on $\mathfrak{E}_R$ edge regions surrounding punctures where $\tilde{f}$ and $\mathcal{G}(f)$ are both trivial holomorphic cylinders. On the rest of our Riemann surface S, the $L^{p,\delta}$ norm is controlled by the L^∞ norm, and f is bounded. First, recall that Lemma 2.2.4 tells us that

$$|d(\exp_u \phi) - du - \nabla\phi| \leq (\|\mathbf{T}\|_\infty |\phi| |du| + |\nabla\phi|) e^{\|\mathbf{T}\|_\infty |\phi|} - |\nabla\phi|$$

By using Lemma 2.2.4 with $u = p$ and $\phi = \psi_\zeta$, noting that ψ_ζ and $d\psi_\zeta$ are bounded, we can get

$$\left| d\exp_p(\psi_\zeta) - d\psi_\zeta \right|$$

as small as we like by choosing ϵ small and recalling that $\|\mathbf{T}\|_\infty$ is proportional to ϵ. Now by using Lemma 2.2.4 with $u = \exp_p(\psi_\zeta)$ and $\phi = \tilde{f} - \psi_\zeta$ noting that every term and its derivative are bounded

$$\left| d\mathcal{G}(f) - d(exp_p(\psi_\zeta) - d(\tilde{f} - \psi_\zeta)) \right|$$

can be made as small as we like by choosing ϵ small. Putting these two expressions together, we can make

$$\left| d\mathcal{G}(f) - d(\tilde{f}) \right|$$

as small as we like, and the result follows after using Lemma 2.6.2.

□

2.7 Banach Structure

Define $\mathcal{B}_{\mathfrak{E}}(S, \mathbb{C}^n) \subset L_1^{p,\delta}(S, \mathbb{C}^n)$ to consist of sections $\phi : S \longrightarrow \mathbb{C}^n$ so that the average of ϕ around each circle at the center of a $\mathfrak{E}$ edge region is 0

$$\oint \phi = 0 \text{ at center of } \mathfrak{E} \text{ edges}$$

This is well defined because $\phi \in L_1^{p,\delta}$ must be continuous.

Our space of maps will be modelled on

$$\begin{array}{ccc}\mathcal{B}_{\mathfrak{E}}(S,\mathbb{C}^n) & \longrightarrow & \mathcal{B}_{\mathfrak{E}} \\ & & \downarrow \\ & & \mathcal{Q}^{\epsilon,\mathfrak{E}}_{g,k,E}\end{array}$$

From a quasi holomorphic graph $u \in \mathcal{Q}^{\epsilon,\mathfrak{E}}_{g,k,E}$ with domain S and a section $\phi \in \mathcal{B}_{\mathfrak{E}}(S,\mathbb{C}^n)$, we can create a map $S \longrightarrow \mathbb{T}^n \rtimes B^n$ by

$$u,\phi \mapsto \exp_{\mathcal{G}(u)}\phi$$

Lemma 2.7.1. *This map is injective restricted to* $u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ *and* $\epsilon(\|\phi\|_{1,p,\delta}+1)$ *small enough.*

Proof:

Suppose that we have $f = \exp_u \phi$ for $\epsilon\|\phi\|_{1,p,\delta}$ small. We need to show that if $f = \exp_{\tilde{u}}\tilde{\phi}$ for $\epsilon\left\|\tilde{\phi}\right\|_{1,p,\delta}$ small, then $u = \tilde{u}$ and $\phi = \tilde{\phi}$. First note that Lemma 2.3.2 tells us that $\|\phi\|_\infty$ is controlled by $\|\phi\|_{1,p,\delta}$. An application of Lemma 2.2.4 gives that

$$\frac{\partial}{\partial t}\exp_{p(t)}\phi \approx \frac{\partial}{\partial t}p(t) \text{ and } \frac{\partial}{\partial t}\exp_p \phi(t) \approx \frac{\partial}{\partial t}\phi(t)$$

for $\epsilon|\phi|$ small enough. This implies that moving the cylinder which we exponentiate out from to obtain our map will change the average of the required ϕ in the opposite direction, so for $\epsilon\|\phi\|_{1,p,\delta}$ small enough, the trivial holomorphic cylinders representing the edges of u and ϕ restricted to these edge regions are uniquely determined by the requirement that the average of ϕ around the middle of edge regions is 0.

As explained in Appendix C.3, for ϵ small enough, once we know the edges of a graph $u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ approximating f in this way, the vertex model curves are determined uniquely, so we have u determined uniquely for $\epsilon(\|\phi\|_{1,p,\delta}+1)$ small enough and hence ϕ is also determined uniquely because of the above estimate.

□

2.8 Gluing map

We now wish to describe a map, which we'll call $d\mathcal{G}_{\mathfrak{E}_R}$ which takes variations $\xi_i \in L_1^{p,\delta}(S_{V_i}, \mathbb{C}^n)$ of the model curves at the vertices V_i of some quasi holomorphic $\mathfrak{E}_R$ graph u with domain S and glues them together to a variation over S in $L_1^{p,\delta}(S, \mathbb{C}^n)$,

$$d\mathcal{G}_{\mathfrak{E}_R} : \oplus_{\text{vertices}} L_1^{p,\delta}(S_{V_i}, \mathbb{C}^n) \longrightarrow L_1^{p,\delta}(S, \mathbb{C}^n)$$

To start off, ξ_i can be considered as a variation over the vertex region $V_i \subset S$ and the edges surrounding it without alteration. If ξ_i and ξ_j are the variations coming from the vertices at either end of a $\mathfrak{E}_R$ edge region, $d\mathcal{G}_{\mathfrak{E}_R}$ averages them over the $\mathfrak{E}$ edge region containing this $\mathfrak{E}_R$ edge using a cutoff function. Choose a cutoff function,

$$\rho_R : [R, R] \longrightarrow [0, 1]$$

so that $\rho_R = 1$ near $-R$ and 0 near R, $|d\rho_R| \leq \frac{1}{R}$ and $\rho_R(t) + \rho_R(-t) = 1$. In cylindrical coordinates centered over the center of an edge traveling from vertex i to vertex j,

$$d\mathcal{G}_{\mathfrak{E}_R}(\oplus\xi)(\theta, t) = \rho_R(t)\xi_i(t, \theta) + \rho_R(-t)\xi_j(\theta, t)$$

Note that these cylindrical coordinates exist, because $\mathfrak{E}_R$ edge regions consist of the subsets $\mathbb{R}/\mathbb{Z} \times (a + R, b - R) \subset \mathbb{R}/\mathbb{Z} \times (a, b)$ of $\mathfrak{E}$ edge regions.

We can define $d\mathcal{G}_{\mathfrak{E}_R}$ on any set of sections $d\xi_i$ of $T^*S_i \otimes \mathbb{C}^n$ analogously. Thus, it makes sense to talk of $d\mathcal{G}_{\mathfrak{E}_R}$ applied to $\oplus d\xi_i$ or $\oplus\bar{\partial}\xi_i$.

The fact that the cutoff function satisfies $d\rho \leq \frac{1}{R}$ implies the following lemma.

Lemma 2.8.1. *If* $\xi_i \in L_1^{p,\delta}$,

$$\|d(d\mathcal{G}_{\mathfrak{E}_R}(\oplus\xi_i)) - d\mathcal{G}_{\mathfrak{E}_R}(\oplus d\xi_i)\|_{p,\delta} \leq \frac{1}{R}\|\oplus\xi_i\|_{p,\delta}$$

We now want to define a cutting map C_0 which is an approximate right inverse to $d\mathcal{G}_{\mathfrak{E}_R}$, which we will then use to construct an exact right inverse, $C_{\mathfrak{E}_R}$. Recall that a subset of the model domain S_V can be identified with the vertex region V and the edges regions surrounding it. Define C_0 of a section $\xi \in L^{p,\delta}$ to be ξ cut off half way along each edge and considered as a section over S_V. So C_0 gives a bounded map

$$C_0 : L^{p,\delta}(S) \longrightarrow \oplus_{\text{vertices}} L^{p,\delta}(S_{V_i})$$

Lemma 2.8.2. *There exists a right inverse $C_{\mathfrak{E}_R}$ to*

$$d\mathcal{G}_{\mathfrak{E}_R} : \oplus_i L^{p,\delta}(S_{V_i}) \longrightarrow L^{p,\delta}(S)$$

with

$$\|C_{\mathfrak{E}_R}\| \leq 2$$

Proof:

$$\|d\mathcal{G}_{\mathfrak{E}_R} \circ C_0(\xi) - \xi\|_{p,\delta} \leq \frac{1}{2} \|\xi\|_{p,\delta}$$

so $\mathcal{G}_{\mathfrak{E}_R} \circ C_0$ is invertible, with $\|(d\mathcal{G}_{\mathfrak{E}_R} \circ C_0)^{-1}\| \leq 2$. Define

$$C_{\mathfrak{E}_R} := C_0 \circ (d\mathcal{G}_{\mathfrak{E}_R} \circ C_0)^{-1}$$

$C_{\mathfrak{E}_R}$ is a right inverse to $d\mathcal{G}_{\mathfrak{E}_R}$ with $\|C_{\mathfrak{E}_R}\| \leq \|C_0\| \, \|d\mathcal{G}_{\mathfrak{E}_R} \circ C_0\| \leq 2$

□

We can construct an approximate left inverse to $\bar{\partial}$ restricted to $\mathcal{B}_{\mathfrak{E}}(S, \mathbb{C}^n)$ using the left inverse Q considered in section 2.4. In particular, consider the map

$$d\mathcal{G} \circ Q \circ C : L^{p,\delta} \longrightarrow L^{p,\delta}_1$$

Lemma 2.8.3.

$$d\mathcal{G} \circ Q \circ C(L^{p,\delta}(\Omega^{0,1}(S, \mathbb{C}^n))) = \mathcal{B}_{\mathfrak{E}}(S, \mathbb{C}^n) \subset L^{p,\delta}_1(S, \mathbb{C}^n)$$

Proof:

Recall that $Q \circ \bar{\partial} = Id$ on $L^{p,\delta}_1$, and the image of C consists of sections in $L^{p,\delta}$ that vanish half way along edges. This means that the image of $Q \circ C$ at each vertex model curve is all sections of $L^{p,\delta}$ that are holomorphic everywhere past half way along the edges. Therefore, if the circle halfway along an edge is given coordinates θ, all the non positive Fourier coefficients of $\phi_i(\theta)$ and nonnegative Fourier coefficients of $\phi_j(\theta)$ will vanish where ϕ_i and ϕ_j denote the sections over the vertex model curves at each end of the edge. Any section in $L^{p,\delta}_1$ over the interior of these vertex model curves which obeys these conditions at its boundary circles can be extended to a section in the image of $Q \circ C$. Applying $d\mathcal{G}$ to these gets a section which restricts to $\frac{1}{2}(\phi_1(\theta) + \phi_2(\theta))$. As ϕ_1 can be anything on one side and ϕ_2 is unrestricted on the other, the only restriction that the image of $d\mathcal{G} \circ Q \circ C$ obeys is the average of sections over these circles at the center of edges is 0. This is the condition that $d\mathcal{G} \circ Q \circ C$ is in $\mathcal{B}_{\mathfrak{E}}(S, \mathbb{C}^n)$.

□

2.9 Self similarity of Q

Lemma 2.9.1. *If S has at least one puncture or $\mathfrak{E}_R$ edge region, there exists a bounded left inverse to $\bar\partial$ restricted to $\mathcal{B}_{\mathfrak{E}_R}(S, \mathbb{C}^n)$,*

$$Q_{\mathfrak{E}_R} : L^{p,\delta}(\Omega^{0,1}(S, \mathbb{C}^n)) \longrightarrow \mathcal{B}_{\mathfrak{E}_R}(S, \mathbb{C}^n)$$

so that $\|Q_{\mathfrak{E}_R}\|$ restricted to connected components of Deligne Mumford space is uniformly bounded. Moreover $\ker Q_{\mathfrak{E}_R}$ can be chosen to consist of smooth one forms supported in $\mathfrak{E}$ vertex regions of S.

Proof:

The proof is by induction on the possible number of internal edges of S, which is $k - 3 + 3g$ where S has genus g and k punctures.

First, note that $\bar\partial$ is injective and Fredholm on $\mathcal{B}_{\mathfrak{E}_R}(S, \mathbb{C}^n)$, so there must be some bounded left inverse $Q_{\mathfrak{E}_R}$. The extra cokernel created by restricting $\bar\partial$ to $\mathcal{B}_{\mathfrak{E}_R}(S, \mathbb{C}^n)$ can be spanned by $\mathbb{C}^n \otimes \text{Span}\{\bar\partial f_i\}$ where f_i is some real valued function that is equal to 1 on the ith internal $\mathfrak{E}$ edge region that contains a $\mathfrak{E}_R$ edge region and 0 on all other edges. So as when we were defining Q, the kernel of $Q_{\mathfrak{E}_R}$ can be chosen to consist of smooth one forms which are supported in $\mathfrak{E}$ vertex regions.

Also note that we can choose $Q_{\mathfrak{E}_R}$ to depend continuously on S when restricted to complex structures where the $\mathfrak{E}_R$ edge regions have fixed combinatorics, so $Q_{\mathfrak{E}_R}$ is uniformly bounded when restricted to any compact subset of Deligne Mumford space with nodal Riemann surfaces removed. We now need to worry about Riemann surfaces with very long internal $\mathfrak{E}$ edges.

The case to start off the induction is the 3 punctured sphere, with $(k-3+3g)=0$. There is only one of these, so we have no trouble choosing $Q_{\mathfrak{E}_R}$ bounded.

Suppose that $\|Q_{\mathfrak{E}_R}\| \leq c_m$ for all Riemann surfaces S with $(k-3+3g) \leq m$ for some R greater than R_{m+1} from axiom 5 in Appendix C. This means that $\mathfrak{E}_R$ edge decompositions will be compatible with cutting and gluing on Riemann surfaces with $(k-3+3g) \leq (m+1)$

Now consider $Q_{\mathfrak{E}_R}$ restricted to surfaces S with $(k-3+3g) = m+1$. First, note that $Q_{\mathfrak{E}_R}$ restricted to these surfaces with all internal $\mathfrak{E}$ edges bounded by $2l$ is uniformly bounded, so we need to consider a surface with at least one edge longer than $2l$. Consider the map

$$d\mathcal{G}_{\mathfrak{E}_l} \circ Q_{\mathfrak{E}_R} \circ C_{\mathfrak{E}_l}$$

for some $l \geq R$. Note that as in the proof of Lemma 2.8.3, the image of this is contained in $\mathcal{B}_{\mathfrak{E}_R}(S, \mathbb{C}^n)$. The $Q_{\mathfrak{E}_R}$ in the above expression acts on sections over Riemann surface with $(k-3+3g) \leq m$, so it is bounded by c_m. Also recall that $\|C\| \leq 2$. Therefore, we can use Lemma 2.8.1 to say

$$\left\|\bar{\partial} \circ d\mathcal{G}_{\mathfrak{E}_l} \circ Q_{\mathfrak{E}_R} \circ C_{\mathfrak{E}_l} - d\mathcal{G}_{\mathfrak{E}_l} \circ \bar{\partial} \circ Q_{\mathfrak{E}_R} \circ C_{\mathfrak{E}_l}\right\| \leq \frac{2}{l} c_m$$

note that $\pi := d\mathcal{G}_{\mathfrak{E}_l} \circ \bar{\partial} \circ Q_{\mathfrak{E}_R} \circ C_{\mathfrak{E}_l}$ is a projection, as the projection $\bar{\partial} \circ Q_{\mathfrak{E}_R}$ preserves the image of C and $C \circ dG$ is the identity restricted to the image of C. Also note that $Q_{\mathfrak{E}_R} \circ C_{\mathfrak{E}_l} \circ \pi = Q_{\mathfrak{E}_R} \circ C_{\mathfrak{E}_l}$. So by choosing $l \geq 4c_m$, we get

$$\left\|\bar{\partial} \circ d\mathcal{G}_{\mathfrak{E}_l} \circ Q_{\mathfrak{E}_R} \circ C_{\mathfrak{E}_l} \circ \pi - \pi\right\| \leq \frac{1}{2}$$

Therefore $\pi \circ \bar{\partial}$ is invertible and bounded below by $\frac{1}{4c_m}$, so $\bar{\partial}$ has a left inverse

$$Q_{\mathfrak{E}_R} = (\pi \circ \bar{\partial})^{-1} \circ \pi$$

bounded by $8c_m^2$.

Note that the $Q_{\mathfrak{E}_R}$ constructed this way satisfies

$$\ker Q_{\mathfrak{E}_R} = d\mathcal{G}_{\mathfrak{E}_l}(\ker Q_{\mathfrak{E}_R})$$

so the $\ker Q_{\mathfrak{E}_R}$ is spanned by smooth one forms with support inside $\mathfrak{E}$ vertex regions. The lemma is now proved by induction, and noting that $\|Q_{\mathfrak{E}_R}\|$ can be chosen smaller than $\|Q_{\mathfrak{E}_{R'}}\|$ for $R' \geq R$.

□

Proposition 2.9.2. *We can choose a left inverse to $\bar{\partial}$,*

$$Q_{\mathfrak{E}_R} : L^{p,\delta}(\Omega^{0,1}(S, \mathbb{C}^n)) \longrightarrow \mathcal{B}_{\mathfrak{E}_R}(S, \mathbb{C}^n)$$

so that

1. $$\ker Q \subset \ker Q_{\mathfrak{E}_{R_2}} \subset \ker Q_{\mathfrak{E}_{R_1}}$$

   for all $R_2 \geq R_1$

2. *$Q_{\mathfrak{E}_R}$ depends smoothly on the complex structure of S in regions where the combinatorics of the $\mathfrak{E}_R$ edge decomposition doesn't change.*

3. *There exists a series of constants $c_m < \infty$ so that if S has genus g and k punctures and $(k - 3 + 3g) \leq m$, then*

   $$\|Q_{\mathfrak{E}_R}\| \leq e^{R\delta} c_m$$

4. *There exists a series of constants $R_m < \infty$ so that if S satisfies $(k-3+3g) \leq m$,*

$\ker Q_{\mathfrak{E}_R}$ *is self similar for* $R \geq R_m$*, in the sense that*

$$\ker Q_{\mathfrak{E}_R} = d\mathcal{G}_{\mathfrak{E}_R}(\ker Q_{\mathfrak{E}_R}) = d\mathcal{G}_{\mathfrak{E}_R}(\ker Q)$$

Proof:

We construct $Q_{\mathfrak{E}_R}$ more carefully from the $Q_{\mathfrak{E}}$ discussed in Lemma 2.9.1. We know that $Q_{\mathfrak{E}_R}$ must equal $Q_{\mathfrak{E}}$ restricted to the image $\bar\partial(\mathcal{B}_{\mathfrak{E}}(S,\mathbb{C}^n))$. Recall that we decided to span the extra cokernel of $\bar\partial$ created by restricting to $\mathcal{B}_{\mathfrak{E}}(S,\mathbb{C}^n)$ by taking $\mathbb{C}^n \otimes \mathrm{Span}\{\bar\partial f_i\}$ where f_i is some smooth real valued function equal to 1 on the ith edge and vanishing on all other $\mathfrak{E}$ edge regions. Actually, we can construct $\ker Q_{\mathfrak{E}}$ to contain $\mathbb{C}^n \otimes \mathrm{Span}\{\bar\partial f_i\}$ where f_i is one on the ith (not necessarily internal) $\mathfrak{E}$ edge, and vanishes on all other $\mathfrak{E}$ edges. These f_i should obey one linear constraint that $\sum f_i = 1$. These can easily be constructed to be self similar in the sense that

$$\{\bar\partial f_i\} \subset \{\, d\mathcal{G}_{\mathfrak{E}_R}(\oplus \bar\partial f_j)\} \text{ for all } R$$

As $\ker Q_{\mathfrak{E}}$ is finite dimensional and just consists of smooth one forms, we can use the L^2 metric on it, and choose $\ker Q_{\mathfrak{E}_R}$ to be the orthogonal compliment in $\ker Q_{\mathfrak{E}}$ of $\ker Q_{\mathfrak{E}} \cap \bar\partial(\mathcal{B}_{\mathfrak{E}_R}(S,\mathbb{C}^n))$. As we chose our f_i to be self similar, there exists some constant c (depending on $(k-3+3g)$) so that if f_i is one on the ith edge which has length R,

$$\|f_i\|_{1,p,\delta} \leq ce^{\delta R}\left\|\bar\partial f_i\right\|_{p,\delta}$$

The projections defined by $\bar\partial \circ Q_{\mathfrak{E}}$ and the complimentary projection onto $\ker Q_{\mathfrak{E}}$ are controlled by $\|Q_{\mathfrak{E}}\|$, which Lemma 2.9.1 tells us is uniformly bounded depending on $(k-3+3g)$. Therefore the $Q_{\mathfrak{E}_R}$ we've defined satisfies

$$\|Q_{\mathfrak{E}_R}\| \leq c_m e^{\delta R} \text{ for } (k-3+3g) \leq m$$

As noted in the end of the proof of Lemma 2.9.1, there exists an $l(m,R)$ so that

$$\ker Q_{\mathfrak{E}_R} = d\mathcal{G}_{\mathfrak{E}_l}(\ker Q_{\mathfrak{E}_R}) \text{ for } l \geq l(m,R) \text{ and } (k-3+3g) \leq m$$

As we chose the functions f_i to be self similar, and the metric for taking the orthogonal compliment is self similar, this tells us that there exists some R_m so that

$$\ker Q_{\mathfrak{E}_R} = d\mathcal{G}_{\mathfrak{E}_R}(\ker Q_{\mathfrak{E}_R}) \text{ for all } R \geq R_m \text{ and } (k-3+3g) \leq m$$

Note that we can define

$$Q = \lim_{R\to\infty} Q_{\mathfrak{E}_R}$$

so $Q = Q_{\mathfrak{E}_R}$ on surfaces with no internal $\mathfrak{E}_R$ edges. Therefore the last property stated is satisfied. The first two properties are satisfied by construction.

□

Define the projection

$$\pi_{Q,\mathfrak{E}_R} := \bar{\partial} \circ Q_{\mathfrak{E}_R}$$

For R large enough, the above proposition implies that $\ker \pi_{Q,\mathfrak{E}_R} = d\mathcal{G}_{\mathfrak{E}_R}(\ker \pi_Q)$. This is important, because then we can use Lemma 2.6.3 to say that we can make $\pi_{Q,\mathfrak{E}_R}\bar{\partial}\mathcal{G}\left(\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}\right)$ as small as we like by choosing R large and ϵ small. The next step is to use the implicit function theorem to modify $\mathcal{G}$ to $\mathcal{G}_\infty$ so that

$$\pi_{Q,\mathfrak{E}_R}\bar{\partial}\mathcal{G}_\infty = 0$$

2.10 Implicit function theorem

We want now to modify the gluing map $\mathcal{G}$ to $\mathcal{G}_\infty$ so that for any quasi holomorphic graph $u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$,

$$\pi_{Q,\mathfrak{E}_R}(\bar{\partial}(\mathcal{G}_\infty(u))) = 0$$

We do this iteratively. Each iteration will be of the form

$$\mathcal{G}_k(u) = \exp_{\mathcal{G}(u)} \phi_k$$

where

$$\phi_0 = 0$$

$$\phi_{k+1} = \phi_k - Q_{\mathfrak{E}_R}(\bar{\partial}(\mathcal{G}_k(u)))$$

In order to prove that this sequence of maps will converge to one with the desired properties, we need to examine how $\bar{\partial}\exp_{\mathcal{G}(u)}\phi$ changes with ϕ.

Lemma 2.10.1. *For quasi holomorphic graphs $u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$,*

$$\left\| \frac{d}{ds} d(\exp_{\mathcal{G}(u)}\phi_s) - \frac{d}{ds}(d\phi_s) \right\|_{p,\delta} \leq c \left\| \frac{d\phi_s}{ds} \right\|_{1,p,\delta}$$

where c can be made arbitrarily small by choosing $\epsilon \left\|\phi_s\right\|_{1,p,\delta}$ small.

Proof:

define

$$F(s,t) = \exp_{\mathcal{G}(u)} t\phi_s$$

$$dF(s,t) = d(\exp_{\mathcal{G}(u)} t\phi_s)$$

We are interested in $\frac{\partial}{\partial s}dF(s,1)$.

Using Lemma 2.2.1

$$\frac{\partial}{\partial t}dF = d(\phi_s) + \mathbf{T}_F(\phi_s, dF)$$

so

$$\begin{aligned} \left| \frac{\partial^2}{\partial t \partial s}dF - \frac{\partial}{\partial s}d(\phi_s) \right| &= \left| \frac{\partial^2}{\partial s \partial t}dF - \frac{\partial}{\partial s}d(\phi_s) \right| \\ &\leq |\nabla \mathbf{T}| \left| \frac{\partial F}{\partial s} \right| |\phi_s| \, |dF| + \|\mathbf{T}\|_\infty \left| \frac{\partial}{\partial s}\phi_s \right| |dF| \\ &\quad + \|\mathbf{T}\|_\infty |\phi_s| \left| \frac{\partial}{\partial s}dF \right| \end{aligned}$$

Applying Lemma 2.2.4 along with the observation $\|\mathbf{T}\|_\infty |d\phi_s| \leq 1$ if $\epsilon \left\|\phi_s\right\|_{1,p,\delta}$ is small enough gives

$$|dF(s,t)| \leq 4(|dF(s,0)| + |d(\phi_s)|)$$

Note that Lemma 2.2.4 is valid regardless of the domain of u. Using this with

$u(s) = F(s,0) = \mathcal{G}(u)$ and $\phi(s) = \phi_s$, and the same assumption that

$$\|\mathbf{T}\|_\infty |\phi_s| \leq 1$$

gives

$$\left|\frac{\partial F}{\partial s}\right| \leq 3\left|\frac{\partial}{\partial s}(\phi_s)\right| = 3\left|\frac{\partial \phi_s}{\partial s}\right|$$

$$\begin{aligned}\left\|\frac{\partial^2}{\partial t \partial s} dF - \frac{\partial}{\partial s} d(\phi_s)\right\|_{p,\delta} \leq\ & 12\|\nabla \mathbf{T}\|_\infty \left\|\frac{\partial \phi_s}{\partial s}\right\|_{p,\delta} \|\phi_s\|_\infty \|dF(s,0)\|_\infty \\ &+ 12\|\nabla \mathbf{T}\|_\infty \left\|\oplus\frac{\partial \phi_s}{\partial s}\right\|_{\infty} \|\phi_s\|_\infty \|d(\phi_s)\|_{p,\delta} \\ &+ 4\|\mathbf{T}\|_\infty \left\|\frac{\partial}{\partial s}\phi_s\right\|_{p,\delta} \|dF(s,0)\|_\infty \\ &+ 4\|\mathbf{T}\|_\infty \left\|\frac{\partial}{\partial s}\phi_s\right\|_{\infty} \|d(\phi_s)\|_{p,\delta} \\ &\quad + \|\mathbf{T}\|_\infty \|\phi_s\|_\infty \left\|\frac{\partial}{\partial s} dF\right\|_{p,\delta}\end{aligned}$$

So by choosing ϵ small enough, we get

$$\left\|\frac{\partial^2}{\partial t \partial s} dF - \frac{\partial}{\partial s} d(\phi_s)\right\|_{p,\delta} \leq c_1 \left\|\frac{\partial \phi_s}{\partial s}\right\|_{p,\delta} + c_2 \left\|\frac{\partial}{\partial s} dF\right\|_{p,\delta}$$

for $c_1 > 0$ and $c_2 > 0$ as small as we like. Integrating this, we can bound $\left\|\frac{\partial}{\partial s} dF\right\|_{p,\delta}$ by $2\left\|\frac{\partial \phi_s}{\partial s}\right\|_{1,p,\delta}$, and then integrating again gives

$$\left\|\frac{\partial}{\partial s} dF(s,1) - \frac{\partial}{\partial s} d(\phi_s)\right\|_{p,\delta} \leq c \left\|\frac{\partial \phi_s}{\partial s}\right\|_{1,p,\delta}$$

for $c > 0$ as small as we like.

□

Recall that we have defined

$$\mathcal{G}_k(u) = \exp_{\mathcal{G}(u)} \phi_k$$

where

$$\phi_0 = 0$$

$$\phi_{k+1} = \phi_k - Q_{\mathfrak{E}_R}(\bar{\partial}(\mathcal{G}_k(u)))$$

Proposition 2.10.2. *For quasi holomorphic graphs $u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$, it is possible to choose $\epsilon > 0$ small enough and R large enough that ϕ_k form a Cauchy sequence. In the limit, $\mathcal{G}_\infty(u) = \exp_{\mathcal{G}(u)} \phi_\infty$ satisfies*

$$\pi_{Q,\mathfrak{E}_R}(\bar{\partial}(\mathcal{G}_\infty(u))) = 0$$

Moreover, this is the unique solution to the above equation for maps of the form $\exp_{\mathcal{G}(u)} \phi$ for $\phi \in \mathcal{B}_{\mathfrak{E}_R}(S, \mathbb{C}^n)$ with $\|\phi\|_{1,p,\delta} \leq 1$. By choosing ϵ and R, it can be arranged that $\|\phi_\infty\|_{1,p,\delta}$ is as small as desired.

Proof:

Given any constant $0 < c < 1$, we can also choose ϵ and R so that Lemma 2.6.3 gives that

$$\left\|\pi_{Q,\mathfrak{E}_R}(\bar{\partial}\mathcal{G}(u))\right\| \leq \frac{c}{2\left\|Q_{\mathfrak{E}_R}\right\|}$$

this is because the left hand side shrinks with R faster than $e^{-(2-\delta)R}$ and Proposition 2.9.2 tells us that $\|Q_{\mathfrak{E}_R}\|$ grows like $e^{\delta R}$ and $\delta < \frac{1}{2}$.

Choose ϵ small enough that Lemma 2.10.1 can be used to show that

$$\left\|\frac{d}{ds} d(\exp_{\mathcal{G}(u)} \phi_s) - \frac{d}{ds}(d\phi_s)\right\|_{p,\delta} \leq \frac{1}{2}\left\|\frac{d\phi_s}{ds}\right\|_{1,p,\delta} \tag{2.6}$$

for $\|\phi\|_{1,p,\delta} \leq 1$. In particular, this means that if $\|\phi\|_{1,p,\delta} + \frac{\|\xi\|_{p,\delta}}{\|Q_{\mathfrak{E}_R}\|} \leq 1$, then

$$\left\|\bar{\partial} \exp_{\mathcal{G}(u)}(\phi + Q_{\mathfrak{E}_R}\xi) - \bar{\partial} \exp_{\mathcal{G}(u)} \phi - \xi\right\|_{p,\delta} \leq \frac{1}{2} \|\xi\|_{p,\delta}$$

This also implies that any solution with $\|\phi\|_{1,p,\delta} \leq 1$ will be unique.

Suppose that

$$\left\|\pi_{Q,\mathfrak{E}_R}(\bar{\partial}\mathcal{G}_k(u))\right\| \leq \frac{2^{-k}c}{2\left\|Q_{\mathfrak{E}_R}\right\|}$$

$$\|\phi_k\|_{1,p,\delta} \leq c(1 - 2^{-k})$$

Then

$$\left\|\pi_{Q,\mathfrak{E}_R}(\bar{\partial}\mathcal{G}_{k+1}(u))\right\| \leq \frac{2^{-k-1}c}{2\left\|Q_{\mathfrak{E}_R}\right\|}$$

$$\|\phi_k\|_{1,p,\delta} \leq c(1 - 2^{-k-1})$$

Therefore the above inequalities hold by induction, and the proposition is proved.

□

To summarize, what we have now is a map $\mathcal{G}_\infty$ which takes quasi holomorphic graphs in $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ and produces curves satisfying the equation

$$\pi_{Q,\mathfrak{E}_R}(\bar{\partial}(\mathcal{G}_\infty(u))) = 0$$

To find genuine holomorphic curves, we also want to solve the equation

$$\bar{\partial}_1 := (Id - \pi_{Q,\mathfrak{E}_R})(\bar{\partial}(\mathcal{G}_\infty(u))) = 0$$

Note that $(Id - \pi_{Q,\mathfrak{E}_R})$ projects onto a finite dimensional space, so if we can prove that $(Id - \pi_{Q,\mathfrak{E}_R})\bar{\partial}\mathcal{G}_\infty$ is continuous, then solutions of the above equation can be found by topological methods.

2.11 Continuity of $\mathcal{G}_\infty$

The strategy for proving that $\mathcal{G}_\infty$ is continuous will be to check that for every $u_0 \in \mathcal{Q}^{\epsilon,\mathfrak{C}_R}_{g,k,E}$ there will exist a neighborhood $u_0 \in U \subset \mathcal{Q}^{\epsilon,\mathfrak{C}_R}_{g,k,E}$ and a continuous family of maps f_u for $u \in U$ so that

$$f_{u_0} = \mathcal{G}_\infty(u_0)$$

$$\text{and } f_u = \exp_{\mathcal{G}(u)} \phi_u \text{ for some } \phi_u \in \mathcal{B}_{\mathfrak{C}}(S, \mathbb{C}^n)$$

Then applying the iteration procedure from the proof of Proposition 2.10.2 will give us that $\mathcal{G}_\infty$ must be continuous.

To say what we mean by 'continuous', we may want to use a weaker metric from the canonical one we have been using up until now so that the space of holomorphic curves can be compactified. An appropriate metric G must satisfy the following properties listed in Appendix B. Essentially, the torus action and J should be well behaved in the G metric. In particular, the canonical frame for the tangent space defined by the torus action and J should be bounded and have its derivatives bounded. The additional assumption that the curvature of G is bounded gives the following estimate:

$$\text{dist}_G(\exp_{p_1} \phi, \exp_{p_2} \phi) \leq \text{dist}_G(p_1, p_2)e^{c|\phi|}$$

Also, the change in the torsion tensor $\mathbf{T}$ measured in the canonical metric should be bounded by distance in G, ie

$$|\mathbf{T}_{p_1} - \mathbf{T}_{p_2}| \leq c\, \text{dist}_G(p_1, p_2)$$

To define the meaning of a 'continuous' family of maps from Riemann surfaces with changing conformal structures, we use the following somewhat ad hoc definition.

Definition 2.11.1. *Say a sequence of maps $f_i : S_i \longrightarrow \mathbb{T}^n \rtimes B^n$ converges to $f : S \longrightarrow \mathbb{T}^n \rtimes B^n$ if*

1. *S_i converges to S in Deligne Mumford space*

2. *There exists a $R \geq 0$ so that all $\mathfrak{E}_R$ edge regions in S are infinite cylinders and a $N \in \mathbb{Z}$ so that for all $i \geq N$ there exist identifications of the $\mathfrak{E}_R$ vertex regions of S_i with the $\mathfrak{E}_R$ vertex regions of S so that in these identifications*

   (a) *the complex structures j_i converge to j in the sense that*

   $$\|j_i - j\|_{1,p,\delta} \text{ converges to } 0$$

   (b) *f_i converges to f in on these $\mathfrak{E}_{R_i}$ vertex regions in the sense that*

   $$\|\mathrm{dist}_G(f_i, f)\|_\infty + \|df_i - df\|_{p,\delta} \text{ converges to } 0$$

   (c) *If a $\mathfrak{E}_R$ edge region of S_i is conformal to $\mathbb{R}/\mathbb{Z} \times (0, 2l)$, then identifying the first half of this with the subset $\mathbb{R}/\mathbb{Z} \times (0, l) \subset \mathbb{R}/\mathbb{Z} \times (0, \infty)$ of the corresponding $\mathfrak{E}_R$ edge region of S, then f_i converges on this subset to f as above in the sense that*

   $$\|\mathrm{dist}_G(f_i, f)\|_\infty + \|df_i - df\|_{p,\delta} \text{ converges to } 0$$

We will call a family of quasi holomorphic graphs u for which the combinatorics of the $\mathfrak{E}$ edge markings is constant continuous if $\mathcal{G}(u)$ is continuous. Note that this means that a continuous family of quasi holomorphic graphs is characterized by

1. The complex structure of the domain changes continuously

2. The position of edge holomorphic cylinders change continuously in the G metric

3. The relative positioning of the ends of edges at a vertex changes continuously in the canonical metric.

We now consider the effect on $\bar{\partial} f$ of small changes of complex structure which can be considered as changing j on the domain of f. We will deal separately with 'large' changes of complex structure which resolve nodes into long edge regions later. Consider a smooth family j_t of complex structures on S.

$$\begin{aligned}\frac{d}{dt}(\bar{\partial}_{j_t} f) &= \frac{d}{dt}(\frac{1}{2}(df + Jdf j_t) \\ &= \frac{1}{2} Jdf \frac{dj_t}{dt}\end{aligned}$$

The following lemma is immediate.

Lemma 2.11.2. *If* $\|df\|_\infty$ *is bounded, then*

$$\left\| \frac{d}{dt}(\bar{\partial}_{j_t} f) \right\|_{p,\delta} \leq \|df\|_\infty \left\| \frac{dj_t}{dt} \right\|_{p,\delta}$$

If $\|df_1 - df_2\|_{p,\delta}$ *is bounded, then*

$$\left\| \frac{d}{dt}(\bar{\partial}_{j_t} f_1) - \frac{d}{dt}(\bar{\partial}_{j_t} f_2) \right\|_{p,\delta} \leq \|df_1 - df_2\|_{p,\delta} \left\| \frac{dj_t}{dt} \right\|_\infty$$

This is enough to control the change in $\bar{\partial}$ of any of the maps which we are interested in for small changes of complex structure that do not change the nodal structure of our domain, as we can choose representatives for these variations in complex structure with bounded L^∞ and $L^{p,\delta}$ norms. Variations of complex structure near the boundary of the Deligne Mumford space are taken care of by Lemma 2.11.5.

Lemma 2.11.3. *Given any* $\mathfrak{E}_R$ *edge region* $\mathbb{R}/\mathbb{Z} \times (0, l)$ *and family of trivial holomorphic cylinders continuous in the G metric,*

$$C_v(\theta, t) := \exp_{p_v}(\theta\alpha + tJ^\epsilon\alpha) \text{ for } (\theta, t) \in \mathbb{R}/\mathbb{Z} \times (0, l)$$

the family

$$\exp_{C_v} \phi$$

is continuous for any $\phi \in L^{p,\delta}_1$

Proof:

define

$$F_v(s) = \exp_{C_v} s\phi$$

$$dF_v(s) = d(\exp_{C_v} s\phi)$$

Using Lemma 2.2.1

$$\frac{\partial}{\partial s} dF_v = \phi + \mathbf{T}_{F_v}(\phi, dF_v)$$

so using the assumptions on G listed in Appendix B

$$\begin{aligned}\left|\frac{\partial}{\partial s} dF_{v_1} - \frac{\partial}{\partial s} dF_{v_2}\right| &\leq \epsilon c \operatorname{dist}_G(F_{v_1}, F_{v_2}) |\phi| |dF_{v_1}| + \|\mathbf{T}\|_\infty |\phi| |dF_{v_1} - dF_{v_2}| \\ &\leq \epsilon c \operatorname{dist}_G(C_{v_1}, C_{v_2}) e^{\epsilon c|\phi|} |\phi| |dF_{v_1}| + \|\mathbf{T}\|_\infty |\phi| |dF_{v_1} - dF_{v_2}|\end{aligned}$$

Integrating this using $dC_{v_1} = dC_{v_2}$ for the initial conditions and noting that $|\phi|$ is bounded by $\|\phi\|_{1,p,\delta}$ gives that $\exp_{C_v} \phi$ is a continuous family.

□

The same proof gives the following

Lemma 2.11.4. *If $p(v)$ is a family of points continuous in the G metric and $\phi \in L_1^{p,\delta}$ is defined on some $\mathfrak{E}_R$ vertex region, then the family of maps defined on this $\mathfrak{E}_R$ vertex region by*

$$\exp_{p(v)} \phi$$

is continuous.

The following lemma deals with 'large' changes in complex structure that resolve nodes in our domain Riemann surfaces.

Lemma 2.11.5. *Take a trivial holomorphic cylinder*

$$C(\theta, t) = \exp_p(\theta\alpha + tJ^\epsilon\alpha) \text{ for } (\theta, t) \in \mathbb{R}/\mathbb{Z} \times (0, \infty)$$

of finite length in the G metric. Choose a smooth cutoff function $\rho : \mathbb{R} \longrightarrow [0, 1]$ so that $\rho(t) = 1$ for $t \leq -1$ and $\rho(t) = 0$ for $t \geq 1$. Then for any $\phi_1, \phi_2 \in L_1^{p,\delta}$, the *family*

$$\exp_{C(\theta,t)}\big(\rho(t-l)\phi_1(\theta,t)+\rho(l-t)\phi_2(2l-t)\big) \text{ for } (t,\theta)\in\mathbb{R}/\mathbb{Z}\times(0,2l)$$

is a continuous family when parametrized by the length of C restricted to $(t,\theta)\in\mathbb{R}/\mathbb{Z}\times(0,2l)$ in the G metric plus $\frac{l}{1+l}$.

Proof:

Because multiplication by ρ and shifting are continuous operations on $L_1^{p,\delta}$ we know that $(\rho(t-l)\phi(\theta,t)+\rho(l-t)\phi_2(2l-t))$ will be continuous in l for l bounded. Note also that the first half of this converges to ϕ_1 on $\mathbb{R}/\mathbb{Z}\times(0,\infty)$ and the second half converges to ϕ_2 on $\mathbb{R}/\mathbb{Z}\times(-\infty,0)$, reparametrizing from the other end of the cylinder. The Lemma then follows from Lemma 2.10.1 and Lemma 2.11.3.

□

Proposition 2.11.6. *For R large enough and ϵ small enough, $\mathcal{G}_\infty$ is continuous at regular points of $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ around which the combinatorics of the $\mathfrak{E}_R$ edge markings do not jump.*

Proof:

Choose some graph $u_0\in\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ so that in a neighborhood of u_0, the combinatorics of $\mathfrak{E}_R$ edge markings do not change. We will show that $\mathcal{G}_\infty$ is continuous at u_0. Consider the map $f_{u_0}:=\mathcal{G}_\infty(u_0)$. We define a continuous family of maps f_u parametrized by quasi holomorphic graphs u in a neighborhood of u_0. We do this as follows. Consider the domain S of u_0. We will only consider quasi holomorphic graphs u with domains that have the same $\mathfrak{E}_R$ edge marking combinatorics, so we have correspondences between the edge and vertex regions of S and those of u. We then choose a family of maps identifying S with all infinite edge regions removed with the corresponding regions in u. By choosing our neighborhood small enough, we can choose this family continuous with the resulting family of complex structures j_u continuous using the $L_1^{p,\delta}$ norm. We also choose these identifications so that they continue as complex maps halfway along the remaining edges of u into the remaining (infinitely long) edges regions of S.

We will define an attempt at f_u, which we will call $\tilde{f}_u$ separately on each vertex and edge region, and then smooth this together into f_u using a cutoff function.

Now define $\tilde{f}_u$ on the internal edge regions of S which aren't infinitely long and all external edge regions as follows:

$$\text{If } f_{u0} = \exp_{C_{u_0}} \phi$$

$$\text{Define } \tilde{f}_u := \exp_{C_u} \phi$$

Above, C_u indicates the corresponding trivial holomorphic cylinder which is an edge of u parametrized so that its center corresponds to the center of our edge region in S. (Note that we might need to extend or shorten C_u to make this fit.)

Define $\tilde{f}_u$ on a neighborhood of vertex regions of S as follows: Choose the location of a family of vertex model curves given by u to be $p(u)$. We can choose $p(u)$ to be continuous when measured in the G metric.

$$\text{If } f_{u0} = \exp_{p(u0)} \phi$$

$$\text{Define } \tilde{f}_u := \exp_{p(u)} \phi$$

Define $\tilde{f}_u$ on edge regions corresponding to internal pairs of infinitely long edge regions of S as follows:

$$\text{If } f_u = \exp_{C_1} \phi_1 \text{ on one cylinder}$$

$$\text{and } f_u = \exp_{C_2} \phi_2 \text{ on its mate}$$

$$\text{Define } \tilde{f}_{u0}(\theta, t) = \exp_{C_{u_0}} (\rho(t-l)\phi_1(\theta, t) + \rho(l-t)\phi_2(\theta, 2l-t))$$

The expression above is for $(\theta, t) \in \mathbb{R}/\mathbb{Z} \times (0, 2l)$ where this parametrizes the correct edge region of u conformally, and C_u is the corresponding trivial holomorphic cylinder.

Lemmas 2.11.2, 2.11.3, 2.11.4, and 2.11.5 tell us that $\tilde{f}_u$ so defined will be continuous families on their respective regions. Note also that $\tilde{f}_u$ will match up approximately in a way that approaches 0 in our canonical metric when u approaches u_0 So we can smooth $\tilde{f}_u$ using a cutoff function in a collar neighborhood of each vertex region to produce a continuous family of maps f_u for which f_{u_0} is our original function. In particular, we know that $\left\|\pi_{Q,\mathfrak{E}}\bar{\partial} f_u\right\|_{1,p,\delta}$ approaches 0 as u approaches u_0. Applying the iteration procedure from Proposition 2.10.2 will converge in a neighborhood of u_0 if R is large enough and ϵ small enough. This will produce a family of functions satisfying $\pi_{Q,\mathfrak{E}}\bar{\partial} = 0$ which must correspond to $\mathcal{G}_\infty$ in some neighborhood of u_0 because of the uniqueness statement in Proposition 2.10.2. Moreover, this family must approach f_{u_0} as u approaches 0. Therefore, $\mathcal{G}_\infty$ is continuous at u_0.

□

Now we want to show that $\bar{\partial} \circ \mathcal{G}_\infty$ is a continuous section

$$\bar{\partial} \circ \mathcal{G}_\infty : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{E}$$

as defined in Definition 1.1.3. We need this special definition because the bundle

$$\begin{array}{rl} \ker Q_{\mathfrak{E}_R} := \mathcal{H}_{\mathfrak{E}_R}(S, \mathbb{C}^n) \longrightarrow & \mathcal{E} \\ & \downarrow \\ & \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \end{array}$$

(which $\bar{\partial} \circ \mathcal{G}_\infty$ is a section of) jumps dimensions in the fiber and base when the combinatorics of $\mathfrak{E}_R$ edge markings changes. Note that we have chosen $Q_{\mathfrak{E}_R}$ so that

$$\ker Q_{\mathfrak{E}_{\tilde{R}}} \subset \ker Q_{\mathfrak{E}_R} \text{ for all } \tilde{R} \geq R$$

as Proposition 2.9.2 tells us we can. The idea is that if $\ker Q_{\mathfrak{E}_R}$ jumps, we can increase R a little to $\tilde{R}$ so that there is locally no jumping of $\ker Q_{\mathfrak{E}_{\tilde{R}}}$. The analysis in this case will then tell us that $\bar{\partial} \circ \mathcal{G}_\infty$ is well behaved. To do this we need a way of relating

$\mathcal{G}_\infty$ on $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ and $\mathcal{Q}^{\epsilon,\mathfrak{E}_{\tilde{R}}}_{g,k,E}$.

Lemma 2.11.7. *If R is large enough and ϵ small enough, there exists a map*

$$\pi_{R,\tilde{R}} : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{Q}^{\epsilon,\mathfrak{E}_{\tilde{R}}}$$

for $R+1 \geq \tilde{R} \geq R$ so that for every $u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ satisfying

$$\pi_{Q,\mathfrak{E}_{\tilde{R}}} \bar{\partial} \mathcal{G}_\infty u = 0$$

$$\mathcal{G}_\infty(u) = \mathcal{G}_\infty(\pi_{R,\tilde{R}} u)$$

Proof:

Note that in order for the above to make sense, we may need to extend $\mathcal{G}_\infty$ to

$$\tilde{\mathcal{Q}}^{\epsilon,\mathfrak{E}_{\tilde{R}}}_{g,k,E} := \{u \in Q^{\epsilon,\mathfrak{E}_{\tilde{R}}} \text{ so that } \left\|\bar{\partial}_0 u\right\| < 2, E(u) \leq E, \text{genus}(u) = g, k \text{ punctures}\}$$

This differs from $\mathcal{Q}^{\epsilon,\mathfrak{E}_{\tilde{R}}}_{g,k,E}$ only in that the requirement on $\bar{\partial}$ of the model curves is weakened so $\left\|\bar{\partial}_0 u\right\| < 2$ rather than $\left\|\bar{\partial}_0 u\right\| < 1$. All proofs concerning $\mathcal{Q}^{\epsilon,\mathfrak{E}_{\tilde{R}}}_{g,k,E}$ work for $\tilde{\mathcal{Q}}^{\epsilon,\mathfrak{E}_{\tilde{R}}}_{g,k,E}$ so long as $\tilde{R}$ is large enough and ϵ small enough. The reason that we may need to do this is the image of $\pi_{R,\tilde{R}}\left(\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}\right)$ may not be in $\mathcal{Q}^{\epsilon,\mathfrak{E}_{\tilde{R}}}_{g,k,E}$, but it can be verified using Lemma 2.2.4 that for ϵ small enough it will be contained in $\tilde{\mathcal{Q}}^{\epsilon,\mathfrak{E}_{\tilde{R}}}_{g,k,E}$.

$\pi_{R,\tilde{R}}$ is defined in Appendix C.3, here we only need to verify its extra properties. With that plan, choose some u so that

$$\pi_{Q,\mathfrak{E}_{\tilde{R}}} \bar{\partial} \mathcal{G}_\infty(u) = 0$$

Choose R large enough and ϵ small enough so that Proposition 2.10.2 works using the $L^{p,\delta}_1$ norm for both $\delta = 0.4$ and $\delta = 0.1$, and we have

$$\mathcal{G}_\infty(u) = \exp_{\mathcal{G}(u)} \phi$$

with $\|\phi\|_{1,p,0.4} < \frac{1}{2}$.

The edges of $\pi_{R,\tilde{R}} u$ correspond to a subset of the edges of u and we use the same trivial holomorphic cylinders. As in Appendix C.3, for every vertex region of $\pi_{R,\tilde{R}}u$, there is some point $p \in \mathbb{T}^n \rtimes B^n$ so that the attached edges start at $\exp_p \zeta_i$ with ζ_i bounded and $\sum_i \zeta_i = 0$. Then consider the map $F : V \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ on the vertex region V

$$u = \exp_{\exp_p(\sum \zeta_i\psi_i)}(F - \sum \zeta_i\psi_i)$$

with notation as in section 2.6. F will be bounded, and we can use Lemma 2.2.4 to prove that for ϵ sufficiently small

$$\left\|\bar{\partial}F - \bar{\partial}\mathcal{G}_\infty u\right\|_{p,0.4}$$

is small. We also have that F is close in $L_1^{p,0.4}$ to the appropriate trivial holomorphic cylinders starting at ζ_i. We can then extend F to all the model domain S_V so that for ϵ small enough,

$$\left\|Q_{\mathfrak{e}_{\tilde{R}}}\bar{\partial}F\right\|_{1,p,0.1} \leq ce^{-0.2\tilde{R}}$$

This uses that the growth of $Q_{\mathfrak{e}_R}$ in the norm with $\delta = 0.1$ is $e^{0.1R}$ as proved in Proposition 2.9.2. Note that in particular, by choosing R large enough, this can be made as small as we like. Our model curve will then be given by

$$[p, f] := [p, F - Q_{\mathfrak{e}_{\tilde{R}}}\bar{\partial}F]$$

We can then use Lemma 2.2.4 to prove that $\mathcal{G}([p, f])$ is close in the $L_1^{p,0.1}$ norm to $\mathcal{G}_\infty(u)$.

The quasi holomorphic graph with these model curves and edges as above will be $\pi_{R,\tilde{R}}u \in \mathcal{Q}^{\epsilon,\mathfrak{e}_{\tilde{R}}}$. We know already that $\mathcal{G}_\infty(u)$ is close to the edge trivial holomorphic cylinders in $L_1^{p,0.4}$, so it is close in $L_1^{p,0.1}$. The above argument gives that it is also close in $L_1^{p,0.1}$ in vertex regions, so for some $\phi \in \mathcal{B}_{\mathfrak{e}_{\tilde{R}}}(S, \mathbb{C}^n)$ with $\|\phi\|_{1,p,0.1} < 1$

$$\mathcal{G}_\infty(u) = \exp_{\pi_{R,\tilde{R}}(u)} \phi$$

The fact that $\pi_{Q,\mathfrak{E}_{\tilde{R}}}\bar{\partial}\mathcal{G}_\infty(u) = 0$ implies that

$$\mathcal{G}_\infty(u) = \mathcal{G}_\infty(\pi_{R,\tilde{R}}u)$$

because of the uniqueness statement in Proposition 2.10.2.

□

Theorem 2.11.8. *For R large enough, and ϵ small enough,*

$$\bar{\partial} \circ G_\infty : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{E}$$

is a continuous section in the sense defined by Definition 1.1.3.

Proof:

At regular points of $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ where there is a neighborhood in which the combinatorics of $\mathfrak{E}_R$ edge regions don't jump, we've seen that for R large enough and ϵ small enough, $\bar{\partial} \circ \mathcal{G}_\infty$ is continuous, as around those points we just use the normal definition.

Now consider an arbitrary graph $u_0 \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$. There exists some $\tilde{R} \geq R$ so that $\ker Q_{\mathfrak{E}_R} = \ker Q_{\mathfrak{E}_{\tilde{R}}}$ at u_0, and $\ker Q_{\mathfrak{E}_{\tilde{R}}}$ doesn't jump in a neighborhood of u_0. We know that if we have chosen R large enough and ϵ small enough,

$$\bar{\partial} \circ \mathcal{G}_\infty : \mathcal{Q}^{\epsilon,\mathfrak{E}_{\tilde{R}}}_{g,k,E} \longrightarrow \ker Q_{\mathfrak{E}_{\tilde{R}}}$$

will be continuous.

We are interested in solutions of the equation

$$\pi_{Q,\mathfrak{E}_{\tilde{R}}}\left(\bar{\partial} \circ G_\infty\right) = 0 \text{ on } \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$$

in some neighborhood U of u_0. Call the set of solutions $U^{\mathfrak{E}_{\tilde{R}}}$. The uniqueness statement in Proposition 2.10.2 implies that this is the same thing as looking for solutions to the equation

$$\pi_{Q,\mathfrak{E}_{\tilde{R}}}\left(\bar{\partial} f\right) = 0$$

for maps f in some open set in the space of maps. Note that Lemma 2.10.1 used as in Proposition 2.10.2 implies that $\bar{\partial} \circ \mathcal{G}_\infty$ is transverse to 0 at $U^{\mathfrak{E}_{\tilde{R}}}$ restricted to variations in $\mathcal{B}_{\mathfrak{E}_{\tilde{R}}}(S, \mathbb{C}^n)$, so $U^{\mathfrak{E}_{\tilde{R}}}$ is transversely cut out.

Lemma 2.11.7 combined with the uniqueness statement of Proposition 2.10.2 tells us that $U^{\mathfrak{E}_{\tilde{R}}}$ will be the intersection of $\mathcal{G}_\infty\left(\tilde{\mathcal{Q}}^{\epsilon,\mathfrak{E}_{\tilde{R}}}_{g,k,E}\right)$ with the above open set. The inverse image of this open set under

$$\mathcal{G}_\infty : \tilde{\mathcal{Q}}^{\epsilon,\mathfrak{E}_{\tilde{R}}}_{g,k,E} \longrightarrow \text{ maps to } \mathbb{T}^n \rtimes B^n$$

will an open set $U' \subset \tilde{\mathcal{Q}}^{\epsilon,\mathfrak{E}_{\tilde{R}}}_{g,k,E}$. Note the proof of Lemma 2.7.1 along with Proposition 2.11.6 tells us that for ϵ small enough $\mathcal{G}_\infty$ will provide a homeomorphism

$$\mathcal{G}_\infty : U' \longrightarrow U^{\mathfrak{E}_{\tilde{R}}}$$

Noting that $\bar{\partial}\mathcal{G}_\infty$ is continuous on U' gives that all the conditions for $\bar{\partial}\mathcal{G}_\infty$ to be continuous at $u_0 \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$.

□

Chapter 3

Convergence to graphs

The goal of this chapter is to show that all bounded energy J^ϵ holomorphic curves are in the image of the gluing map $\mathcal{G}_\infty$ which is the subject of the previous chapter. This is achieved by showing that any bounded energy J^ϵ holomorphic curve in $\mathbb{T}^n \rtimes B^n$ will be close to a J^ϵ quasi holomorphic graph of the type described in section 1.1.4 for ϵ small enough. The first task is to describe the properties of the taming form ω which will keep our holomorphic curves well behaved.

3.1 Taming form

The class of manifolds under consideration are Lagrangian fibrations of the form

$$\begin{array}{c} \mathbb{T}^n \longrightarrow \mathbb{T}^n \rtimes B^n \\ \downarrow \pi \\ B^n \end{array}$$

ω is a symplectic form on $\mathbb{T}^n \rtimes B^n$ which is symmetric with respect to the torus rotations, and which vanishes when restricted to torus fibers. The first requirement is that ω is positive on J holomorphic planes, this together with the condition that torus fibers are Lagrangian implies that ω is positive on any J^ϵ holomorphic plane.

Lemma 3.1.1. *ω is positive on J^ϵ holomorphic planes for any $\epsilon \in (0, 1]$*

Proof:

This is a simple computation. Represent an arbitrary vector as $v_1 + J^\epsilon v_2$ where v_1 and v_2 are vertical vectors tangent to torus fibers. Then

$$\omega(v_1 + J^\epsilon v_2, J^\epsilon v_1 - v_2) = \epsilon(\omega(v_1, Jv_1) + \omega(v_2, Jv_2)) + \epsilon^2(\omega(Jv_2, Jv_1))$$

By assumption $\omega(v_1, Jv_1)$ and $\omega(v_2, Jv_2)$ are both positive, so the only way the above expression can be negative is if $\omega(Jv_2, Jv_1)$ is negative. In that case, the above expression is greater than

$$\epsilon(\omega(v_1, Jv_1) + \omega(v_2, Jv_2)) + \omega(Jv_2, Jv_1)) = \epsilon\omega(v_1 + Jv_2, J(v_1 + Jv_2)) \geq 0$$

□

As we are in the non compact setting, we have some extra requirements on ω (these are listed in appendix B and checked for some spaces). First, we require that the ω energy,

$$E_\omega := \int_S u^*\omega$$

is constant for any continuous family of holomorphic curves u. Actually, we want the same to be true for slightly more flexible families of maps. We will say a smooth map from a punctured Riemann surface $u : S \longrightarrow \mathbb{T}^n \rtimes B^n$ has *holomorphic ends* if there is some open neighborhood of the punctures of S on which u is holomorphic. If the energy of u is finite, this will place some restrictions on the behavior of u near punctures; for example, if $\mathbb{T}^n \rtimes B^n$ has cylindrical ends, the punctures will either be removable singularities or converge to cylinders over Reeb orbits, if $\mathbb{T}^n \rtimes B^n$ arises as a dense open set in some compact symplectic manifold, then punctures will need to be removable singularities in this compact manifold. We assume that the ω energy is constant on any continuous family of maps with J^ϵ holomorphic ends.

The non compactness of our problem and the need to tame a degenerating family of complex structures requires us to have a little more flexibility in our taming forms. Call $\Lambda_{\omega,E}$ the set of taming forms $\tilde{\omega}$ which are positive on holomorphic planes, vanish restricted to torus fibers, give constant integrals restricted to connected components of the space of maps with holomorphic ends, and so that $E_{\tilde{\omega}}(u) = E_\omega(u)$ for any J^ϵ holomorphic map u with $E_\omega(u) \leq E$. If $D \subset S$ is part of the domain of a holomorphic map u, define the energy of the map restricted to D to be

$$E_D(u) := \sup_{\tilde{\omega} \in \Lambda_{\omega, E_\omega(u)}} \int_D u^* \tilde{\omega}$$

Note that $E_D(u) \leq E_\omega(u)$.

The final condition on our manifold and ω is that for any energy bound E there exists an $r > 0$ smaller than the injectivity radius of B^n and $c_E > 0$ so that for every $p \in \mathbb{T}^n \rtimes B^n$ there exists a taming form $\omega_{p,E} \in \Lambda_{\omega,E}$ so that

$$\omega_p (v, Jv) \geq c_E |v|^2 \tag{3.1}$$

for all tangent vectors v based at points within a distance r from p. This gives a way of controlling the local area of our holomorphic curves.

The above assumptions are listed and checked for some spaces in appendix B.

Note that if we want a local area bound for J^ϵ holomorphic curves, the above forms will give us the estimate

$$\omega_{p,E}(v, J^\epsilon v) \geq \epsilon c_E |v|^2$$

This is not good enough. We will need to make an exact adjustment to ω_p to concentrate it further around the point p in order to get a good local area bound for J^ϵ holomorphic curves.

First, use Lemma 2.2.4 to find a radius $r > 0$ smaller than the injectivity radius of B^n and the r used in the definition of the $\omega_{p,E}$, so that in a ball of radius r around any point $p \in \mathbb{T}^n \rtimes B^n$,

$$\left|(d \exp_p)(v) - v\right| \leq \frac{1}{2} |v|$$

We now describe an exact alteration to $\omega_{p,E}$ in coordinates centered at p. If the coordinates for $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ are given by x_i, then use the coordinates

$$\exp_p(x + \sum y_i J\partial_{x_i}) \mapsto (x, y) \in \frac{\mathbb{R}^n}{\mathbb{Z}^n} \times \mathbb{R}^n$$

Note that in these coordinates our above estimate becomes

$$|J\partial_{x_i} - \partial_{y_i}| \leq \frac{1}{2} \tag{3.2}$$

Now consider the two-form

$$\theta_{p,\epsilon} := d\left(\sum -f_i^\epsilon dx_i\right)$$

Here f_i^ϵ is a smooth function of y supported in the ball of radius r satisfying the following conditions involving the constant c_E from the definition of $\omega_{p,E}$.

$$\frac{c_E}{4n\epsilon} \geq \frac{\partial f_i^\epsilon}{\partial y_i} \geq -\frac{c_E}{4n}$$

$$\left|\frac{\partial f_i^\epsilon}{\partial y_j}\right| < \frac{c_E}{4n} \text{ for } i \neq j$$

and, for $|y_i| \leq \frac{r\epsilon}{3}$ and $|y| \leq \frac{r}{3}$

$$\frac{\partial f_i^\epsilon}{\partial y_i} = \frac{c_E}{4n\epsilon}$$

$$\frac{\partial f_i^\epsilon}{\partial y_j} = 0, i \neq j$$

The following lemma holds in the metric rescaled to be preserved by J^ϵ.

Lemma 3.1.2. *For $\epsilon < \frac{1}{6n}$*

$$\theta_{p,\epsilon}(v, J^\epsilon v) \geq -\epsilon c_E |v|^2$$

and within $\frac{r\epsilon}{3}$ of the torus fiber at p,

$$\theta_{p,\epsilon}(v, J^\epsilon v) \geq \frac{c_E}{16n} |v|^2$$

Proof:

Decompose v as $v_1 + J^\epsilon v_2$ where v_1 and v_2 are torus directions. Note that the stated properties of f_i together with our estimate 3.2 are enough to tell us:

$$\theta(v, w) \geq -|v|\,|w|\,\frac{3c_E}{8\epsilon}$$

$$\theta(v_1, Jv_1) \geq -\frac{3c_E}{8}\,|v_1|^2$$

$$\theta(v_1, Jv_1) \geq \frac{c_E}{8n\epsilon}\,|v_1|^2$$

With the above estimates, we can show that $\theta(v, J^\epsilon v)$ can't be too negative

$$\begin{aligned}\theta(v, J^\epsilon v) &= \theta(v_1, \epsilon J v_1) + \theta(v_2, \epsilon J v_2) + \theta(\epsilon J v_2, \epsilon J v_1)\\ &\geq -(|v_1|^2 + |v_2|^2 + |v|_1\,|v|_2)\epsilon\frac{3c_E}{8}\\ &\geq -\epsilon c_E\,|v|^2\end{aligned}$$

Also, within $\frac{r\epsilon}{3}$ of the torus fiber at p,

$$\begin{aligned}\theta(v, J^\epsilon v) &= \theta(v_1, \epsilon J v_1) + \theta(v_2, \epsilon J v_2) + \theta(\epsilon J v_2, \epsilon J v_1)\\ &\geq (|v_1|^2 + |v_2|^2)\frac{c_E}{8n} - |v|_1\,|v|_2\,\epsilon\frac{3c_E}{8}\\ &\geq \frac{c_E}{16n}\,|v|^2\end{aligned}$$

□

Define the set of taming forms $\Lambda_{\omega,E,\epsilon}$ to be the set of closed two forms that are nonnegative on J^ϵ holomorphic planes, vanish when restricted to torus fibers, and give the same integral as ω when restricted to J^ϵ holomorphic curves of energy less than E. We have that

$$\omega_{p,E,\epsilon} := \omega_{p,E} + \theta_{p,\epsilon} \in \Lambda_{\omega,E,\epsilon}$$

We can use the forms $\omega_{p,E,\epsilon}$ to prove the following lemma.

Lemma 3.1.3. *Given an energy bound E, there exists an $r > 0$ and $c_E > 0$ so that* *for all positive* $\epsilon \leq 1$ *and* $p \in \mathbb{T}^n \rtimes B^n$*, there exists a taming form* $\omega_{p,E,\epsilon} \in \Lambda_{\omega,E,\epsilon}$ *so that in the metric corresponding to* J^ϵ*,*

$$\omega_{p,E,\epsilon}(v, J^\epsilon v) > c_E \left|v\right|^2$$

for tangent vectors v *within* r *of the torus fiber at* p*. Moreover,* $\omega_{p,E,\epsilon}$ *can be chosen so that*

$$\lim_{\epsilon \to 0} (\exp_p^\epsilon)^* \omega_{p,E,\epsilon} = c_E \omega_0$$

where $\exp_p^\epsilon(x, y) := \exp_p(x + \sum y_i J^\epsilon \partial_{x_i})$ *and* ω_0 *is a standard taming form on* $\mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^n$ *given by*

$$\omega_0 = \sum d\left(-f(y_i) dx_i\right)$$

where f *is some increasing function.*

A corollary of the above lemma is that if some component of the space of quasi holomorphic graphs $\mathcal{Q}^{\epsilon, \mathfrak{C}_R}_{g,k,E}$ contains a graph u so that $\mathcal{G}_\infty u$ is holomorphic, then the energy of the individual vertex model curves in that component is bounded.

3.2 Derivative bounds

In what follows we will obtain a first derivative bound for our holomorphic maps by a bubbling type argument. First, we will apply a standard elliptic regularity lemma for the linear $\bar{\partial}$ operator to show that bounds on the first derivatives of J^ϵ holomorphic curves will imply second derivative bounds independent of ϵ.

Lemma 3.2.1. *For a given integer* k *and* $1 < p < \infty$*, there exists a constant* c *so that*

$$\|u\|_{L^p_{k+1}(D(\frac{1}{2}))} \leq c \left(\left\|\bar{\partial} u\right\|_{L^p_k(D(1))} + \|u\|_{L^p(D(1))} \right)$$

where $L^p_k(D(1))$ *indicates the* L^p *norm on the first* k *derivatives in the unit complex disk.*

Lemma 3.2.2. *If* $u : D(1) \longrightarrow \mathbb{T}^n \rtimes B^n$ *is holomorphic, then remembering the identification of* $T_p(\mathbb{T}^n \rtimes B^n)$ *with* $\mathbb{C}^n$ *and using coordinates* (x, y) *for the unit disk,*

$u_x : D(1) \longrightarrow \mathbb{C}^n$ *satisfies*

$$\bar{\partial}(u_x) = \frac{1}{2} J\mathbf{T}(u_y, u_x)$$

Proof:

$$\begin{aligned}\bar{\partial}(u_x) &= \frac{1}{2}(\nabla_{u_x} u_x + J\nabla_{u_y} u_x) \\ &= \frac{1}{2}(\nabla_{u_x}(u_x + Ju_y) + J\mathbf{T}(u_y, u_x)) \\ &= \frac{1}{2} J\mathbf{T}(u_y, u_x)\end{aligned}$$

□

Note that the $\bar{\partial}$ operator on derivatives is just the usual linear $\bar{\partial}$ operator in our trivialization. In particular we can apply Lemma 3.2.1 to u_x in the case that du is bounded to obtain

Lemma 3.2.3. *for every* $1 < p < \infty$*, there exists a constant* c *so that for any holomorphic map* $u : D(1) \longrightarrow \mathbb{T}^n \rtimes B^n$*,*

$$\|du\|_{L^p_1(D(\frac{1}{2}))} \leq c(\|\mathbf{T}\|_\infty \|du\|^2_{L^\infty(D(1))} + \|du\|_{L^\infty(D(1))})$$

Following a bubbling type argument, we are now ready to get a derivative bound for J^ϵ holomorphic curves disks with bounded energy when ϵ is close to 0

Proposition 3.2.4. *For a given energy bound* E*, there exists an* $\epsilon_E > 0$ *so that for all* $0 < \epsilon < \epsilon_E$*, all* J^ϵ *holomorphic curves with* ω *energy less than* E *satisfy a uniform derivative bound.*

In particular, given any domain in a bounded energy J^ϵ *holomorphic curve* u *conformal to the complex unit disk,* $u : D \longrightarrow \mathbb{T}^n \rtimes B^n$ *satisfies*

$$|\nabla u(0)| \text{ bounded}$$

Proof:

Suppose to the contrary that there exists a sequence $u_i : S_i \longrightarrow \mathbb{T}^n \rtimes B^n$ of bounded energy J^{ϵ_i} holomorphic curves with a sequence of holomorphic inclusions of the unit disc $D \subset S_i$, so that $\lim_{i\to\infty} \epsilon_i = 0$ and

$$u_i : D \longrightarrow \mathbb{T}^n \rtimes B^n$$

$$\text{satisfies } \lim_{i\to\infty} |\nabla u_i(0)| = \infty$$

We will use a minor modification of the proof of the bubbling lemma 5.11 from [1] to obtain a nontrivial holomorphic plane with bounded energy and a contradiction. Note that all these maps satisfy

$$\sup_{\tilde{\omega}\in\Lambda_{\omega,E,\epsilon_i}} \int_D u_i^*\tilde{\omega} \le E$$

First, we obtain a sequence of rescaled J^{ϵ_i} holomorphic maps $\tilde{u}_i : D(R_i) \longrightarrow \mathbb{T}^n \rtimes B^n$ as in the proof of lemma 5.11 in [1] so that

$$|\nabla \tilde{u}_i| \le 2$$

$$|\nabla \tilde{u}_i(0)| = 1$$

$$\lim_{i\to\infty} R_i = \infty$$

Suppose that R_i is also chosen so that

$$\lim_{i\to\infty} \epsilon_i R_i = 0$$

Write $\tilde{u}_i$ in coordinates centered on $\tilde{u}_i(0)$:

$$\tilde{u}_i = \exp_{\tilde{u}_i(0)} \phi_i$$

Identifying $(T_{\tilde{u}_i(0)}(\mathbb{T}^n \rtimes B^n), J^{\epsilon_i})$ with $\mathbb{C}^n$, we get a sequence of functions

$$\phi_i : D(R_i) \longrightarrow \mathbb{C}^n$$

Note that $|\phi_i| \leq 2R_i$ so Lemma 2.2.4 tells us that there exists a constant c so that

$$|d\tilde{u}_i - d\phi_i| \leq c\epsilon_i R_i e^{c\epsilon_i R_i} |d\phi_i|$$

so for i large enough, $|d\phi_i| \leq 3$, and

$$\lim_{i\to\infty} \left|\bar{\partial}\phi_i\right| = 0$$

Lemma 3.2.3 implies that as $|d\tilde{u}_i| \leq 2$, the L^p_1 norm of $d\tilde{u}_i$ is bounded on compact subsets of $\mathbb{C}$. As the inclusion of L^p_1 into C^0 is compact for $p > 2$, we can choose a subsequence of $\tilde{u}_i$ so that $d\tilde{u}_i$ converges in C^0_{loc}, and hence ϕ_i converges in C^1_{loc} to a holomorphic map $\phi : \mathbb{C} \longrightarrow \mathbb{C}^n$ with energy less than E and $|\nabla\phi(0)| = 1$. Recall that Lemma 3.1.3 tells us that

$$E \geq \int_{\mathbb{C}} c_E \phi^* \omega_0 = \int_{\mathbb{C}} c_E \phi^*(-\sum df(y_i) dx_i)$$

However, at least one component of ϕ is a nonconstant, entire holomorphic map $\mathbb{C} \longrightarrow \mathbb{C}$, and hence must cover all of that component of $\mathbb{C}^n$. Therefore the energy of ϕ must be infinite, a contradiction.

□

3.3 Convergence of cylinders

In this section, we use the derivative bound from Proposition 3.2.4 to prove that the parts of bounded energy J^ϵ holomorphic curves conformal to long cylinders will be close to trivial holomorphic cylinders for ϵ small.

Consider a J^ϵ holomorphic curve u with energy less than E that has a subset of its domain conformal to $S^1 \times [-1, R+1]$. Proposition 3.2.4 tells us for ϵ small enough that the restriction of u to the cylinder

$$u : S^1 \times [0, R] \longrightarrow \mathbb{T}^n \times B^n$$

satisfies a derivative bound depending only on the energy E.

Now consider a nearby trivial holomorphic cylinder $C : S^1 \times [0, R] \longrightarrow \mathbb{T}^n \rtimes B^n$ so that $u_*([S^1]) = C_*([S^1]) \in H_1(\mathbb{T}^n)$. Note that our derivative bound tells us that for ϵ small enough, $u(S^1)$ must be contained in a ball in the base B^n which is smaller than the injectivity radius of B^n, so $u_*([S^1]) = C_*([S^1]) \in \pi_1(\mathbb{T}^n \rtimes B^n)$. Recall that trivial cylinders are holomorphic cylinders of the form $C(\theta, t) = \exp_p(\theta v + tJ^\epsilon v)$. Write u in coordinates about this cylinder:

$$u(\theta, t) = \exp_{C(\theta,t)} \phi(\theta, t)$$

The main lemma we will use will be Lemma 2.2.4 which tells us that

$$|du - d\phi - dC| \leq \| \mathbf{T}\|_\infty |\phi| \, |dC| \, e^{\|\mathbf{T}\|_\infty \|\phi\|} + |d\phi| \, (e^{\|\mathbf{T}\|_\infty |\phi|} - 1)$$

Note that for $\|\mathbf{T}\|_\infty |\phi|$ small, this gives a bound on $|d\phi|$ as we already have bounds on $|du|$ and $|dC|$. From now on, assume that $\|\mathbf{T}\|_\infty |\phi|$ is small enough so we have a bound on $|d\phi|$. Note that this will give us a bound for $|\phi(\theta_1, t) - \phi(\theta_2, t)|$. Simplifying the above expression, we have for $\|\mathbf{T}\|_\infty |\phi|$ small enough,

$$|du - d\phi - dC| \leq c \, \|\mathbf{T}\|_\infty |\phi| \tag{3.3}$$

We have

$$\int \frac{\partial \phi}{\partial \theta}(\theta, t) d\theta = 0$$

so

$$\begin{aligned} \left| \frac{d}{dt} \int \phi d\theta \right| &\leq \int \left| \bar{\partial} \phi \right| d\theta \\ &\leq c \, \|\mathbf{T}\|_\infty \max_{\theta \in S^1} |\phi(\theta, t)| \\ &\leq c \, \|\mathbf{T}\|_\infty \left(\left| \int \phi d\theta \right| + \max_{\theta_1, \theta_2 \in S^1} |\phi(\theta_1, t) - \phi(\theta_2, t)| \right) \end{aligned} \tag{3.4}$$

$C(\theta, t)$ can be chosen so that $\int \phi(\theta, 0) d\theta = 0$, so for ϵ small enough there is a constant c_R so that integrating the above expression gives,

$$\left|\bar{\partial}\phi\right| \leq c_R \epsilon \max |\phi(\theta_1, t) - \phi(\theta_2, t)| \tag{3.5}$$

and

$$\|\phi\|_\infty \leq (c_R \epsilon + 1) \max |\phi(\theta_1, t) - \phi(\theta_2, t)| \tag{3.6}$$

Now we can use the Cauchy integral formula on ϕ:

$$\begin{aligned}\phi(\theta_0, t_0) = &\int \frac{\phi(\theta, 0)}{1 - e^{-2\pi t_0} e^{2\pi i(\theta - \theta_0)}} d\theta - \int \frac{\phi(\theta, R)}{1 - e^{2\pi (R - t_0)} e^{2\pi i(\theta - \theta_0)}} d\theta \\ &+ \int\int \frac{\bar{\partial}\phi}{1 - e^{2\pi(t - t_0)} e^{2\pi i(\theta - \theta_0)}} d\theta dt\end{aligned} \tag{3.7}$$

We can now use this to bound $\left|\phi(\theta, \frac{R}{2})\right|$:

$$\begin{aligned}\left|\phi(\theta, \frac{R}{2})\right| \leq &\, 3e^{-\pi R}(c_R \epsilon + 1) \max |\phi(\theta_1, t) - \phi(\theta_2, t)| \\ &+ \tilde{c}_R c_R \epsilon \max |\phi(\theta_1, t) - \phi(\theta_2, t)|\end{aligned} \tag{3.8}$$

where $\tilde{c}_R = \max \int \frac{1}{\left|1 - e^{2\pi(t - t_0)} e^{2\pi i(\theta - \theta_0)}\right|} d\theta dt$

Lemma 3.3.1. *For ϵ small enough, given any J^ϵ holomorphic curve u of energy less than E so that part of its domain is conformal to $S^1 \times [-1, 3]$,*

$$u : S^1 \times [0, 2] \longrightarrow \mathbb{T}^n \rtimes B^n$$

satisfies the following.

There exists a trivial holomorphic cylinder $C : S^1 \times [0, 2] \longrightarrow \mathbb{T}^n \rtimes B^n$ so

$$u(\theta, t) = \exp_{C(\theta, t)} \phi(\theta, t)$$

$$\int \phi(\theta, 0) d\theta = 0$$

$$\max|\phi(\theta_1,1)-\phi(\theta_2,1)|\leq e^{-2}\max_{t\in[0,2]}|\phi(\theta_1,t)-\phi(\theta_2,t)|$$

$$|\phi(\theta,1)|\leq e^{-2}\max_{t\in[0,2]}|\phi(\theta_1,t)-\phi(\theta_2,t)|$$

moreover, if $\tilde{C}$ is a close by trivial holomorphic cylinder and $u=\exp_{\tilde{C}}\tilde{\phi}$,

$$\max\left|\tilde{\phi}(\theta_1,1)-\tilde{\phi}(\theta_2,1)\right|\leq e^{-2}\max_{t\in[0,2]}|\phi(\theta_1,t)-\phi(\theta_2,t)|$$

Now consider a bounded energy J^ϵ holomorphic curve u with part of its domain conformal to $S^1\times[-1,2N+1]\longrightarrow\mathbb{T}^n\rtimes B^n$. For ϵ small enough, there exist a series of trivial holomorphic cylinders $C_0,\ldots,C_N$ and ϕ_i so that $u=\exp_{C_i}\phi_i$ and $\int\phi_i(\theta,i)d\theta=0$. Repeated application of Lemma 3.3.1 gives that

$$\max_{t\in[k,2N-k]}|\phi_i(\theta_1,t)-\phi_i(\theta_2,t)|\leq e^{-2k}c$$

$$\operatorname{dist}\left(C_{i-1}(\theta,i),C_i(\theta,i)\right)\leq ce^{-2i}$$

For $\|\mathbf{T}\|_\infty$ small enough, Lemma 2.2.4 tells us that the growth of the distance between trivial cylinders is small enough so that

$$\max_{t\in[0,i]}\operatorname{dist}(C_{i-1}(\theta,t),C_i(\theta,t))\leq ce^{-i}$$

Thus, we have

Proposition 3.3.2. *Given an energy bound E, there exists a constant c so that for ϵ small enough, for each part of a J^ϵ holomorphic curve conformal to $S^1\times[-1,2R+1]$ there exists a unique trivial holomorphic cylinder C so that in coordinates $u=\exp_C\phi$,*

$$\int_{S^1}\phi(\theta,R)d\theta=0$$

$$|\phi(\theta,t)|\leq ce^{-\min\{t,2R-t\}}$$

and

$$\left|\bar{\partial}\phi(\theta,t)\right|\leq ce^{-\min\{t,2R-t\}}$$

Lemma 3.2.1 then gives us the immediate corollary that $\phi \in L_1^{p,\delta}$ for any weight $\delta \leq 1$ is uniformly bounded.

3.4 Convergence to model curves

Proposition 3.4.1. *Given an energy bound E, genus g, and number of punctures k for a surface S there exists an R and $\epsilon > 0$ so that the following is true. Any J^ϵ holomorphic map $u : S \longrightarrow \mathbb{T}^n \rtimes B^n$ with energy bounded by E can be approximated on each $\mathfrak{E}_R$ vertex region $V \subset S$ by a quasi-holomorphic model curve $f : S_V \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ in the sense that*

$$u = \exp_{\mathcal{G}(f)} \phi \text{ on } S_V$$

$$\text{and } \|\phi\|_{1,p,\delta} \leq c$$

for $c > 0$ arbitrarily small.

Proof:

First, Proposition 3.3.2 can be used on the ends to show that for ϵ small enough and R large enough u must be close in $L_1^{p,\delta}$ to trivial holomorphic cylinders on $\mathfrak{E}_R$ edge regions. In fact, as the exponential decay of u is like e^{-R} and our exponential weights grow like $e^{\delta R}$, u must be within some constant times $e^{-(1-\delta)R}$ of trivial holomorphic cylinders on $\mathfrak{E}_R$ edges in $L_1^{p,\delta}$.

Each of these trivial holomorphic cylinders starts at a point p_i. Next, note that the derivative bound from Proposition 3.2.4 and the bounded diameter of $\mathfrak{E}_R$ vertex regions $V \subset S$ imply that $u(V)$ has bounded diameter. For ϵ small, we can choose a point p so that each $p_i = \exp_p \zeta_i$, $\sum \zeta_i = 0$ and

$$u = \exp_{\exp_p(\sum \zeta_i \psi_i)}(F - \sum \zeta_i \psi_i)$$

for some map $F : V \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$. Note that the above expression is chosen to look a lot like the expression for $\mathcal{G}$ introduced in section 2.6. Note also that F is bounded, so we can use Lemma 2.2.4 to prove that for ϵ sufficiently small

$$\left\|\bar{\partial}F - \bar{\partial}u\right\|_{p,\delta}$$

is small. We also have that F is close in $L_1^{p,\delta}$ to the appropriate trivial holomorphic cylinders starting at ζ_i. We can therefore extend F to all of the model domain S_V which is asymptotic to these cylinders and has

$$\left\|(\bar{\partial}F)\right\|_{p,\delta} \leq ce^{-(1-\delta)R}$$

$$\text{and } \left\|Q_{\mathfrak{E}_R}(\bar{\partial}F)\right\|_{1,p,\delta} \leq ce^{-(1-2\delta)R}$$

This is because Lemma 2.9.2 tells us that the norm of $Q_{\mathfrak{E}_R}$ grows like $e^{\delta R}$. We can make this as small as we like by choosing R large as $\delta < \frac{1}{2}$. Now consider the quasi-holomorphic model curve f given by

$$f := F - Q_{\mathfrak{E}_R}\bar{\partial}F$$

Lemma 2.2.4 can be used to prove that

$$u = \exp_{\mathcal{G}(f)} \phi \text{ on } V$$

$$\text{and } \|\phi\|_{1,p,\delta} \text{ is small.}$$

Note that $\pi_Q(\bar{\partial}f) = 0$ and $(Id - \pi_Q)\bar{\partial}f$ is small.

□

3.5 Convergence to holomorphic graphs

Now all the pieces are in place for us to prove that for ϵ small enough, any J^ϵ holomorphic curve with small enough energy will be close to a glued together holomorphic graph.

The idea of what follows is to approximate the $\mathfrak{E}_R$ edge regions of our holomorphic curve with edges of a holomorphic graph using Proposition 3.3.2, and approximate what remains using Proposition 3.4.1.

Proposition 3.5.1. *Given an energy bound E, genus g and number of punctures k for a surface S, for R large enough and ϵ small enough (dependent on R), the following is true. Any J^ϵ holomorphic curve $f : S \longrightarrow \mathbb{T}^n \times B^n$ with energy less than E is in the image of*

$$\mathcal{G}_\infty : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \text{ maps to } \mathbb{T}^n \rtimes B^n$$

Proof:

Recall that the edge regions of $\mathfrak{E}_R$ consist of the subsets $\mathbb{R}/\mathbb{Z} \times (a+R, b-R) \subset \mathbb{R}/\mathbb{Z} \times (a,b)$ of $\mathfrak{E}$ edge regions. Proposition 3.3.2 tells us that for ϵ small enough and R large enough, our holomorphic curve f restricted to any of the $\mathfrak{E}_R$ edge regions can be approximated by a unique trivial holomorphic cylinder C so that

$$f = \exp_C \phi \text{ so that } \oint \phi = 0$$

on the circle at the center of the edge, and $\|\phi\|_{1,p,\delta}$ is small. There exist quasi holomorphic model curves $f_i : S_{V_i} \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ that match up with the ends of these trivial holomorphic cylinder edges as in Proposition 3.4.1 so that f restricted to the vertex region V_i

$$f = \exp_{\mathcal{G}(f_i)} \phi$$

where $\|\phi\|_{1,p,\delta}$ and $\left\|\bar\partial f_i\right\|$ are small if ϵ is small enough and R large enough. These model curves and edges fit together to form a J^ϵ quasi holomorphic graph $u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}$ so that

$$u = \exp_{\mathcal{G}(u)} \phi$$

with $\|\phi\|_{1,p,\delta}$ small. Note that the average of ϕ over all the circles at the center of $\mathfrak{E}_R$ edge regions is zero, which is exactly the condition that $\phi \in \mathcal{B}_{\mathfrak{E}_R}(S, \mathbb{C}^n)$. If we choose R large enough and ϵ small enough that $\|\phi\|_{1,p,\delta} < 1$, and Proposition 2.10.2 holds, then Proposition 2.10.2 tells us that this must be the unique ϕ satisfying the above conditions so that $\pi_{Q_{\mathfrak{E}_R}} \bar\partial \exp_{\mathcal{G}(u)} \phi = 0$, therefore f is in the image of $\mathcal{G}_\infty$. □

Chapter 4

J^ϵ holomorphic graphs

Combining Propositions 2.10.2, 3.5.1 and Theorem 2.11.8, we have now proved Theorem 1.1.4. In particular, there exists continuous section

$$\bar{\partial} \circ \mathcal{G}_\infty := \bar{\partial}_1 : \mathcal{Q}^{\epsilon, \mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{E}$$

of the bundle

$$\begin{array}{rl} \ker Q_{\mathfrak{E}_R} := \mathcal{H}_{\mathfrak{E}_R}(S, \mathbb{C}^n) \longrightarrow & \mathcal{E} \\ & \downarrow \\ & \mathcal{Q}^{\epsilon, \mathfrak{E}_R}_{g,k,E} \end{array}$$

and an embedding of the corresponding moduli space of J^ϵ holomorphic curves

$$\mathcal{M}^\epsilon_{g,k,E} \longrightarrow \mathcal{Q}^{\epsilon, \mathfrak{E}_R}_{g,k,E}$$

as the intersection of $\bar{\partial}_1$ with the zero section.

Now we want to prove Theorem 1.1.6. In particular, we want to define a C^1 smooth section

$$[\bar{\partial}_0] : \mathcal{Q}^{\epsilon, \mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{E}$$

so that the moduli space $\mathcal{M}^{\epsilon, [\bar{\partial}_0], \mathfrak{E}_R}_{g,k,E}$ which is the intersection of $[\bar{\partial}_0]$ with the zero section is cobordant in $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ to $\mathcal{M}^{\epsilon}_{g,k,E}$.

$[\bar\partial_0]$ is a smoothing of $\bar\partial_0$ defined by an averaging of $\bar\partial_0$. In particular, for ϵ small enough, dependent on E, g, and R', there is a well defined projection

$$\pi_{R,R'} : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{Q}^{\epsilon,\mathfrak{E}_{R'}}$$

for all $0 \leq R \leq R'$. This is defined in Appendix C.3.

This preserves the domain Riemann surface, and satisfies

$$\bar\partial_0 \circ \pi_{R,R'} = \bar\partial_0$$

when the $\mathfrak{E}_{R'}$ edge regions are in one to one correspondence with the $\mathfrak{E}_R$ edge regions.

Moreover,

$$\pi_{R,R'} : \{u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \text{ so that } \bar\partial_0 u \in \mathcal{H}_{\mathfrak{E}_{R'}}(S,\mathbb{C}^n) \subset \mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n)\} \longrightarrow \mathcal{Q}^{\epsilon,\mathfrak{E}_{R'}}$$

is a diffeomorphism onto its image whenever restricted to regions where $\ker Q_{\mathfrak{E}_{R'}}$ or equivalently $\mathcal{H}_{\mathfrak{E}_{R'}}(S,\mathbb{C}^n)$ does not jump.

Now define the section

$$[\bar\partial_0] := \int_0^1 \bar\partial_0 \circ \pi_{R,(R+t)} dt$$

It is easily verified using Lemma 2.2.4 that $[\bar\partial_0]$ is a C^1 section in the sense of Definition 1.1.3. If we want $[\bar\partial_0]$ transverse to the zero section we can take $[\bar\partial_0] - \nu$ where ν is some smooth section with the property that $\pi_{Q,\mathfrak{E}_{R+1}}\nu = 0$. (We can either take a finite cover to get rid of automorphisms or use a multi-section here. The fact that we can take $\nu \in \ker Q_{\mathfrak{E}_{R+1}}$ follows from Lemma C.3.1.) We can do the same thing to make $\bar\partial_1$ transverse to the zero section. This will correspond to solutions of some perturbed $\bar\partial$ equation.

Now we want to define a family of continuous sections $[\bar\partial_t]$ so that the intersection of $[\bar\partial_t]$ with the zero section for $t \in [0,1]$ defines a cobordism between $\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}_R}_{g,k,E}$ and $\mathcal{M}^{\epsilon}_{g,k,E}$.

Define

$$[\bar\partial_t] := \int_0^1 ((1-s)\bar\partial_0 + s\bar\partial_1) \circ \pi_{R,R+s} ds$$

Note that Lemma 2.11.7 tells us that the intersection of $[\bar\partial_1]$ with the zero section is the same as the intersection of $\bar\partial_1$ with the zero section, or $\mathcal{M}^{\epsilon}_{g,k,E}$. For R large enough, and ϵ small enough, $[\bar\partial_t]$ can be shown to be continuous restricted to $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ in the sense of Definition 1.1.3 using Lemmas 2.2.4, 2.10.1 and Proposition 2.11.6.

As Proposition 2.10.2 tells us that we can make $\bar\partial_1$ as close as we like to $\bar\partial_0$ by making R large and ϵ small, this cobordism will be contained in the interior of $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$. We can also perturb this family to be transverse to the zero section. In many cases, this tells us that in order to compute invariants of the moduli space of holomorphic curves, we can calculate these invariants using the moduli space of J^ϵ holomorphic curves.

Recall that a smooth family of graphs in $u_s \in \mathcal{Q}^{\epsilon,\mathfrak{E}}_{g,k,E}$ is characterized by the existence of a smooth family of gluings. This is equivalent to the existence of a family of sections $\phi_s \in \mathcal{B}_{\mathfrak{E}}(S,\mathbb{C}^n)$ which vanish on $\mathfrak{E}$ edge regions so that

$$\exp_{\mathcal{G}(u_s)} \phi_s$$

gives a smooth family of maps. For R large enough, and ϵ small enough, we can choose a gluing of $\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}_R}_{g,k,E}$ which is 'small' in the sense that $\|\phi_s\|_{1,p,\delta} < 1$. (This follows from Lemma 2.10.1 and Lemma 2.6.3.)

Theorem 4.0.2. *For R large enough and ϵ small enough, any smooth small gluing of $\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}_R}_{g,k,E}$ is the moduli space of solutions of some perturbed $\bar\partial$ equation. Moreover, if $\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}_R}_{g,k,E}$ is transversely cut out, so is its gluing.*

Proof:

First, note that for any given R we can choose ϵ small enough so that the map

$$u, \phi \mapsto \exp_{\mathcal{G}(u)} \phi$$

is injective for $u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ and $\|\phi\|_{1,p,\delta} \leq 5$ as in section 2.7. Then the set of smooth small gluings of any smooth family in $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ is contractible. Therefore, we can extend our small gluing of $\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}_R}_{g,k,E}$ to a small gluing of $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ so that the restriction to any smooth family of solutions to $\pi_{Q,\mathfrak{E}_{R+1}}[\bar\partial_0]u = 0$ is a smooth family of gluings, and the restriction to subsets where $\ker Q_{\mathfrak{E}_R}$ doesn't jump is smooth. The information contained in this gluing consists of an assignment of a section $\phi_u \in \mathcal{B}_{\mathfrak{E}_R}(S, \mathbb{C}^n)$ to every $u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$. So our gluing is given by

$$u \mapsto \exp_{\mathcal{G}_u} \phi_u$$

Note that we can get related gluings of $\mathcal{Q}^{\epsilon,\mathfrak{E}_{R+s}}_{g,k,E}$ for $s \in [0,1]$ by considering $\mathcal{Q}^{\epsilon,\mathfrak{E}_{R+s}}_{g,k,E}$ as the solutions of $\pi_{Q,\mathfrak{E}_{R+s}}[\bar\partial_0]u$ inside $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$.

We now want to define a perturbation of the $\bar\partial$ equation so that the image of $\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}_R}_{g,k,E}$ under this gluing is the solution space. Choose some cutoff function $\rho : [0,5] \longrightarrow [0,1]$ so that $\rho(t) = 1$ for $t \leq 4$ and ρ vanishes in a neighborhood of 5. Define

$$\bar\partial^R_0 \left(\exp_{\mathcal{G}(u)} \phi\right) := \rho(\|\phi\|_{1,p,\delta}) \left(\bar\partial_0 u + \bar\partial(\phi - \phi_u)\right) + (1 - \rho(\|\phi\|_{1,p,\delta}))\bar\partial \left(\exp_{\mathcal{G}(u)} \phi\right)$$

Extend $\bar\partial^R_0$ to be $\bar\partial$ on any other map. We can define $\bar\partial^{R+s}_0$ analogously. Now define

$$[\bar\partial_0]^R f := \int_0^1 \bar\partial^{R+s}_0 f ds$$

Suppose that R has been chosen large enough and ϵ small enough so that Propositions 2.10.2 and 3.5.1 and Lemma C.3.1 work using $\mathfrak{E}_R$ and $\mathfrak{E}_{R+1}$, and so that Lemma

2.10.1 tells us that

$$\left\| \frac{d}{ds} d(\exp_{\mathcal{G}(u)} \phi_s) - \frac{d}{ds}(d\phi_s) \right\|_{p,\delta} \leq \frac{1}{2} \left\| \frac{d\phi_s}{ds} \right\|_{1,p,\delta} \text{ for } \|\phi_s\| \leq 5$$

This ensures that the solution set to $[\bar{\partial}_0]^R f = 0$ is exactly the image of $\mathcal{M}^{\epsilon,[\bar{\partial}_0],\mathfrak{E}_R}_{g,k,E}$ under our gluing. Note that $[\bar{\partial}_0]^R$ is transverse to the zero section if $[\bar{\partial}_0]$ is.

□

Chapter 5

Examples

Definition 5.0.3. *The use of the qualifier **virtually** when describing a moduli space will mean that there is some perturbation of the $\bar{\partial}$ equation so that the moduli space is transversely cut out as described.*

5.1 Holomorphic spheres in $T^*\mathbb{T}^n$

Any embedded Lagrangian torus in a symplectic manifold has a neighborhood symplectomorphic to a neighborhood of the zero section in $T^*\mathbb{T}^n$. We can give this neighborhood a complex structure which gives it a cylindrical end. This is talked of as 'stretching the neck' of the boundary of the neighborhood in symplectic field theory, [2]. Giving $T^*\mathbb{T}^n$ coordinates $(x, y) \in \mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^n$, a model for this kind of end is given by the complex and symplectic structures

$$J\frac{\partial}{\partial x} = f(|y|)\frac{\partial}{\partial y}$$

$$\omega = d(-\frac{y}{f(|y|)}dx)$$

where $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is 1 in a neighborhood of 0, asymptotic to the identity, and has derivative bounded by 1. Note that this gives coordinates for a unit neighborhood of the zero section in $T^*\mathbb{T}^n$ with the standard symplectic structure. The verification that this satisfies the assumptions listed in Appendix B is the same as in section B.2.

Recall that the moduli space of J^ϵ holomorphic spheres with k punctures and energy less than E, $\mathcal{M}^\epsilon_{0,k,E}$ is given by the intersection of $\bar\partial_1 : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{0,k,E} \longrightarrow \mathcal{E}$ with the zero section. This is cobordant inside $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{0,k,E}$ to the moduli space $\mathcal{M}^{\epsilon,[\bar\partial_0]}_{0,k,E}$ consisting of the intersection of our section $[\bar\partial_0]$ with the zero section. In this case, if we do not constrain the moduli space further, it will have codimension one boundary, maybe with corners, so cobordisms need to be treated more delicately to extract invariants. To avoid this, we shall put more constraints on the moduli spaces that we look for. In particular, we will fix the complex structure and the image of a marked point. Note that for these problems, there is no dimension jumping (which always is associated with changes of complex structure.) Thus without complicating matters by using the smoothed section $[\bar\partial_0]$, we can just consider the intersection of $\bar\partial_0$ with the zero section, $\mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}}_{g,k,E}$. Theorem 4.0.2 tells us that in this case, for R large enough and ϵ small enough the image under the gluing map, $\mathcal{G}\left(\mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}_R}_{g,k,E}\right)$ is the moduli space of solutions to some perturbed $\bar\partial$ equation.

Lemma 5.1.1. *Given any point $p \in \mathbb{T}^n \times \mathbb{R}^n$, k homology classes in $H_1(\mathbb{T}^n, \mathbb{Z})$ adding up to zero, and a Riemann sphere S with k punctures and a marked point z, there exists a unique graph $u \in \mathcal{M}^{\epsilon,\bar\partial_0,\mathfrak{E}_R}$ with underlying Riemann surface S so that the marked point is sent to p,*

$$\mathcal{G}(u)(z) = p$$

and the k ends are sent to the k specified homology classes.

Proof:

The underlying Riemann surface S, homology classes of the ends, and the requirement that all model curves are holomorphic, specifies the types of model curves and trivial holomorphic cylinders that must be glued together, all that remains is a choice of where to exponentiate them out from. Concentrate first on the component of S that contains the marked point z. If z is on an edge, sending z to p fixes the location of the edge's trivial holomorphic cylinder. If z is in a vertex region, we must send z to p using the gluing map $\mathcal{G}$ described in section 2.6. This fixes the location from which the vertex region is exponentiated out from.

After fixing the location of the component containing z, the location of all other components are fixed by the need to attach to the fixed component. As the genus is zero, there is only one condition fixing the location of each component. Note that this proof only works if there are no infinitely long internal edges with nonzero homology classes.

□

This tells us what the answer for holomorphic spheres should be virtually. (Recall the special sense in which 'virtually' is used, definition 5.0.3)

Lemma 5.1.2. *Given a point p in $\mathbb{T}^n \times \mathbb{R}^n$, k homology classes in $H_1(\mathbb{T}^n, \mathbb{Z})$ adding up to zero, and a Riemann sphere S with k punctures and a marked point z, for ϵ small enough, there is virtually one J^ϵ holomorphic map $u : S \longrightarrow \mathbb{T}^n \times \mathbb{R}^n$ so that $u(z) = p$ and loops around the k punctures are sent to the specified k homology classes.*

Proof:

First, note that as proved in section 2.11, $\mathcal{G}_\infty$ is continuous and close to $\mathcal{G}$. Therefore Lemma 5.1.1 tells us that counting multiplicities, there is one J^ϵ quasi holomorphic graph u with holomorphic model curves for which $\mathcal{G}_\infty(u)$ satisfies all the requirements above apart from being holomorphic. The same proof as Lemma 5.1.1 gives that the same is true for quasi holomorphic model curves with $\bar{\partial}_0$ specified. As $\bar{\partial}_1$ is a perturbation of $\bar{\partial}_0$, for ϵ small enough, there will be virtually one J^ϵ holomorphic curve meeting the requirements.

□

Note that the above lemma only applies to holomorphic spheres with distinct punctures. For nodal holomorphic spheres, we have to consider the compactifications. This involves studying holomorphic curves in the cylindrical end of our manifold, $\mathbb{T}^n \times (\mathbb{R}^n - \{0\})$ with the complex structure introduced in example 1.0.2, we do this in section 5.3.

5.2 Curves with boundary on the zero section of $T^*\mathbb{T}^n$

The zero section of $T^*\mathbb{T}^n$ is a Lagrangian submanifold. One scheme for studying holomorphic curves with boundary on an embedded Lagrangian torus inside an arbitrary symplectic manifold is to use symplectic field theory to isolate a standard neighborhood of the Lagrangian torus and study the holomorphic curves with boundary in that setting. The advantage of this is that $T^*\mathbb{T}^n$ with the complex structure we consider has an antiholomorphic involution

$$\Phi : T^*\mathbb{T}^n \longrightarrow T^*\mathbb{T}^n$$

$$\Phi(x, y) = (x, -y)$$

which preserves the zero section. Instead of studying holomorphic curves with boundary on the zero section, we can study holomorphic maps

$$u : (S, \phi) \longrightarrow (T^*\mathbb{T}^n, \Phi)$$

$$u \circ \phi = \Phi \circ u$$

Here (S, ϕ) denotes a Riemann surface with an antiholomorphic involution. Such a map can be constructed from a curve with boundary on the zero section by reflecting. To reverse the operation, note that the fixed point set of (S, ϕ) must be sent to the zero section, so taking half the curve gives a holomorphic map with boundary on the zero section.

We can find these holomorphic curves with involutions by considering quasi holomorphic graphs with involutions, and making sure that our gluing procedure preserves involutions. Note that as we chose our edge and vertex markings to depend only on the conformal structure of S, the $\mathfrak{C}$ edge-vertex decomposition is conserved by these involutions. The original gluing map $\mathcal{G}$ preserves any involution. The map which may not preserve involutions is the model left inverse Q. Given a Riemann surface with involution (S, ϕ), define the equivariant model left inverse

$$Q_\phi(\nu) := \frac{1}{2}(Q\nu + (\phi, \Phi) \circ Q \circ (\phi, \Phi)\nu)$$

Here (ϕ, Φ) indicates the action induced on a map by acting on the domain by ϕ and the range by Φ.

Lemma 5.2.1.

$$Q_\phi : L^{p,\delta}(\Omega^{0,1}(S, \mathbb{C}^n)) \longrightarrow L_1^{p,\delta}(\Omega^0(S, \mathbb{C}^n))$$

is a (ϕ, Φ) *equivariant left inverse to*

$$\bar{\partial} : L_1^{p,\delta}(\Omega^0(S, \mathbb{C}^n)) \longrightarrow L^{p,\delta}(\Omega^{0,1}(S, \mathbb{C}^n))$$

Proof:

First note that $\bar{\partial}$ is (ϕ, Φ) equivariant,

$$(\phi, \Phi) \circ \bar{\partial} = \bar{\partial} \circ (\phi, \Phi)$$

If $\xi \in L_1^{p,\delta}(S, \mathbb{C}^n)$ then

$$\begin{aligned} Q_\phi \bar{\partial} \xi &= \frac{1}{2}(Q\bar{\partial}\xi + (\phi, \Phi)Q(\phi, \Phi)\bar{\partial}\xi) \\ &= \frac{1}{2}(\xi + (\phi, \Phi)Q\bar{\partial}(\phi, \Phi)\xi) \\ &= \frac{1}{2}(\xi + (\phi, \Phi)(\phi, \Phi)\xi) \\ &= \xi \end{aligned}$$

so Q_ϕ is a left inverse to $\bar{\partial}$.

$$Q_\phi \circ (\phi, \Phi) = \frac{1}{2}(Q \circ (\phi, \Phi) + (\phi, \Phi) \circ Q) = (\phi, \Phi) \circ Q_\phi$$

so Q_ϕ is (ϕ, Φ) equivariant.

□

We can do the same to define an equivariant version of $Q_{\mathfrak{e}_R}$, $Q_{\mathfrak{e}_R, \phi}$.

Define the equivariant gluing map $\mathcal{G}_{\infty,\phi}$ in the same way as $\mathcal{G}_\infty$ was defined in section 2.8, using instead of $Q_{\mathfrak{E}_R}$, the equivariant left inverse $Q_{\mathfrak{E}_R,\phi}$.

We can define $\mathcal{Q}^{\epsilon,\mathfrak{E}}_{g,k,E,\phi} \subset \mathcal{Q}^{\epsilon,\mathfrak{E}}_{g,k,E,}$ to be the subset of quasi holomorphic graphs in $\mathcal{Q}^{\epsilon,\mathfrak{E}}_{g,k,E}$ which are preserved by the action of (ϕ,Φ)

As in chapter 4, there is a bundle $\mathcal{E}$ over $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E,\phi}$ with the sections $[\bar\partial_0]$ and $\bar\partial_1$. We can take the (ϕ,Φ) invariant part of this bundle, $\mathcal{E}_\phi$. Constructing $[\bar\partial_0]$ and $\bar\partial_1$ equivariantly, they give sections

$$\bar\partial_1, [\bar\partial_0] : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E,\phi} \longrightarrow \mathcal{E}_\phi$$

Theorem 5.2.2. *For R large enough, and ϵ small enough, there is an embedding*

$$\mathcal{M}^{\epsilon}_{g,k,E,\phi} \longrightarrow \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E,\phi}$$

of the moduli space of holomorphic maps with involutions so that the image is the intersection of the section

$$\bar\partial_1 : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E,\phi} \longrightarrow \mathcal{E}_\phi$$

with the zero section. Moreover $M^{\epsilon}_{g,k,E,\phi}$ is cobordant inside $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E,\phi}$ to the moduli space $M^{\epsilon,[\bar\partial_0]}_{g,k,E,\phi}$ which is the intersection of $[\bar\partial_0]$ with the zero section. (If each moduli space is transversely cut out.)

Proof:

First we recall the map from holomorphic curves to $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$. This relies on Proposition 3.3.2, which, for ϵ small enough, provides unique trivial holomorphic cylinders approximating the thin parts of a Riemann surface S of length greater than R_0. As the normalizing condition used to choose these cylinders is (ϕ,Φ) invariant, if a holomorphic curve is (ϕ,Φ) invariant, the set of these approximating trivial holomorphic cylinders will be invariant too. We can then implement Proposition 3.4.1 equivariantly using our equivariant left inverse Q_ϕ to obtain vertex model curves. As every step is (ϕ,Φ) invariant, we must obtain a graph in $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E,\phi}$. The fact that this map gives an embedding follows from Propositions 2.10.2 and 3.5.1.

The (ϕ, Φ) equivariant gluing map, $\mathcal{G}_{\infty,\phi}$ maps $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E,\phi}$ to (ϕ, Φ) invariant quasi holomorphic curves. Taking $\bar\partial$ of the resulting curves defines our section

$$\bar\partial_1 : \mathcal{Q}_{g,k,E,\phi} \longrightarrow \mathcal{E}_\phi$$

Holomorphic curves map to the intersection of $\bar\partial_1$ with the zero section of $\mathcal{E}$. The cobordism to $M^{\epsilon,[\bar\partial_0]}_{g,k,E,\phi}$ is defined as in as section 4 by the intersection of

$$[\bar\partial_t] = \int_0^1 ((1-t)\bar\partial_0 + t\bar\partial_1) \circ \pi_{R+t,R} dt$$

with the zero section. $\pi_{R+t,R}$ is also defined equivariantly, so this defines an equivariant family of sections.

□

Lemma 5.2.3. *Given a punctured Riemann surface with antiholomorphic involution* (S, ϕ)*, any bounded energy map*

$$u : S \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$$

so that $u_* : H_1(S, \mathbb{Z}) \longrightarrow H_1(\mathbb{C}^n/\mathbb{Z}^n, \mathbb{Z})$ *and* $\bar\partial u$ *are both* (ϕ, Φ) *invariant satisfies*

$$\Phi \circ u \circ \phi = u + constant$$

Proof:

$\bar\partial(\Phi \circ u \circ \phi) = \phi \circ (\bar\partial u) \circ \phi = \bar\partial u$. Bounded energy maps $u : S \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ are determined by $u_* : H_1(S, \mathbb{Z}) \longrightarrow H_1(\mathbb{C}^n/\mathbb{Z}^n, \mathbb{Z})$ and $\bar\partial u$ up to addition of constants.

□

Lemma 5.2.4. *Given a complex disk* D *with* k *punctures and a choice of homology class in* $H_1(\mathbb{T}^n, \mathbb{Z})$ *for each puncture, for* ϵ *small enough (dependent on the choice of homology classes), there exists a* J^ϵ *holomorphic map*

$$u : D \longrightarrow T^*\mathbb{T}^n$$

so that u sends δD to the zero section of $T^*\mathbb{T}^n$, and the punctures of D have the prescribed homology classes. This map is virtually (definition 5.0.3) unique up to translation in the direction of $\mathbb{T}^n$.

Proof:

Doubling the disk D gives a Riemann surface with involution (S, ϕ). We first need to verify the above statement for (ϕ, Φ) invariant quasi holomorphic graphs u with underlying Riemann surface S satisfying $\bar{\partial}_0 = \nu$ for any small $\nu \in \mathcal{E}_\phi$. The lemma will then follow as $\bar{\partial}_1 u$ is a perturbation of $\bar{\partial}_0$ when we restrict to a single complex structure.

If a vertex component of S contains some of δD, then Lemma 5.2.3 tells us that the model curve for this vertex will automatically be (ϕ, Φ) invariant. This model curve can be exponentiated out from any point on the zero section of $T^*\mathbb{T}^n$.

As ϕ preserves the conformal structure of S, it preserves the $\mathfrak{E}$ edge markings. Thus any $\mathfrak{E}$ edge region of S containing some of δD must be cut exactly in half by δD, either longways or in the middle. An edge cut in the middle by δD contains all of δD. The placement of such an edge is determined up to $\mathbb{T}^n$ translation. An edge cut longways by δD must have homology class zero. The trivial holomorphic cylinder associated with such an edge will just be a constant map to some point of the zero section of $T^*\mathbb{T}^n$. A vertex attached to such an edge must also contain some of δD. The puncture associated to our edge must be preserved by ϕ, and thus as the model curve is (ϕ, Φ) invariant, the model curve must map this puncture to some point on the zero section. Thus all the vertices and edges that contain some of δD and must be located on the zero section can be attached together without leaving the zero section.

As in Lemma 5.1.1, after we have chosen the placement of one of these components which must be on the zero section, the placement of all other components is determined by the complex structure of S and $\bar{\partial}_0 u$. The fact that S has zero genus means that there are no further conditions that need to be fulfilled. Any such quasi holomorphic graph which has $\bar{\partial} u \in \mathcal{E}_\phi$ will automatically be (ϕ, Φ) invariant due to the symmetry of the construction.

We now have that there exists a J^ϵ quasi holomorphic graph satisfying $\bar{\partial}_0 u = \nu$ for any small $\nu \in \mathcal{E}_\phi$, and this graph is unique up to $\mathbb{T}^n$ translation. The same is true for $\mathfrak{E}_R$ markings with any R. For R large enough and ϵ small enough, $\bar{\partial}_1$ is a perturbation of $\bar{\partial}_0$ when restricted to a single complex structure, and the Lemma follows.

□

5.3 Holomorphic spheres in $\mathbb{T}^n \times (\mathbb{R}^n - \{0\})$

We want to consider $\mathbb{T}^n \times (\mathbb{R}^n - \{0\})$ with the cylindrical complex structure which the end of the structure on $\mathbb{T}^n \times \mathbb{R}^n$ considered earlier is asymptotic to.

$$J\frac{\partial}{\partial x} = |y|\frac{\partial}{\partial y}$$

This manifold has two ends. The end as $|y| \to \infty$ will be referred to as the positive end and the one around $y = 0$ the negative end. The curves relevant to the compactification of the space of holomorphic spheres in $\mathbb{T}^n \times \mathbb{R}^n$ will be holomorphic spheres with several punctures, one of which is special and sent to a particular orbit in the negative end.

Lemma 5.3.1. *Given a closed geodesic α in $\mathbb{T}^n$ with class $[\alpha] \in H_1(\mathbb{T}^n, \mathbb{Z})$ and k classes in $H_1(\mathbb{T}^n, \mathbb{Z})$ summing to $-[\alpha]$, for any sphere S with $k+1$ punctures, for ϵ small enough, there is a J^ϵ holomorphic map $u : S \longrightarrow \mathbb{T}^n \times (\mathbb{R}^n - \{0\})$ so that the first puncture is asymptotic to α at the negative end, and the other ends have the appropriate homology classes. Up to scaling in the y direction and translating in the direction of $[\alpha]$, there is virtually (definition 5.0.3) one of these maps.*

Note that depending on the complex structure of S, the other ends of u may be either positive or negative. Generically, they will be positive.

Proof:

As we are restricting to a fixed complex structure, $\bar{\partial}_1$ is a perturbation of $\bar{\partial}_0$. We first solve the problem for quasi holomorphic graphs with $\bar{\partial}_0$ specified. The requirement that the first edge is asymptotic to α fixes its position up to translation in the $[\alpha]$ direction or scaling in the y direction, each of which is a symmetry. The other components of u are then fixed as in the proof of Lemma 5.1.1. Note also that such solutions of $\bar{\partial}_0 u = 0$ will be transversely cut out.

Now note that the ends of a graph u and $\mathcal{G}_\infty(u)$ are the same. Thus as $\bar{\partial}_1$ is a perturbation of $\bar{\partial}_0$, we have for ϵ small enough the required result.

□

Appendix A

Notation

J	:=	almost complex structure
ω	:=	symplectic form
∇	:=	unique connection preserving J and torus action
$\mathbf{T}$	:=	torsion tensor of ∇
$\mathbb{T}^n \rtimes B^n$	:=	Lagrangian torus fibration which is the ambient space.
$\mathcal{Q}^{\epsilon,\mathfrak{E}}_{g,k,E}$	:=	space of J^ϵ quasi holomorphic graphs u with genus g, k punctures, and energy less than E with $\left\lVert\bar\partial_0 u\right\rVert < 1$. See sections 1.1.4 and 2.5.
$\mathcal{E}$	:=	a finite dimensional bundle over $\mathcal{Q}^{\epsilon,\mathfrak{E}}_{g,k,E}$ defined in section 1.1.4.
$\mathcal{H}_{\mathfrak{E}}(S,\mathbb{C}^n)$	:=	the fiber of $\mathcal{Q}^{\epsilon,\mathfrak{E}}_{g,k,E}$. Defined in section 1.1.4. Equal to $\ker Q_{\mathfrak{E}}$, discussed in sections 2.4 and 2.9
$\bar\partial_0$	:=	Section of $\mathcal{E}$. A way of taking $\bar\partial$ of quasi holomorphic graphs involving taking $\bar\partial$ of each individual model curve. See section 1.1.4.
$[\bar\partial_0]$	:=	A smoothing of $\bar\partial_0$. See sections 1.1.6 and 4.
$\mathcal{M}^{\epsilon,[\bar\partial_0],\mathfrak{E}_R}_{g,k,E}$	:=	Intersection of $[\bar\partial_0]$ with the zero section. Cobordant to space of J^ϵ holomorphic curves. See chapter 4.
$\mathfrak{E}$, $\mathfrak{E}_R$	:=	A decomposition of Riemann surfaces into edge and vertex regions. See appendix C.

$L^{p,\delta}_k$	:=	Weighted L^p space on k derivatives. The weight δ is chosen so that $0 < \delta < \frac{1}{2}$. p is larger than 2. Defined in section 2.3.
$\lVert\phi\rVert_{1,p,\delta}$	:=	The norm of ϕ in $L^{p,\delta}_1$. Note that this norm is always measured in the rescaled metric in which J^ϵ is an isometry and torus fibers have the standard metric on $\mathbb{R}^n/\mathbb{Z}^n$.
$\mathcal{B}_{\mathfrak{C}}(S,\mathbb{C}^n)$	:=	The subspace of $L^{p,\delta}_1(S,\mathbb{C}^n)$ which consists of sections which average to 0 over the center of $\mathfrak{C}$ edge regions. See section 2.7.
$\mathcal{B}_{\mathfrak{C}}$	:=	Bundle over $\mathcal{Q}^{\epsilon,\mathfrak{C}}_{g,k,E}$ with fiber $\mathcal{B}_{\mathfrak{C}}(S,\mathbb{C}^n)$ on which the space of maps is locally modelled. Defined in section 2.7.
Q	:=	Model left inverse to $\bar\partial$ restricted to $L^{p,\delta}_1(S,\mathbb{C}^n)$. Defined in section 2.4
$Q_{\mathfrak{C}}$	:=	Model left inverse to $\bar\partial$ restricted to $\mathcal{B}_{\mathfrak{C}}(S,\mathbb{C}^n)$. Defined in section 2.9.
$\pi_{Q,\mathfrak{C}}$	:=	Projection onto the image of $\bar\partial\,(\mathcal{B}_{\mathfrak{C}}(S,\mathbb{C}^n))$ with kernel given by $\ker Q$. Defined by $\pi_{Q,\mathfrak{C}} := \bar\partial \circ Q_{\mathfrak{C}}$. Introduced in section 2.9.
$D_{\bar\partial,u}$	:=	Linearization of the $\bar\partial$ operator at a map u. Introduced in section 2.2.
$\mathcal{G}$	:=	'Gluing' map. Takes quasi holomorphic graphs and produces maps to $\mathbb{T}^n \rtimes B^n$. Introduced in section 2.6
$\mathcal{G}_\infty$	:=	Map which takes quasi holomorphic graphs in $\mathcal{Q}^{\epsilon,\mathfrak{C}_R}_{g,k,E}$ and produces quasi holomorphic curves f satisfying $\pi_{Q,\mathfrak{C}}\bar\partial f = 0$. Discussed in section 2.10.
$\bar\partial_1$	:=	Section of $\mathcal{E}$ defined by $\bar\partial_1 := \bar\partial \circ \mathcal{G}_\infty$. The intersection of $\bar\partial_1$ with the zero section gives the moduli space of holomorphic curves.
$\pi_{R,R'}$	:=	Local submersion from $\mathcal{Q}^{\epsilon,\mathfrak{C}_R}_{g,k,E}$ to $\mathcal{Q}^{\epsilon,\mathfrak{C}_{R'}}$. Defined in appendix C.3.
$\Lambda_{\omega,E}$	:=	Set of taming forms equivalent to ω for holomorphic curves of energy less than E. Defined in section 3.1.

Appendix B

Technical assumptions

1. Torus fibers are Lagrangian

2. The symplectic form and J are preserved by the torus action on the fibers.

3. ω is positive on holomorphic planes

4. The ω energy is constant on continuous families of maps with holomorphic ends

5. The energy provides a bound on the local area of holomorphic maps. More precisely, for any energy bound E there exist constants $r > 0$ and $c_E > 0$ so that for any point p in the base manifold B^n, there exists a taming form $\omega_{p,E} \in \Lambda_{\omega,E}$ so that in the fibration over the ball of radius r around p

   $$\omega_p(v, Jv) \geq c_E \left|v\right|^2$$

   (Recall that $\Lambda_{\omega,E}$ is defined as the set of closed two-forms which are nonnegative on holomorphic planes, vanish restricted to torus fibers and have the same integral as ω on any J^ϵ holomorphic curve with ω energy less than E.)

6. $\left\|\mathbf{T}\right\|_\infty$ and $\left\|\nabla\mathbf{T}\right\|_\infty$ are bounded.

7. $\mathbb{T}^n \rtimes B^n$ is complete and has injectivity radius bounded below.

8. If a weaker metric G is to be used to compactify the space of holomorphic curves, then the curvature of G and the derivatives in G's Levi Civita connection of the canonical frame should be bounded so that

$$\operatorname{dist}_G(\exp_{p_1}\phi, \exp_{p_2}\phi) \leq \operatorname{dist}_G(p_1, p_2)e^{c|\phi|}$$

   Also, the change in the torsion tensor $\mathbf{T}$ measured in the canonical metric should be bounded by distance in G, ie

$$|\mathbf{T}_{p_1} - \mathbf{T}_{p_2}| \leq c \operatorname{dist}_G(p_1, p_2)$$

B.1 $\mathbb{C}^n/\mathbb{Z}^n$

$\mathbb{C}^n/\mathbb{Z}^n$ has an obvious torus action which preserves the usual complex structure. In this case, g is the euclidean metric and ∇ the corresponding trivial connection. The torsion tensor $\mathbf{T} = 0$. To choose a taming form compatible with our requirements, we consider $\mathbb{C}^n/\mathbb{Z}^n$ as the subset of a toric manifold where the $\mathbb{T}^n$ action is free. For example, we could consider $\mathbb{C}/\mathbb{Z} = \mathbb{C}^* \subset \mathbb{C}P^1$, and consider some rotationally symmetric symplectic form on $\mathbb{C}P^1$. All assumptions apart from 4, 5, and 8 are immediate.

To understand 4, we must first understand what it means for a map to have holomorphic ends. A map f from a punctured Riemann surface has holomorphic ends if there exists a neighborhood of each puncture so that f is holomorphic. As we can consider f as a map to our toric manifold, the removable singularity theorem tells us that if f has finite energy, then f extends to a map of the entire Riemann surface to the toric manifold. As ω is a closed differential form on this toric manifold, the ω energy of f depends only on its homology class, so assumption 4 is satisfied.

To understand 5 we first recall that the set of taming forms $\Lambda_{\omega,E}$ is defined as the set of closed two forms which which are nonnegative on holomorphic planes, zero on torus fibers, and have the same energy as ω on any J^ϵ holomorphic curve. In this case, any translation of ω as a form on $\mathbb{C}^n/\mathbb{Z}^n$ will represent the same cohomology class on the compactification, so any translation of ω is in $\Lambda_{\omega,E}$. Assumption 5 is then easily seen to be satisfied.

The final assumption we need to check is assumption 8. Note that any holomorphic curve with bounded energy in the taming form pulled back from some toric manifold will have bounded energy for any other taming form pulled back from a compact toric manifold. Thus the moduli space of holomorphic curves is independent of the choice of compactification. We want a compactification of this moduli space that corresponds to the moduli space of holomorphic curves on our toric manifold. For this we want to use a weaker metric G pulled back from a metric on the toric manifold. It is natural in this case to use the metric

$$G(v_1, v_2) = \omega(v_1, Jv_2)$$

The fact that the torus action extends to a smooth action on the toric manifold and the derivative of J in the Levi-Civita connection of G is bounded gives us assumption 8.

B.2 Symplectization of unit cotangent bundle of $\mathbb{T}^n$

Give $\mathbb{T}^n$ the standard metric on $\mathbb{R}^n/\mathbb{Z}^n$. The unit cotangent bundle has a $\mathbb{T}^n$ invariant contact form. Identify the symplectization of the unit cotangent bundle with

$$\mathbb{T}^n \times (\mathbb{R}^n - \{0\})$$

Giving $\mathbb{T}^n$ coordinates $x \in \mathbb{R}^n/\mathbb{Z}^n$ and $(\mathbb{R}^n - \{0\})$ standard coordinates y, we have a torus invariant almost complex structure J defined by

$$J\frac{\partial}{\partial x_i} = |y| \frac{\partial}{\partial y_i}$$

The metric g in this case is the product of the standard symmetric metrics on

$\mathbb{T}^n \times S^{n-1} \times \mathbb{R}$, so assumption 7 is satisfied. Note that this structure is cylindrical because translation in the $\mathbb{R}$ direction preserves J and the metric. Unlike the last example, J only gives an almost complex structure, and the torsion $\mathbf{T}$ of our flat connection ∇ is non zero. Recall that ∇ is defined by declaring the frame $\{\partial_{x_i}, J\partial_{x_i}\}$ to be constant.

$$\begin{aligned}\mathbf{T}(J\partial_{x_i}, J\partial_{x_j}) &= -[|y|\,\partial_{y_i}, |y|\,\partial_{y_j}] \\ &= -\frac{\partial\,|y|}{\partial y_i}\,|y|\,\partial_{y_j} + \frac{\partial\,|y|}{\partial y_j}\,|y|\,\partial_{y_i}\end{aligned}$$

$\mathbf{T}$ vanishes in the torus fiber directions. We see that $|\mathbf{T}|$ and $|\nabla\mathbf{T}|$ are bounded and assumption 6 is satisfied. Of course, this has to be the case for smooth choices of J on manifolds with cylindrical ends which are otherwise compact.

A choice of taming form which satisfies our conditions is given by

$$\omega = \sum dx_i \wedge df_i(y)$$

where $f : (\mathbb{R}^n - \{0\}) \longrightarrow \mathbb{R}^n$ is given by

$$f(y) = \frac{h(|y|)}{|y|}y$$

and $h : \mathbb{R}^+ \longrightarrow (1,2)$ is some surjective smooth increasing function. Assumptions 1, 2, and 3 are immediately seen to be satisfied.

To understand maps with holomorphic ends, we can use the modified removable singularity theorem from [1] to see that any bounded energy map with holomorphic ends will either extend as a holomorphic map over its punctures or converge to a Reeb orbit at an end of our manifold. Our taming form is exact

$$\omega = d(\sum f_i(y)dx_i)$$

$$\lim_{|y|\to 0} \sum f_i(y)dx_i = \sum \frac{y}{|y|}dx_i \text{ and } \lim_{|y|\to\infty} \sum f_i(y)dx_i = \sum \frac{2y}{|y|}dx_i$$

So the ω energy of a map with holomorphic ends just depends on the integral of the above one forms over the Reeb orbits at its ends. The Reeb vector field is given by

$$\sum \frac{y_i}{|y|}\frac{\partial}{\partial x_i}$$

so families of Reeb orbits all occur over the same point in S^{n-1}, and have a constant integral. Therefore, a continuous family of maps with holomorphic ends will always have the same energy, and assumption 4 is satisfied.

Assumption 5 is seen to be satisfied when we note that translations of ω in the $\mathbb{R}$ direction are in $\Lambda_{\omega,E}$.

An appropriate choice for G that satisfies assumption 8 is

$$G(v,w) = \frac{1}{2}(\omega(v,Jw) + \omega(w,Jv))$$

This metric squashes down the $\mathbb{R}$ factor to a finite interval. The change in $\mathbf{T}$ is still dominated by G distance, as translation in this $\mathbb{R}$ direction doesn't affect $\mathbf{T}$. A more extreme choice for G is actually a pseudo metric which crunches down the $\mathbb{R}$ factor to a point. This 'metric' comes in useful when considering the moduli space of holomorphic curves or graphs mod translations in the $\mathbb{R}$ factor.

B.3 Contact three manifolds with a $\mathbb{T}^2$ action

Take a compact contact three manifold M that has a $\mathbb{T}^2$ action preserving the contact structure. Assume that there is a contact form λ_0 preserved by the $\mathbb{T}^2$ action so that each point in M either has a neighborhood on which the $\mathbb{T}^2$ action is free or the neighborhood is modeled on an open neighborhood of $\{r = 0\}$ in

$$(r, \theta_1, \theta_2) \in D^2 \times S^1$$

$$\lambda_0 = d\theta_2 + r^2 d\theta_1$$

with $\mathbb{T}^2$ acting by rotating the θ_1 and θ_2 coordinates. If we quotient M by the $\mathbb{T}^2$ action we either get S^1 and M is some multiple cover of the unit cotangent bundle of

$\mathbb{T}^2$, or we get an interval $[0,1]$ where the $\mathbb{T}^2$ action degenerates as above at each end, with the coordinate that is part of the interval given by r^2. M can be a sphere, a Lens space, or $S^1 \times S^2$ depending on how the $\mathbb{T}^2$ action degenerates at the ends. We will only consider the latter cases, as covers of the unit cotangent bundle are covered by section B.2.

As the Reeb vector field of this contact form may not be generic enough to fit some needs, we will work with a more flexible model for the contact form λ near where the $\mathbb{T}^2$ action degenerates

$$\lambda = (1 + h(r^2))d\theta_2 + r^2 d\theta_1$$

where h is some smooth function which vanishes at 0, and λ describes a contact form in some neighborhood of $\{r = 0\}$. The contact distribution is spanned by the vector fields

$$r\frac{\partial}{\partial r}, \left((1 + h(r^2))\frac{\partial}{\partial \theta_1} - r^2 \frac{\partial}{\partial \theta_2}\right)$$

for $r \neq 0$. We can choose some complex structure J on the contact distribution, for example,

$$J\left((1 + h(r^2))\frac{\partial}{\partial \theta_1} - r^2 \frac{\partial}{\partial \theta_2}\right) = r\frac{\partial}{\partial r}$$

The direction of the Reeb vector field is given by the kernel of

$$d\lambda = dr \wedge (\frac{r}{2}d\theta_1 + \frac{r}{2}h' d\theta_2)$$

$$\text{Reeb vector field } \mathbf{R} = c(r^2)\left(-h'\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2}\right)$$

$$\text{where } c(r^2) := \frac{1}{(1 + h(r^2)) - r^2 h'(r^2)}$$

The symplectization of M can be identified with $M \times \mathbb{R}$ with symplectic form $de^s\lambda$ where the $\mathbb{R}$ coordinate is s. The complex structure that behaves well in a symplectization pairs the Reeb vector field with $\frac{\partial}{\partial s}$. So we have J given in coordinates close to where the $\mathbb{T}^2$ action degenerates as

$$J\begin{pmatrix} \frac{\partial}{\partial\theta_1} \\ \frac{\partial}{\partial\theta_2} \end{pmatrix} = A\begin{pmatrix} r\frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{pmatrix}$$

$$A := \begin{pmatrix} (1+h) & -r^2 \\ -ch' & c \end{pmatrix}^{-1}$$

Note that

$$\lim_{r\to 0}\frac{|A - Id|}{r} = 0$$

The size of the torsion tensor $\mathbf{T}$ as $r \to 0$ is controlled by the derivatives of A with respect to $r\frac{\partial}{\partial r}$, and $|\nabla\mathbf{T}|$ is controlled by the derivative of this with respect to $r\frac{\partial}{\partial r}$, both of which approach 0. So assumption 6 is satisfied by this complex structure. Note also that the vector fields $J\frac{\partial}{\partial\theta_1}$ and $J\frac{\partial}{\partial\theta_2}$ are smooth vector fields on $M \times \mathbb{R}$, independent of s, and the torsion tensor depends smoothly on position in M. This tells us that any metric which is the product of a smooth metric on M times any metric on $\mathbb{R}$ will satisfy assumption 8.

An appropriate taming form is given by

$$\omega = d(\phi(s)\lambda)$$

where ϕ is some smooth surjective function $\mathbb{R} \longrightarrow (1,2)$ with positive derivative. It is shown in [1] that a bounded ω-energy holomorphic map of a punctured disk to $M\times\mathbb{R}$ must converge to a Reeb orbit at $s = \pm\infty$ or extend to a holomorphic map over the puncture. Therefore, the ω energy of any map with holomorphic ends will depend on the integrals of $\phi\lambda$ over the Reeb orbits at its ends. As continuous families of Reeb orbits always have the same λ integral, the ω energy will be constant on continuous families of maps with holomorphic ends, and assumption 4 is satisfied.

The final assumption to check is assumption 5. First note that by translating ω in the $\mathbb{R}$ direction, we can satisfy assumption 5 away from where the $\mathbb{T}^2$ action degenerates. We can now specialize to our neighborhood where

$$\lambda = (1 + h(r^2))d\theta_2 + r^2 d\theta_1$$

For a generic contact form of the above type, for any given energy E, there will be a radius $r_E > 0$ so that no Reeb orbits appearing in holomorphic curves of energy less that E exist in the region with $r \leq r_E$. The idea now is to alter λ in the neighborhood where $r \leq r_E$. Our complex structure is such that the following forms are both non negative on holomorphic planes

$$d\lambda = dr \wedge (\frac{r}{2}d\theta_1 + \frac{r}{2}h' d\theta_2)$$

$$\text{and } d\left((\phi - 1)\lambda\right)$$

We can change λ to

$$\tilde{\lambda} = f(r^2)d\theta_2 + g(r^2)d\theta_1$$

where f and g satisfy

$$g(0) = 0$$

$$g(r^2) = r^2 \text{ for } r \geq r_E$$

$$f(r^2) = (1 + h(r^2)) \text{ for } r \geq r_E$$

$$(f', g') = c(h', 1) \text{ for some } c(r) \geq 0$$

$d\tilde{\lambda}$ will have the same integral as λ on any J^ϵ holomorphic curve with energy less than E. By making (f', g') large in some region, we concentrate $d\tilde{\lambda}$ in the direction of $r\frac{\partial}{\partial r}$. By adding some translate of $d(\phi - 1)\lambda$ to this, we can get taming forms satisfying assumption 5.

Appendix C

Vertex-edge decomposition, $\mathfrak{E}$

We need a family of decompositions $\mathfrak{E}_R$ of our domain Riemann surface S into subsets labeled as vertices or edges which is reminiscent of the decomposition into thick and thin parts given by the uniformisation theorem. Our decomposition should obey the following axioms:

1. The $\mathfrak{E}$ edge regions are disjoint open subsets of S conformal to $\mathbb{R}/\mathbb{Z} \times (a, b)$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{\infty\}$ and $a < b$. $\mathfrak{E}_R$ edge regions for $R \geq 0$ consist of the subsets $\mathbb{R}/\mathbb{Z} \times (a + R, b - R) \subset \mathbb{R}/\mathbb{Z} \times (a, b)$ of $\mathfrak{E}$ edge regions.

2. $\mathfrak{E}$ vertex regions are the closed subsets of S which are connected components of the compliment of the edges.

3. The $\mathfrak{E}$ edge and vertex regions depend only on the conformal structure of S. The edge regions which are conformal to punctured disks change smoothly with a smooth change of conformal structure.

4. For each edge, there exists some distance r so that the edge is the unique edge contained in some connected component of the set of points in S with injectivity radius less than r in the metric given by the uniformisation theorem.

5. there exists a sequence R_m so that the $\mathfrak{E}_R$ decomposition obeys the following axioms for all $R \geq R_m$ on Riemann surfaces with $(3g - 3 + k) \leq m$ where k is the number of punctures and g is the genus.

(a) Take a $\mathfrak{E}_R$ vertex region V and replace each $\mathfrak{E}_R$ edge region $\mathbb{R}/\mathbb{Z} \times (0, R)$ surrounding it with $\mathbb{R}/\mathbb{Z} \times (0, \infty)$. This creates a new Riemann surface S_V. The $\mathfrak{E}_R$ edges of S_V are exactly these subsets $\mathbb{R}/\mathbb{Z} \times (0, \infty)$.

(b) Take a Riemann surface S with a node. The $\mathfrak{E}_R$ edge regions surrounding each side of this node will be conformal to $\mathbb{R}/\mathbb{Z} \times (0, \infty)$ on one side and $\mathbb{R}/\mathbb{Z} \times (-\infty, 0)$ on the other. Resolve this node by gluing these cylinders together with the identification

$$(\theta, t) \cong (\theta - \Theta, t - l)$$

These cylinders are identified along a cylinder of length l. This gluing produces a new Riemann surface with our two infinite cylinders replaced by this new cylinder. The $\mathfrak{E}_R$ edge regions of our new Riemann surface consist of this new cylinder and the images under this construction of all the other $\mathfrak{E}_R$ regions of S.

(c) There exists a distance $r(m, R)$ so that if S has genus g and k punctures so that $(k + 3g - 3) = m$, the regions in S with injectivity radius less than $r(m, R)$ in the metric given by the uniformisation theorem are all contained inside the $\mathfrak{E}_R$ edges of S.

By $\mathfrak{E}$ edge regions moving smoothly in axiom 3, we mean that there exists locally a smooth family of diffeomorphisms which are the identity on these edge regions. For changes of complex structure resolving a node, the other edge regions are given by axiom 5b. This counts as 'smooth'.

The important axioms here that require us to modify the usual thick-thin decomposition given by the uniformisation theorem are axioms 5a and 5b. These ensure that our markings of edges and vertices are compatible with the operations of gluing.

An alternative description of quasi holomorphic graphs and the gluing procedure uses as $\mathfrak{E}$ regions on a Riemann surface the thick-thin decomposition coming from the uniformisation theorem. For compatibility with gluing we then have to have the $\mathfrak{E}$ regions on vertex model curves depending on the global structure of the original

Riemann surface. The results will then hold with only minor modifications after the equivalent of Proposition 2.9.2 has been proved.

C.1 Construction of $\mathfrak{E}$

We construct $\mathfrak{E}$ inductively as follows:

Pick some $\rho > 0$ so that the regions of injectivity radius less than ρ in the constant curvature -1 metric provided by the uniformisation theorem are always annuli. The $\mathfrak{E}$ edge regions we construct will always be contained in these ρ-thin regions. There is a holomorphic identification of each ρ-thin region surrounding a puncture with the unit complex disk D, punctured at 0. This identification is unique up to rotations of D. Note that the space of regions in D that are star shaped around 0 is convex. More than this, the space of collections of star shaped regions in the ρ-thin regions surrounding punctures that are preserved by automorphisms is also convex. We shall construct $\mathfrak{E}$ edge regions to be star shaped in these ρ-thin regions surrounding punctures. This has the advantage of there being no topological obstructions to defining our $\mathfrak{E}$ decomposition in patches on Deligne-Mumford space and then making them match up using a partition of unity.

Define the $\mathfrak{E}$ edge regions of a three punctured sphere to be the standard disks of radius $\frac{1}{2}$ inside the ρ-thin regions. (Identifying the ρ-thin regions as above with the standard complex unit disk punctured at 0). These are star shaped and preserved by automorphisms. This also trivially satisfies the axioms of a $\mathfrak{E}$ decomposition.

Suppose that we have constructed our $\mathfrak{E}$ decomposition on parts of Deligne Mumford space with $(3g-3+k) \leq m$. Suppose that this decomposition satisfies the above axioms and also has that each $\mathfrak{E}$ edge region is contained inside some ρ-thin region, and the $\mathfrak{E}$ regions surrounding punctures are star shaped in their ρ-thin region.

We now want to define our $\mathfrak{E}$ decomposition on a component of Deligne Mumford space with $(3g-3+k) = (m+1)$. Due to axiom 5a, for R large enough the $\mathfrak{E}_R$ decomposition has to be the decomposition already defined on the boundary strata, which consists of nodal curves with components satisfying $(3g-3+k) \leq m$. The

$\mathfrak{E}_R$ decomposition in some neighborhood of the boundary strata is then determined using axiom 5b.

We will start off on the boundary strata with the $\mathfrak{E}$ decomposition already defined for Riemann surfaces with $(3g-3+k) \leq m$. We will alter this by removing $\mathfrak{E}$ edge regions shorter than some size R_{m+1} and extend it to a neighborhood U of the boundary strata so that it satisfies axiom 5b. We will then patch this together with the $\mathfrak{E}$ decomposition defined outside of a neighborhood of the boundary strata by taking disks of radius $\frac{1}{2}$ inside the ρ-thin disks surrounding punctures.

In particular, suppose we want to take a nodal Riemann surface and resolve some node. Axiom 5c tells us that there must be a $\mathfrak{E}$ edge region surrounding each half of the node. Identify the $\mathfrak{E}$ edge region of one half with $\mathbb{R}/\mathbb{Z} \times (0, \infty)$ and the other half with $\mathbb{R}/\mathbb{Z} \times (-\infty, 0)$. (To do this, there is an unimportant choice of angular parameter). A gluing involves an identification of parts of these two cylinders. These identifications can be parametrized coordinates $(\Theta, l) \in \mathbb{R}/\mathbb{Z} \times (0, \infty)$. In particular, take the identification

$$(\theta, t) \cong (\theta - \Theta, t - l)$$

This identifies our two cylinders on a cylinder of length l. The gluing given by this identification replaces our pair of infinitely long cylinders with this one. Define the $\mathfrak{E}$ edge regions of this new surface for small r to consist of this new cylinder and the image of all other $\mathfrak{E}$ edge regions under this construction. Lemma C.1.1 below shows that for some R_0, this construction gives a well defined $\mathfrak{E}$ decomposition for some neighborhood U of the boundary strata corresponding to all gluings with l greater than some R_0. Moreover, on this neighborhood U, the edge regions will be contained in the ρ-thin parts and the edge regions surrounding punctures will be star shaped.

We will alter this decomposition on this neighborhood U. First, set

$$R_{m+1} = \min\left(\frac{R_0 + 1}{2}, R_m\right)$$

Then remove all edge regions shorter than $2R_{m+1}$. This decomposition locally defined on U will obey the above axioms.

Now define a second $\mathfrak{E}$ decomposition, by taking the disks of radius $\frac{1}{2}$ inside the

ρ-thin disks surrounding punctures. This gives a well defined $\mathfrak{E}$ decomposition outside any neighborhood of the boundary strata, which satisfies all our axioms. Note that the length of the longest internal edge before we deleted edges above is still well defined on our neighborhood. We interpolate between these two locally defined $\mathfrak{E}$ decompositions using this as a parameter, using our disks of radius $\frac{1}{2}$ outside of our neighborhood U and if the length of the longest internal edge before deletion is shorter than $R_0 + \frac{1}{4}$, and using our previously defined decomposition if the length of the longest internal edge before deletion is longer than $R_0 + \frac{3}{4}$.

This gives a globally defined $\mathfrak{E}$ decomposition on the components of Deligne Mumford space with $(3g - 3 + k) = (m + 1)$, which obeys all our axioms and the addition assumptions. We can continue inductively to define this $\mathfrak{E}$ decomposition on all of Deligne Mumford space.

The following is based on the exposition given in [3]. We want to show that the gluing procedure described above gives us a well defined $\mathfrak{E}$ decomposition on some neighborhood of the boundary strata. We will do this locally.

Suppose we have some nodal Riemann surface S with automorphism group G_S. The $\mathfrak{E}_R$ edge regions will be preserved as a set by G_S, but they may be permuted or twisted by the action of G_S. We can choose a uniformising neighborhood (U_S, G_S) for S inside the boundary strata that keeps all the nodes of S as nodes. A 'uniformising neighborhood' means an open subset $U_S \in \mathbb{R}^n$ with a group action G_S and a diffeomorphism identifying U_S/G_S with a neighborhood of S. These are the local charts for orbifolds.

We can choose smooth identifications of the $\mathfrak{E}$ edges surrounding our nodes of surfaces in U_S with $\mathbb{R}/\mathbb{Z} \times (0, \infty)$ on one side and $\mathbb{R}/\mathbb{Z} \times (-\infty, 0)$ on the other side. This may not be be well defined down in Deligne Mumford space, as our automorphisms may act nontrivially on these identifications. This is called a small disk structure in [3]. If there are k of these nodes, we then have a map defined by gluing on our k nodes as described above

$$\Phi : U_S \times D^k \longrightarrow \text{ Deligne Mumford space}$$

D indicates the complex unit disk with coordinates $e^{-2\pi(l+i\Theta)}$ with $(\Theta, l) \in \mathbb{R}/\mathbb{Z} \times (0, \infty)$. Our gluing map is defined as above by making the identification

$$(\theta, t) \cong (\theta - \Theta, t - l)$$

on each of our k nodes. Note that as our group action G_S may act nontrivially on our coordinates on the nodal $\mathfrak{E}$ edges, the action of G_S extends naturally to $U_S \times D^k$.

The following is proved in [3]:

Lemma C.1.1. *There exists some open neighborhood $\tilde{U}_S$ of $(S, 0) \subset U_S \times D^k$ so that $\left(\tilde{U}_S, G_S\right)$ together with the map Φ is a uniformising neighborhood of S.*

To get that our gluing procedure gives a well defined $\mathfrak{E}$ decomposition in a neighborhood of the boundary strata, we just need to patch together a finite number of these neighborhoods (noting that the boundary strata are compact.) Note that as our $\mathfrak{E}$ decomposition that we start with on the boundary strata satisfies axiom 5b, gluings on k nodes close to the strata where there are $k + 1$ nodes will give the same $\mathfrak{E}$ decomposition as gluings starting from the strata with $k + 1$ nodes. Restricting to a smaller neighborhood if necessary, we see that it is possible to keep the resulting $\mathfrak{E}$ edge regions inside the ρ-thin regions, and make the $\mathfrak{E}$ edge regions surrounding punctures star shaped inside their ρ-thin disks.

C.2 Metric

We want a family of metrics on our domain compatible with gluing and our complex structure and so that Proposition 3.2.4 gives a uniform derivative bound. For this we use the decomposition of a Riemann surface into edges and vertices discussed in section C, and choose a metric so that the following are satisfied.

1. The metric is in the conformal class given by the complex structure on the Riemann surface.
2. The metric on edges is given by the standard metric on $\mathbb{R}/\mathbb{Z} \times (a,b)$.
3. The metric on a vertex region V depends only on the Riemann surface S_V created by replacing all edges surrounding V with semi infinite cylinders $\mathbb{R}/\mathbb{Z} \times (0,\infty)$.
4. The metric depends continuously on the complex structure of the Riemann surface.
5. For every point $p \in S$ there exists a holomorphic embedding of the unit disk $D \longrightarrow S$ sending 0 to p and with derivative at 0 greater than $\frac{1}{2}$.

C.3 $\pi_{R,R'} : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{Q}^{\epsilon,\mathfrak{E}_{R'}}$

The goal of this section is to define the map

$$\pi_{R,R'} : \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E} \longrightarrow \mathcal{Q}^{\epsilon,\mathfrak{E}_{R'}} \text{ for } R' \geq R$$

$\pi_{R,R'}$ will preserve the domain Riemann surface of a quasi holomorphic graph, but as we are changing the $\mathfrak{E}_R$ edge markings, some things will need to be altered. Recall that the edge regions of $\mathfrak{E}_R$ consist of subsets $\mathbb{R}/\mathbb{Z} \times (a+R, b-R) \subset \mathbb{R}/\mathbb{Z} \times (a,b)$ of the $\mathfrak{E}$ edge regions parametrized conformally. If a $\mathfrak{E}$ edge region is conformal to a cylinder shorter than $2R$, then it will not contribute to the $\mathfrak{E}_R$ edge regions. Two things can happen to a $\mathfrak{E}_R$ edge when we change to $\mathfrak{E}_{R'}$ markings for $R' \geq R$, it can get shorter, or it can disappear entirely. The trivial holomorphic cylinders of the edges of $\pi_{R,R'}u$ will consist of the restriction of the trivial holomorphic cylinders from the edges of u restricted to $\mathfrak{E}_{R'}$ edge regions.

If no edge regions disappear, then we can simply take the map on vertex model curves $[p,f]$ to be the identity. Thus $\pi_{R,R'}$ is practically the identity when no edges disappear (or said another way, when $Q_{\mathfrak{E}_R} = Q_{\mathfrak{E}_{R'}}$ or when $\mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n) = \mathcal{H}_{\mathfrak{E}_{R'}}(S,\mathbb{C}^n)$).

What we need to define $\pi_{R,R'}$ in general is a way of assigning model curves when some $\mathfrak{E}_R$ edge region contains no $\mathfrak{E}_{R'}$ edge region.

Recall that the condition on joining a vertex model curve given by $[p, f]$, $f : S_V \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ to an adjacent edge in a quasi holomorphic graph is given by considering the edge as a subset $\mathbb{R}/\mathbb{Z} \times (0, R) \subset \mathbb{R}/\mathbb{Z} \times (0, \infty)$ of the edge region surrounding a puncture in S_V, and then parameterizing the edge as

$$C(\theta, t) = \exp_{\exp_p \zeta}(\theta\alpha + tJ^\epsilon\alpha)$$

$$\text{where } \lim_{t\to\infty} f(\theta, t) = \zeta + \theta\alpha + tJ\alpha$$

The above expression $\exp_p \zeta$ is to be understood after identifying $(T_p(\mathbb{T}^n \rtimes B^n), J^\epsilon)$ with $\mathbb{C}^n$. Note that this identification multiplies the part of ζ that projects to the base manifold by ϵ. Recall also that a normalizing condition we put on f was that $\sum \zeta_i$ is in the real (torus) subspace of $\mathbb{C}^n/\mathbb{Z}^n$.

Because the torsion of our flat connection ∇ is bounded, there exists a radius r_0 smaller than the injectivity radius of the base manifold, so that given any finite set of points b_i inside a ball in our base manifold of radius r_0 there is a unique average b so that

$$b_i = \exp_b v_i \text{ for } |v_i| \leq 2r_0$$

$$\text{and } \sum v_i = 0$$

ζ is bounded for model curves appearing in $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$, and the edges that disappear have length bounded by $2(R' - R)$. The number of internal edges is bounded by $3g - 3 + k$. We can choose ϵ so that $\epsilon(\max |\zeta| + (6g - 6 + 2k + 2)(R' - R)) < \frac{r_0}{2}$. This means that all the ends of the edges of $\pi_{R,R'}u$ attached to a $\mathfrak{E}_{R'}$ vertex region V will be contained in a ball of radius r_0 in the base, and have a unique average. We choose a point p in the torus fiber over this average point, and have the new model curve $[p, f]$ living at this point p. Once we have chosen p we can choose the ζ_i which get us from p to the ends of the attached edges. We can then define a function

$F : S \longrightarrow \mathbb{C}^n/\mathbb{Z}^n$ so that

$$\exp_{\exp_p(\sum \psi_i\zeta_i)}\left(F - \sum \psi_i\zeta_i\right) \text{ on } V \subset S_V$$

and F continues as the appropriate holomorphic cylinders. The notation above is explained in Section 2.6. All that is important for now is that this makes F become the correct trivial holomorphic cylinders near punctures. If we want our model curve to be quasi holomorphic, then we take

$$f := F - Q\bar{\partial}F$$

This defines the map $\pi_{R,R'}$. Note that we needed to choose ϵ small depending on R'.

Note $\ker d\pi_{R,R'}u$ can be identified with movements of the edge holomorphic cylinders of u that are conformal to cylinders of length less than $2(R'-R)$. If there are m of these, by tracking a marked point on the center of each edge, we can identify

$$\ker d\pi_{R,R'} = (\mathbb{C}^n)^m$$

Note also that the complex dimension of $\mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n)/\mathcal{H}_{\mathfrak{E}_{R'}}(S,\mathbb{C}^n)$ is also nm.

Lemma C.3.1. *There exists a canonical isomorphism*

$$I_S : (\mathbb{C}^n)^m = \ker d\pi_{R,R'} \longrightarrow \mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n)/\mathcal{H}_{\mathfrak{E}_{R'}}(S,\mathbb{C}^n)$$

so that the derivative of $\bar{\partial}_0$ *projected to* $\mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n)/\mathcal{H}_{\mathfrak{E}_{R'}}(S,\mathbb{C}^n)$ *and restricted to* $\ker d\pi_{R,R'}$

$$D_{\bar{\partial}_0} : \ker d\pi_{R,R'} \longrightarrow \mathcal{H}_{\mathfrak{E}_R}(S,\mathbb{C}^n)/\mathcal{H}_{\mathfrak{E}_{R'}}(S,\mathbb{C}^n)$$

satisfies

$$\left\| D_{\bar{\partial}_0} - I_S \right\| \leq \frac{1}{2\left\| I_S^{-1} \right\|}$$

for ϵ *small enough dependent on* R' *and* $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$

Proof:

In fact, $I_S = D_{\bar{\partial}_0}$ in the integrable case. Recall that the way we defined $Q_{\mathfrak{E}_R}$ in Proposition 2.9.2, $\ker Q_{\mathfrak{E}_R}$ is spanned by $\ker Q_{\mathfrak{E}_{R'}}$ and $\mathbb{C}^n \otimes \operatorname{Span}\{\bar{\partial} f_i\}$ where f_i is some function that is equal to 1 on the ith $\mathfrak{E}_R$ edge region that contains no $\mathfrak{E}_{R'}$ edge region, and 0 on all other $\mathfrak{E}_R$ edge regions.

In the integrable case, shifting the ith edge edge by $v_i \in \mathbb{C}^n$ will change $\bar{\partial}_0$ by $v \otimes \bar{\partial} f_i$. Considering this map on movements of $\mathfrak{E}_R$ edges that don't contain $\mathfrak{E}_{R'}$ edge regions defines the isomorphism I_S. An application of Lemma 2.2.4 shows that because the length of these edges and the size of model curves in $\mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$ is bounded, that $D_{\bar{\partial}_0}$ converges to this integrable case as ϵ approaches 0

□

Note that the fact that $\bar{\partial}_0$ is C^1 smooth when the $\mathfrak{E}_R$ decomposition doesn't jump and the above lemma implies that for ϵ small enough, $\pi_{R,R'}$ provides a C^1 smooth diffeomorphism from solutions of the equation

$$\bar{\partial}_0 u \in \mathcal{H}_{\mathfrak{E}_{R'}}(S, \mathbb{C}^n) \text{ for } u \in \mathcal{Q}^{\epsilon,\mathfrak{E}_R}_{g,k,E}$$

to its image in $Q^{\epsilon,\mathfrak{E}_{R'}}$ when restricted to sets where the $\mathfrak{E}_{R'}$ decomposition doesn't jump.

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