SINGULARITIES WITH $\mathbb{G}_m$-ACTION AND THE LOG MINIMAL MODEL PROGRAM FOR $\overline{M}_g$

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Abstract. We give a precise formulation of the modularity principle for the log canonical models $\overline{M}_g(\alpha) := \text{Proj} \bigoplus_{d \geq 0} H^0(\overline{M}_g, [d(K_{\overline{M}_g} + \alpha \delta)])$. We define a new invariant of Gorenstein curve singularities with $\mathbb{G}_m$-action which can be used to predict the critical $\alpha$-value at which a singularity first arises in the modular interpretation of $\overline{M}_g(\alpha)$. We compute these critical $\alpha$-values for large classes of singularities, including all ADE, toric planar, and unibranch Gorenstein singularities, and use these results to give a conjectural outline of the log MMP for $\overline{M}_g$.

Contents

1. Introduction 1
2. Curve singularities with $\mathbb{G}_m$-action 8
3. Character calculations 13
4. Outline of the log MMP for $\overline{M}_g$ 26
5. Connections to Intersection Theory 27
6. Connections to GIT 31
References 32

1. Introduction

In [Has05], Hassett and Keel initiated a program to give modular interpretations to the log canonical models

\begin{equation}
\overline{M}_g(\alpha) := \text{Proj} \bigoplus_{d \geq 0} H^0(\overline{M}_g, [d(K_{\overline{M}_g} + \alpha \delta)]),
\end{equation}

for all $\alpha \in [0,1] \cap \mathbb{Q}$ such that $K_{\overline{M}_g} + \alpha \delta$ is effective. Subsequent contributions to the Hassett-Keel program have established the modularity of various log canonical models by realizing them as GIT quotients of Hilbert schemes of pluricanonically embedded curves [HH09, HH13, HL10]. These works have established the modularity of the log canonical models $\overline{M}_g(\alpha)$ for all $\alpha$ when $g = 2,3$, and for $\alpha \geq 7/10 - \epsilon$ when $g \geq 4$. For additional partial results in genera four, five, and six see [Fed12, CMJL12a, CMJL12b, FS13b, Mül13]. Unfortunately, the GIT approach is limited by the absence of natural GIT quotient descriptions of $\overline{M}_g(\alpha)$ for low values of $\alpha$, as well as by increasing difficulty of the GIT stability analysis even in cases where quotient descriptions do exist. It is therefore natural to seek an abstract formulation of the aim of the Hassett-Keel program.

2000 Mathematics Subject Classification. Primary: 14H10; Secondary: 14D23, 14H20.
1. Modularity Principle for $\overline{M}_g(\alpha)$. Informally, the Hassett-Keel program suggests the existence of stacks of singular curves whose moduli spaces are the log canonical models $\overline{M}_g(\alpha)$. Before we make this precise, we need to introduce some terminology. First, we let $U_g$ denote the stack of complete Gorenstein\(^1\) curves of arithmetic genus $g$ with ample dualizing line bundle. We let $\pi : C_g \to U_g$ denote the universal curve and consider the following line bundles on $U_g$:

$$
\lambda_m := c_1(\pi_*\omega_m^n), \quad \text{for } m \geq 1,
$$

$$
K := 2\lambda_2 - 13\lambda_1, 
$$

$$
\delta := 13\lambda_1 - \lambda_2.
$$

Remark 1.1. A Grothendieck-Riemann-Roch computation shows that the restriction of $K$ and $\delta$ to $\overline{M}_g$ is the usual canonical class and boundary divisor class, respectively; see [HM82, Section 2] and [HM98, Section 3.E] for details of this computation.

Recall from [Alp08] that a morphism $\phi : X \to X$ from an algebraic stack to an algebraic space is a good moduli space if $\phi_*\mathcal{O}_X = \mathcal{O}_X$ and $\phi_* : \text{QCoh}(X) \to \text{QCoh}(X)$ is an exact functor. The formal properties of good moduli spaces will not play a significant role in this paper, but we encourage the reader to keep in mind that the canonical example of a good moduli space is the morphism $[X^{ss}/G] \to X^{ss}/G$ from a GIT quotient stack to the GIT quotient.

Finally, since $\overline{M}_g(\alpha)$ is defined as the Proj of the graded section ring of the $\mathbb{Q}$-line bundle $K_{\overline{M}_g} + \alpha\delta$, there is a natural $\mathbb{Q}$-line bundle $\mathcal{O}_{\overline{M}_g(\alpha)}(1)$ on $\overline{M}_g(\alpha)$ such that $f_*(K_{\overline{M}_g} + \alpha\delta) = \mathcal{O}_{\overline{M}_g(\alpha)}(1)$ under the rational map $f : \overline{M}_g \dashrightarrow \overline{M}_g(\alpha)$. With all this notation, we may consider the goal of the Hassett-Keel program as the verification of the following modularity principle.

Principle 1.2 (Modularity principle for the log MMP for $\overline{M}_g$). For all $\alpha \in [0, 1] \cap \mathbb{Q}$ such that $K_{\overline{M}_g} + \alpha\delta$ is effective, there exists an open normal substack $\overline{M}_g(\alpha) \subseteq U_g$ and a morphism $\phi : \overline{M}_g(\alpha) \to \overline{M}_g(\alpha)$ satisfying:

1. $\phi$ is a good moduli space.
2. $\phi^*\mathcal{O}_{\overline{M}_g(\alpha)}(1)|_{\overline{M}_g(\alpha) \cap \overline{M}_g} \simeq (K_{\overline{M}_g} + \alpha\delta)|_{\overline{M}_g(\alpha) \cap \overline{M}_g}$.

To verify the modularity principle in this abstract setting, one would like a natural recipe for defining the open substacks $\overline{M}_g(\alpha) \subset U_g$. One step in this recipe should be a method for predicting which singular curves arise at a given value of $\alpha$. In this paper, we construct a new invariant of curve singularities with $\mathbb{G}_m$-action, namely the $\alpha$-invariant, which is tailored to answer this question. We compute the $\alpha$-invariant for several natural classes of singularities, and use our computations to give a conjectural outline of the log MMP down to $\alpha = 5/9$. In fact, these computations have already been deployed as a useful guiding heuristic in recent works on the log minimal model program [CMJL12b, AFSvdW13].

1.2. Curves with $\mathbb{G}_m$-action in the log MMP for $\overline{M}_g$. Before defining the $\alpha$-invariant, we should explain why singularities with $\mathbb{G}_m$-action play such a special role in our analysis. In fact, there is good reason to expect that all reduced curve singularities arising in the log MMP

\(^1\)Our decision to focus on Gorenstein curves is motivated in part by experience (we have yet to see a non-Gorenstein curve appear in the Hassett-Keel program) and in part by technical convenience (the Hodge line bundles which play a central role in this paper are naturally defined over the locus of Gorenstein curves).
admit a $\mathbb{G}_m$-action. Indeed, at each ‘threshold’ value of $\alpha$, we expect a diagram

$$
\begin{array}{c}
\mathcal{M}_g(\alpha+\epsilon) \xrightarrow{i_+} \mathcal{M}_g(\alpha) \xrightarrow{i_-} \mathcal{M}_g(\alpha-\epsilon) \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{M}_g(\alpha+\epsilon) \to \mathcal{M}_g(\alpha-\epsilon) \\
\downarrow \quad \downarrow \\
\mathcal{M}_g(\alpha) \\
\end{array}
$$

where $i_+, i_-$ are open immersions, the vertical maps are good moduli spaces, and $j_+, j_-$ are projective morphisms induced by $i_+, i_-$. Crucially, the closed points of

$$\mathcal{M}_g(\alpha) - (\mathcal{M}_g(\alpha+\epsilon) \cup \mathcal{M}_g(\alpha-\epsilon))$$

must all have infinite stabilizers, i.e. must correspond to curves admitting a $\mathbb{G}_m$-action. Indeed, the only way an open immersion of stacks can induce a projective morphism of good moduli spaces is for all new points of the larger stack to be identified with certain points of the smaller stack, and this requires the new closed points of the larger stack to have infinite stabilizers.

There are two reasons for believing that each transition of the log MMP should entail a diagram of this kind. First, we believe the log MMP for $\mathcal{M}_g$ should be modeled after a variation of GIT problem, and VGIT problems always give rise to such diagrams [DH98, Tha96]. Indeed, as one varies the polarization in VGIT, the semistable locus ‘jumps’ at certain walls, and the corresponding open inclusions of the semistable locus induce projective proper morphisms of the corresponding quotients. Second, the work of Hassett-Hyeon and Hyeon-Morrison produces precisely such a picture at the first two critical thresholds. Indeed, at $\alpha = 9/11$, the points of $\mathcal{M}_g(\alpha) - (\mathcal{M}_g(\alpha+\epsilon) \cup \mathcal{M}_g(\alpha-\epsilon))$ correspond to curves with rational cuspidal tails (Figure 1) which contain all elliptic tails and cuspidal curves in their basins of attraction [HM10]; see [HH13, Definition 5.2] for the definition of the basins of attraction. At $\alpha = 7/10$, the points of $\mathcal{M}_g(\alpha) - (\mathcal{M}_g(\alpha+\epsilon) \cup \mathcal{M}_g(\alpha-\epsilon))$ correspond to curves with rational tacnodal bridges, and contain all elliptic bridges and tacnodal curves in their basins of attraction [HH13].

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node at (0,0) {\includegraphics[width=2cm]{figure1.png}};
\node at (3,0) {\includegraphics[width=2cm]{figure2.png}};
\node at (1.5,0) {\includegraphics[width=0.5cm]{figure3.png}};
\end{tikzpicture}
\caption{Rational cuspidal tail is an isotrivial degeneration of both elliptic tails and cuspidal curves}
\end{figure}

\footnote{Since these curves correspond to closed points of a stack with a good moduli space, their stabilizers must be linearly reductive. But if $C$ is any reduced curve with positive dimensional linearly reductive automorphism group $\text{Aut}(C)$, the connected component of the identity of $\text{Aut}(C)$ is a non-trivial torus, i.e. $C$ admits a $\mathbb{G}_m$-action.}
1.3. The case of marked curves. We note that there is a simple way to generalize our results to the case of pointed curves. Namely, let $U_{g,n}$ denote the stack of Gorenstein curves $C$ of arithmetic genus $g$ with smooth marked points $p_1, \ldots, p_n \in C$ such that $\omega_C(p_1 + \cdots + p_n)$ is ample. We let $\pi: \mathcal{C}_{g,n} \to U_{g,n}$ denote the universal curve and $\sigma_\chi: U_{g,n} \to \mathcal{C}_{g,n}$ denote the universal sections. Then we can define

$$\lambda_m := c_1 \left( \pi_* \omega^m \left( \sum_{i=1}^n \sigma_i \right) \right), \quad \text{for } m \geq 1,$$

(1.3)

$$\psi_i := \sigma_i^\ast (\omega), \quad \psi := \sum_{i=1}^n \psi_i,$$

$$K := 2\lambda_2 - 13\lambda_1 - \psi, \quad \delta := 13\lambda_1 - \lambda_2 + \psi.$$

With this notation, we will be able to define and study the invariants of marked curves with $\mathbb{G}_m$-action, and relate them to the invariants of unmarked curves.

1.4. $\alpha$-invariants of curves with $\mathbb{G}_m$-action. In moduli theory, the presence of objects with infinite automorphism groups is usually considered bad news, but the key idea of this paper is to make a virtue of necessity by exploiting this feature of the log MMP to obtain a priori information on the definition of the stacks $\overline{\mathcal{M}}_g(\alpha)$. To see how this works, recall that if $[C] \in U_g$ is any point and $L$ is a line bundle defined in a neighborhood of $[C]$, then the natural action of $\text{Aut}(C)$ on the fiber $L|_C$ induces a character $\text{Aut}(C) \to \mathbb{G}_m$. If $\eta: \mathbb{G}_m \to \text{Aut}(C)$ is any one-parameter subgroup, then there is an induced character $\mathbb{G}_m \to \mathbb{G}_m$ which is necessarily of the form $z \mapsto z^n$ for some integer $n \in \mathbb{Z}$. For a given curve $C$, a non-trivial one-parameter subgroup $\eta: \mathbb{G}_m \to \text{Aut}(C)$, and a line bundle $L \in \text{Pic}(U_g)$, we denote this integer by $\chi_L(C, \eta)$. If $\text{Aut}(C) \simeq \mathbb{G}_m$, we write simply $\chi_L(C)$, where $\eta: \mathbb{G}_m \to \text{Aut}(C)$ is understood to be the identity.

Note that $\chi_L(C)$ is only defined up to sign due to the choice of isomorphism $\text{Aut}(C) \simeq \mathbb{G}_m$, but the ratios $\chi_L(C)/\chi_M(C)$, for any two line bundles $L$ and $M$, are well-defined and this is all we ultimately use. If $L = \lambda_m$, we write simply $\chi_m(C, \eta)$ or $\chi_m(C)$ if $\eta$ is understood. These characters are connected to the modularity principle by the following observation.

**Proposition 1.3 (Key Observation).** Assume the modularity principle holds. Suppose in addition that the locus of worse-than-nodal curves in $\overline{\mathcal{M}}_g(\alpha)$ has codimension at least 2. Then for any $[C] \in \overline{\mathcal{M}}_g(\alpha)$ and any one-parameter subgroup $\eta: \mathbb{G}_m \to \text{Aut}(C)$, we have

$$\chi_{(K + \alpha \delta)}(C, \eta) = 0.$$

**Proof.** If the modularity principle holds, then $\phi^\ast \mathcal{O}_{\overline{\mathcal{M}}_g(\alpha)}(1)|_{\overline{\mathcal{M}}_g(\alpha) \cap \overline{\mathcal{M}}_g} \simeq (K_{\overline{\mathcal{M}}_g} + \alpha \delta)|_{\overline{\mathcal{M}}_g(\alpha) \cap \overline{\mathcal{M}}_g}$, where $\phi: \overline{\mathcal{M}}_g(\alpha) \to \overline{\mathcal{M}}_g(\alpha)$ is a good moduli space. On the other hand, since $\text{codim}(\overline{\mathcal{M}}_g(\alpha) \setminus \overline{\mathcal{M}}_g(\alpha)) \geq 2$ and $\overline{\mathcal{M}}_g(\alpha)$ is normal, the line bundle $K + \alpha \delta = (2 - \alpha)\lambda_2 + (13\alpha - 13)\lambda_1$ is the unique locally-free extension of $(K_{\overline{\mathcal{M}}_g} + \alpha \delta)|_{\overline{\mathcal{M}}_g(\alpha) \cap \overline{\mathcal{M}}_g}$ to $\overline{\mathcal{M}}_g(\alpha)$. Here, we use the fact that the Hodge line bundles $\lambda_m$ are defined over all of $U_g$ and hence over $\overline{\mathcal{M}}_g(\alpha)$. It follows that $K + \alpha \delta \simeq \phi^\ast \mathcal{O}_{\overline{\mathcal{M}}_g(\alpha)}(1)$. However, for any point $[C] \in \overline{\mathcal{M}}_g(\alpha)$, the induced action of $\text{Aut}(C)$ on the fiber of $\phi^\ast \mathcal{O}_{\overline{\mathcal{M}}_g(\alpha)}(1)$ is necessarily trivial. This finishes the proof. \qed

What makes this observation useful is the fact that the character $\chi_{(K + \alpha \delta)}(C, \eta)$ can be computed in terms of the intrinsic geometry of $C$, without knowing the definition of $\mathcal{M}_g(\alpha)$ a priori. Indeed, by definition

$$\chi_{(K + \alpha \delta)}(C, \eta) = (2 - \alpha)\chi_2(C, \eta) + (13\alpha - 13)\chi_1(C, \eta).$$
Now the characters on the right are manifestly computable in terms of the intrinsic geometry of \((C, \eta)\), indeed \(\chi_m(C, \eta)\) is simply the determinant of the action of \(\eta\) on \(H^0(C, \omega_C^2)\). Using this, we may define the \(\alpha\)-invariant of a curve with \(G_m\)-action to be the unique rational number \(\alpha(C, \eta)\) such that this character is zero.

**Definition 1.4** (\(\alpha\)-invariant). The \(\alpha\)-invariant of a curve \(C\) with \(G_m\)-action \(\eta\): \(G_m \to \text{Aut}(C)\) is the rational number

\[
\alpha(C, \eta) := \frac{13 - 2 \left( \frac{\chi_2(C, \eta)}{\chi_1(C, \eta)} \right)}{13 - \left( \frac{\chi_2(C, \eta)}{\chi_1(C, \eta)} \right)}.
\]

If \(\chi_1(C, \eta) = 0\), the \(\alpha\)-invariant is undefined. If \(\text{Aut}(C) \simeq G_m\), we write \(\alpha(C)\) for the \(\alpha\)-invariant \(\alpha(C, \eta)\) where \(\eta: G_m \to \text{Aut}(C)\) is understood to be the identity.

We conclude that, as long as the \(\alpha\)-invariant of a curve with \(G_m\)-action is defined, it is the only \(\alpha\)-value at which the curve can show up in \(\overline{\mathcal{M}}_g(\alpha)\).

**Corollary 1.5.** Assume the modularity principle holds and let \([C] \in \overline{\mathcal{M}}_g(\alpha)\) be any point. If \(C\) admits a \(G_m\)-action \(\eta\) such that \(\chi_1(C, \eta) \neq 0\), then \(\alpha = \alpha(C, \eta)\).

**Proof.** By Proposition 1.3, we must have

\[
\chi_{(K + \alpha\delta)}(C, \eta) = (2 - \alpha)\chi_2(C, \eta) + (13\alpha - 13)\chi_1(C, \eta) = 0.
\]

Solving for \(\alpha\), we obtain the desired statement. \(\square\)

While the \(\alpha\)-invariant \(\alpha(C, \eta)\) can be computed for any explicitly given curve with \(G_m\)-action, certain classes of curves play a distinguished role in ‘announcing’ critical thresholds of the log minimal model program. In this paper, we isolate three classes of complete curves with \(G_m\)-action, namely \(\hat{O}\)-atoms, \(\hat{O}^S\)-atoms, and \(A_i/A_j\)-chains. These will be defined carefully in Section 2 (Definitions 2.7, 2.10, and 2.13), but we give here an informal description of these curves and of their geometric significance in the context of the log minimal model program.

We have already seen that \(\overline{\mathcal{M}}_g(9/11)\) contains curves of the form \(T \cup E\), where \(E\) is a rational cuspidal tail, and that \(\overline{\mathcal{M}}_g(7/10)\) contains curves of the form \(T \cup E\), where \(E\) is a rational tacnodal bridge. The \(\hat{O}\)-atom is simply a generalization of this construction to an arbitrary Gorenstein curve singularity \(\hat{O}\) with \(G_m\)-action. Indeed, if the singularity \(\hat{O}\) has \(b\) branches and \(\delta\)-invariant \(\delta(p)\), an \(\hat{O}\)-atom is simply a curve of the form

\[
C = E_1 \cup \ldots \cup E_b \cup C_0,
\]

where \(C_0\) is any smooth curve of genus \(g - \delta(p) - b + 1\) and \(E_1, \ldots, E_b\) are rational curves attached to \(C_0\) nodally and meeting in a singularity analytically isomorphic to \(\hat{O}\) (see Figure 2 for an example of \(J_{10}\)-atom).

The \(G_m\)-action on \(\hat{O}\) extends to define a one-parameter subgroup \(\eta: G_m \to \text{Aut}(C)\), and we define the \(\alpha\)-invariant of the \(\hat{O}\)-atom to be \(\alpha(C, \eta)\). Using this definition, the \(\alpha\)-invariant of a cusp and a tacnode are \(9/11\) and \(7/10\) respectively, and in general we expect the \(\alpha\)-value associated to a singularity to be the first \(\alpha\)-value at which the generic form of this singularity appears in the modular interpretation of \(\overline{\mathcal{M}}_g(\alpha)\). (We discuss non-generic forms of singularities in the next paragraph.) We compute the \(\alpha\)-invariants of a broad range of singularities and the results are collected in Table 1 of Section 4.
Figure 2. An atom of the $y^3 = x^6$ singularity (also called $J_{10}$-atom)

There are two natural generalizations of $\hat{O}$-atoms which we expect to play an important role in future stages of the log minimal model program for $\overline{M}_g$. To motivate these constructions, let us consider which loci on $\overline{M}_g$ must be modified at the critical value $\alpha = 19/29$. The divisor $\Delta_2$ is covered by curves of slope $19/29$, and we expect curves with genus 2 tails to be replaced by ‘dangling oscnodes,’ using the blow-up/blow-down procedure pictured in Figure 3. The distinguished curve with $\mathbb{G}_m$-action to which this dangling oscnode isotrivially specializes is the curve pictured in Figure 5.

Figure 3. Given a smoothing of a curve with a genus 2 tail attached at an arbitrary point $p$, after blowing up the conjugate point of $p$ and contracting the genus 2 curve, one obtains a dangling $\mathbb{P}^1$ attached at an oscnode.

Figure 4. Dangling $A_5$-atom.

Note that this curve is not an oscnodal atom because one of the branches of the oscnode ‘dangles,’ i.e. is not nodally attached to the rest of the curve. This motivates a generalization of the construction of the $\hat{O}$-atom, where some subset $S$ of the branches of $\hat{O}$ are allowed to dangle, and we call such curves $\hat{O}^S$-atoms. The $\alpha$-invariant of an $\hat{O}^S$-atom depends on $S$, and
this gives rise to important subtleties in the unfolding of the log minimal model program. For example, while the \(\alpha\)-invariant of the dangling oscnodal atom is 19/29, the \(\alpha\)-invariant of the standard oscnodal atom is 17/28, reflecting the fact that genus 2 bridges attached at conjugate Weierstrass points are not replaced until \(\alpha = 17/28\). The \(\alpha\)-invariants for a wide selection of \(\hat{O}^S\)-atoms are displayed in Table 2.

A second locus that must be modified at \(\alpha = 19/29\) is the locus of genus two tails attached tacnodally at a Weierstrass point. Of course, this locus does not appear in \(\mathcal{M}_g\), but it appears in \(\mathcal{M}_g(7/10)\) and subsequent models, and one can check that it is covered by curves of slope 19/29. We expect these curves to be replaced by tacnodally attached ramphoid tails. The corresponding distinguished curve with \(\mathbb{G}_m\)-action to which this curve specializes is pictured in Figure 5. This motivates consideration of \(A_i/A_j\)-chains constructed by concatenating \(A_i\) and \(A_j\)-singularities along rational branches (Definition 2.13). The \(\alpha\)-invariants associated to \(A_i/A_j\)-chains are listed in Table 2.

![Figure 5. A3/4-atom.](image)

In summary, using the computations of \(\alpha\)-invariants of \(\hat{O}\)-atoms, \(\hat{O}^S\)-atoms, and \(A_i/A_j\)-chains, we can assemble a fairly comprehensive set of prediction for the log minimal model program. While there are innumerable further variations on these constructions, we expect that these three varieties of singular curves with \(\mathbb{G}_m\)-action are sufficient to give a skeletal outline of log minimal model program for \(\alpha \geq 5/9\).

1.5. Outline of the paper: Let us now give a roadmap for the rest of the paper. In Section 2, we discuss singularities with \(\mathbb{G}_m\)-action, and define \(\hat{O}\)-atoms, \(\hat{O}^S\)-atoms, and \(A_i/A_j\)-chains, and the associated \(\alpha\)-invariants. In Section 3, we explain how to compute these \(\alpha\)-invariants: In 3.1, we compute the \(\alpha\)-invariants of a wide range of singularities, including ADE, toric planar, and unibranch Gorenstein singularities. In 3.2, we give an alternative method for computing \(\chi_K\) and \(\chi_\delta\), under additional assumptions on the deformation space of the curve. While the results of these calculations are ultimately derivable from the calculations in 3.1, they help to illuminate the geometric significance of \(\alpha\)-invariants.

In Section 4, we collect our computations from Section 3 and display them in Tables 1 and 2. In addition, we explain how to use these \(\alpha\)-invariants to generate predictions for the log MMP, and give a complete outline of the transitions of the log MMP for \(\alpha \geq 5/9\) in Table 3.

Finally, in Sections 5 and 6, we explain the connection of our \(\alpha\)-invariant calculations to important invariants coming from intersection theory and geometric invariant theory. In Section 5, we explain how the \(\alpha\)-invariant is related to slopes of curves on the locus of stable limits of a singularity. In Section 6, we explain how Hilbert-Mumford indices are related to the characters. We use this connection to prove some new instability results for ribbons; we should mention that these computations were a key inspiration for the results of [AFS13].
Notation. We work over an algebraically closed field $k$ of characteristic $0$ and denote by $\mathbb{G}_m$ the multiplicative group of units in $k$. A curve is a finite type scheme over $k$ of pure dimension 1.

Acknowledgements. We thank Anand Deopurkar, David Hyeon, and Fred van der Wyck for stimulating discussions. We thank David Hyeon for sharing an early version of his preprint [Hye10]. We also thank the referee for a careful reading and remarks that significantly improved the exposition.

2. Curve singularities with $\mathbb{G}_m$-action

In this section, we collect definitions and preliminary results concerning curve singularities with $\mathbb{G}_m$-action; all curves are assumed to be reduced. We begin by recalling the definition of the dualizing sheaf of a singular curve in Section 2.1. We use this description in Section 2.2 to show that the analytic isomorphism type of a Gorenstein curve singularity with $\mathbb{G}_m$-action uniquely determines an affine finite type curve with $\mathbb{G}_m$-action. This leads to a precise definition in Section 2.3 of an $\hat{\mathcal{O}}$-atom for any Gorenstein singularity $\hat{\mathcal{O}}$ with $\mathbb{G}_m$-action.

2.1. Dualizing sheaf of a singular curve. Given a reduced curve $C$, denote by $\nu: \tilde{C} \to C$ the normalization of $C$ and consider the sheaf $\Omega_{\tilde{C}} \otimes k(\tilde{C})$ of rational differentials on $\tilde{C}$; here $k(\tilde{C})$ stands for the sheaf of rational functions on $\tilde{C}$. By [BHPVdV04, Prop.6.2] (see also [Ser88, Ch.IV] for a general discussion of duality on singular curves) the dualizing sheaf of $C$ can be defined as the subsheaf $\omega_C \subset \nu^* (\Omega_{\tilde{C}} \otimes k(\tilde{C}))$ satisfying the following condition: For every $p \in C$ and every $f \in \mathcal{O}_{C,p}$

\[
\sum_{t \in \nu^{-1}(p)} \text{Res}_t (f \omega) = 0.
\]

Differentials satisfying (2.1) are called Rosenlicht differentials.

Example 2.1. Suppose $p \in C$ is a curve singularity with $b$ branches. Let $t_1, \ldots, t_b$ be uniformizers of the normalized branches. We have a collection of Rosenlicht differentials at $p$ defined for every pair of branches $i \neq j \in \{1, \ldots, b\}$ by

$$\omega(i,j) := \left( \cdots, \frac{dt_i}{t_i}, \cdots, -\frac{dt_j}{t_j}, \cdots \right) \in \omega_C,p.$$ 

Example 2.2. In the notation of Example 2.1, if the branch of $p \in C$ with the uniformizer $t_i$ is singular, then \( \left( \cdots, \frac{dt_i}{t_i^2}, \cdots \right) \in \omega_C,p. \)

We say that a curve singularity $p \in X$ is decomposable if

$$\hat{\mathcal{O}}_{X,p} \cong k[[x_1, \ldots, x_n, y_1, \ldots, y_m]]/(I, y_1, \ldots, y_m) \cap (J, x_1, \ldots, x_n),$$

where $I \subseteq (x_1, \ldots, x_n) \cap k[x_1, \ldots, x_n]$, and $J \subseteq (y_1, \ldots, y_m) \cap k[y_1, \ldots, y_m]$ are ideals.

We use Examples 2.1 and 2.2 to show that Gorenstein singularities are generally indecomposable:

Proposition 2.3. Every Gorenstein curve singularity, with the exception of the ordinary node $xy = 0$, is indecomposable.
Proof. Suppose \( \hat{O}_{X,p} \simeq k[[x_1, \ldots, x_n, y_1, \ldots, y_m]]/(I, y_1, \ldots, y_m) \cap (J, x_1, \ldots, x_n) \) is a decomposable Gorenstein curve singularity with \( b \) branches. Let \( k[[t_1]] \oplus \cdots \oplus k[[t_b]] \) be the normalization of \( \hat{O}_{X,p} \). By decomposability assumption, there are two branches, say those with uniformizers \( t_1 \) and \( t_2 \), such that \( (x_1, \ldots, x_n) \not\rightarrow 0 \in k[[t_1]] \) and \( (y_1, \ldots, y_m) \not\rightarrow 0 \in k[[t_2]] \). We now analyze differentials \( \omega(i, j) \) from Example 2.1. Note that \( \omega(1, 2) \notin (x_1, \ldots, x_n, y_1, \ldots, y_m) \omega_{X,p} \). Since \( X \) is Gorenstein, we conclude that \( \omega_{X,p} \) must be generated by \( \omega(1, 2) \). If \( b \geq 3 \), then \( \omega(1, 3) \) is not a multiple of \( \omega(1, 2) \) and we arrive at a contradiction. If \( b = 2 \), then the fact that \( (dt_1/t_1^2, 0) \) is not a multiple of \( \omega(1, 2) \), hence not in \( \omega_{X,p} \), implies that one of the \( x_i \)'s has valuation 1 in \( k[[t_1]] \). Similarly, one of the \( y_j \)'s has valuation 1 in \( k[[t_2]] \). It follows that both branches of \( p \) are smooth and intersect transversally. Thus \( p \in X \) is a node. \( \square \)

2.2. Curve singularities with \( \mathbb{G}_m \)-action. Observe that if \( C \in \mathcal{U}_g \) is a curve with \( \mathbb{G}_m \)-action and \( p \in C \) is a singular point, then there is an affine open \( \mathbb{G}_m \)-invariant subcurve \( X \subset C \) such that \( p \in X \).\(^3\) For this reason, we focus our attention on affine singular curves with \( \mathbb{G}_m \)-action.

Given an affine curve \( X \) with a distinguished singular point \( p \in X \) and \( \mathbb{G}_m \)-action on \( X \) fixing \( p \), we can choose coordinates so that:

1. \( X \simeq \text{Spec } k[x_1, \ldots, x_n]/I \) where \( p \) corresponds to the maximal ideal \( (x_1, \ldots, x_n) \), and
2. \( \mathbb{G}_m = \text{Spec } k[\lambda, \lambda^{-1}] \) acts by \( x_i \mapsto \lambda^{d_i} x_i \) and \( I \) is quasi-homogeneous with respect to this action.

We say that the \( \mathbb{G}_m \)-action is good if the weights \( d_i \) are either all negative or all positive. If the \( \mathbb{G}_m \)-action on \( (X, p) \) is good, we can find a minimal \( \mathbb{G}_m \)-invariant open affine subcurve of \( X \) containing \( p \). Henceforth, we assume that any good \( \mathbb{G}_m \)-action acts freely on \( X - p \).

Evidently, if \( (X, p) \) is an affine curve with good \( \mathbb{G}_m \)-action, then the induced action on the normalization \( \hat{X} \) is non-trivial on every branch and hence \( \hat{X} = \text{Spec } k[t_1] \oplus \cdots \oplus k[t_b] \), where \( b \) is the number of the branches of \( p \in X \), and so we can describe \( \mathcal{O}_{X,p} \) as a \( \mathbb{G}_m \)-invariant subring of \( k[t_1] \oplus \cdots \oplus k[t_b] \). The following proposition says that a \( \mathbb{G}_m \)-action on an indecomposable curve singularity must be good.

Lemma 2.4. Suppose \( p \in X \) is an indecomposable singular curve with \( \mathbb{G}_m \)-action. Then the \( \mathbb{G}_m \)-action is good.

Proof. Suppose \( X \simeq \text{Spec } k[x_1, \ldots, x_n, y_1, \ldots, y_m]/I \) is an affine reduced curve with \( \mathbb{G}_m \)-action fixing the origin such that \( \mathbb{G}_m \) acts with positive weights on \( (x_1, \ldots, x_n) \), and with non-positive weights on \( (y_1, \ldots, y_m) \). The \( \mathbb{G}_m \)-action lifts to the normalization of \( X \) at \( p \). Let \( t_1, \ldots, t_k \) be the uniformizers of the branches on which \( \mathbb{G}_m \) acts non-trivially with positive weight, and let \( t_{k+1}, \ldots, t_b \) be the uniformizers of all other branches. Then \( (x_1, \ldots, x_n) \not\rightarrow 0 \in k[[t_{k+1}]] \oplus \cdots \oplus k[[t_b]] \) and \( (y_1, \ldots, y_m) \not\rightarrow 0 \in k[[t_1]] \oplus \cdots \oplus k[[t_k]] \). It follows that \( x_i y_j = 0 \) on the normalization, for all \( i \) and \( j \). Since \( X \) is reduced, it follows that \( (x_1, \ldots, x_n)(y_1, \ldots, y_m) \subset I \), which in turn implies that \( X \) is decomposable. A contradiction. \( \square \)

A \( \mathbb{G}_m \)-action on \( (X, p) \) induces a \( \mathbb{G}_m \)-action on \( \hat{O}_{X,p} \).\(^4\) Our first goal is to show that the pair \( (X, p) \) can be essentially recovered from \( \hat{O}_{X,p} \) when \( \hat{O}_{X,p} \) is Gorenstein. To begin, we record an immediate consequence of Lemma 2.4.

\( ^3 \)As the example of a nodal plane cubic illustrates, the condition that \( \omega_{ti} \) is ample cannot be omitted from this observation.

\( ^4 \)In this paper, we define a \( \mathbb{G}_m \)-action on a complete local ring \( \hat{O} \) to be the action induced by an algebraic action on a finite-type \( k \)-algebra \( A \) fixing a maximal ideal \( m \) and such that \( A_m = \hat{O} \).
Corollary 2.5. A non-trivial $G_m$-action on a Gorenstein non-nodal singularity is necessarily good.

The next result shows that any given analytic isomorphism type of a Gorenstein curve singularity $O$ comes from a unique affine curve $(X, p)$, with a uniquely determined (up to scaling) $G_m$-action.

Proposition 2.6. Suppose $(X, p)$ and $(Y, q)$ are affine curves with $G_m$-action such that $O_{X, p}$ and $O_{Y, q}$ are abstractly isomorphic Gorenstein complete local rings. Then, after renormalizing the $G_m$-actions, there is a $G_m$-equivariant isomorphism $(X, p) \simeq (Y, q)$ of pointed affine schemes.

Proof. The claim is obvious for the node, so we assume that $O_{X, p}$ is not a node. Then by Corollary 2.5, the $G_m$-actions on $(X, p)$ and $(Y, q)$ are good. Let $(X, p) = \text{Spec} \, k[x_1, \ldots, x_n]/I$ and $(Y, q) = \text{Spec} \, J[y_1, \ldots, y_m]/J$. Let $k[t_1] \oplus \cdots \oplus k[t_b]$ be the $G_m$-equivariant normalization of $(X, p)$ and $k[s_1] \oplus \cdots \oplus k[s_b]$ be the $G_m$-equivariant normalization of $(Y, q)$, such that the $G_m$-actions are given by $\lambda \cdot t_j = \lambda^{-\alpha_j} t_j$ and $\lambda \cdot s_i = \lambda^{-\beta_i} s_i$. (We are using the fact that the number of branches is an analytic invariant of a singularity.)

Consider an isomorphism $i : O_{X, p} \to O_{Y, q}$. It induces the isomorphism

$$f : k[t_1] \oplus \cdots \oplus k[t_b] \to k[s_1] \oplus \cdots \oplus k[s_b]$$

of complete local rings of the normalizations. After renumbering the branches and scaling, we can assume that $f(t_i) = s_i \pmod{(s_i^2)}$.

Our first goal will be to show that the weights $(\alpha_i)_{i=1}^b$ and $(\beta_i)_{i=1}^b$ are the same (up to scaling and reordering). Since $\omega_{X, p}$ is a free rank one $O_{X, p}$-module, we can choose a $G_m$-semi-invariant generator $\omega_0(X)$ of $\omega_{X, p}$, of the form

$$\omega_0(X) = \left( \frac{dt_1}{t_1^{m_1}}, \ldots, \frac{dt_b}{t_b^{m_b}} \right).$$

Clearly, $m_i \geq 2$ for all $i$. (Otherwise, $\omega(1, 2)$ is a scalar multiple of $\omega$, so $p \in X$ is an ordinary node.) It follows that $\alpha_1(m_1 - 1) = \cdots = \alpha_b(m_b - 1)$ and so the $(\alpha_1, \ldots, \alpha_b)$ is determined (up to scaling) by the orders of vanishing of $\omega_0(X)$ on the normalization $X$. However, the latter is clearly an analytic invariant. Hence, after renormalization, we have $(\alpha_i)_{i=1}^b = (\beta_i)_{i=1}^b$.

For any $\lambda \in G_m$, we consider the following isomorphism

$$f_\lambda := \lambda \circ f \circ \lambda^{-1} : k[[t_1]] \oplus \cdots \oplus k[[t_b]] \to k[[s_1]] \oplus \cdots \oplus k[[s_b]].$$

Since $O_{X, p}$ and $O_{Y, q}$ are $G_m$-invariant subrings of $k[[t_1]] \oplus \cdots \oplus k[[t_b]]$ and $k[[s_1]] \oplus \cdots \oplus k[[s_b]]$, respectively, we conclude that $f_\lambda$ restricts to an isomorphism $O_{X, p} \to O_{Y, q}$. Since $\lambda$ acts with equal weights on $t_i$ and $s_i$, we have $f_\lambda(t_i) = s_i \pmod{(\lambda s_i^2)}$. Consider the isomorphism $f_0 : s_i \to t_i$ of the normalizations. Since $f_\lambda(O_{X, p}/m_{X, p}^N) = O_{Y, q}/m_{Y, q}^N$ for $\lambda \neq 0$, we must also have $f_0(O_{X, p}/m_{X, p}^N) = O_{Y, q}/m_{Y, q}^N$. By letting $N \to \infty$ and taking the inverse limit, we conclude that $f_0$ restricts to an isomorphism $O_{X, p} \to O_{Y, q}$. The claim follows. \hfill \Box

2.3. Atoms associated to curve singularities with $G_m$-action. Suppose $O$ is a Gorenstein non-nodal curve singularity with $G_m$-action. By Proposition 2.6, there exists a unique up to isomorphism pointed affine curve $(X, p)$ with $G_m$-action, free on $X - p$, such that $O \simeq O_{X, p}$. Moreover, the action is good and is unique (up to scaling). The affine scheme $X$ admits a canonical compactification $\tilde{X}$, constructed as follows: Let $\{E_i\}_{i=1}^b$ be the branches of $X$ at
p, where the enumeration is chosen once and for all. Since $\tilde{E}_i \simeq A^1$ and the $G_m$-action on $E_i$ has a unique fixed point $p$, we can compactify $\tilde{E}_i$ to $\mathbb{P}^1$ by adding a single point $p_i$ at infinity. Compactifying each branch of $X$, we obtain a complete pointed curve $(\hat{X};\{p_i\}_{i=1}^b)$ whose irreducible components have normalization isomorphic to $\mathbb{P}^1$, and such that the $G_m$ acts on each $E_i$ by scaling with fixed points $p$ and $p_i$. We call $(\hat{X};\{p_i\}_{i=1}^b)$ a $G_m$-invariant compactification of $\hat{O}$.

**Definition 2.7** ($\hat{O}$-atoms). Suppose $\hat{O}$ is a Gorenstein curve singularity with $G_m$-action and $G_m$-invariant compactification $(\hat{X};\{p_i\}_{i=1}^b)$. Let $\delta(p)$ be the $\delta$-invariant. An $\hat{O}$-atom of genus $g$ is any curve $C$ obtained by nodally attaching a smooth curve $C_0$ of genus $g - \delta(p) - b + 1$ to each of the $b$ branches of $\hat{X}$ at the points $\{p_1, \ldots, p_b\}$.

Let $\eta$: $G_m \to \text{Aut}(C)$ be the unique $G_m$-action on an $\hat{O}_{X,p}$-atom $C$ extending the $G_m$-action on $X$. Then by the construction of $C$ and Proposition 2.6, the characters $\chi_1(C, \eta)$ and $\chi_2(C, \eta)$ (defined in Section 1.4) depend only on the complete local ring $\hat{O}_{X,p}$. When a singularity $\hat{O}_{X,p}$ has a name, e.g. $E_6 : y^3 - x^4 = 0$, we use that name to denote the corresponding atom, e.g. 'E$_6$-atom.'

**Definition 2.8.** The $\alpha$-invariant $\alpha(\hat{O})$ of a Gorenstein singularity with $G_m$-action $\hat{O}$ is defined to be the $\alpha$-invariant $\alpha(C, \eta)$ of any $\hat{O}$-atom $C$.

**Remark 2.9.** It would have been arguably more natural to define the $\alpha$-invariant of $\hat{O}_{X,p}$-singularity in terms of characters of the complete curve $\hat{X}$. We chose the above definition so that the $\alpha$-invariant agrees with the $\alpha$-values at which $A_2$ and $A_3$-singularities first appear in $\overline{M}_g(\alpha)$ for $g \geq 3$.

The $\alpha$-invariant is an intrinsically computable object. In fact, we compute the $\alpha$-invariants for all ADE, planar toric, and monomial unibranch Gorenstein singularities, as well as of ribbons, in Section 3 (the results are displayed in Table 1 of Section 4). As explained in the introduction, we expect these $\alpha$-invariants to be the $\alpha$-values at which the given singularities first appear in $\overline{M}_g(\alpha)$. Note that our predictions for when $A_k$ singularities arise agree with the computations of Hyeon, who uses different heuristics [Hye10].

There is a natural variant of the construction of the $\hat{O}$-atom, which we expect to play an important role in the future stages of the program.

**Definition 2.10** ($\hat{O}^S$-atoms). Suppose $\hat{O}$ is a Gorenstein curve singularity with $G_m$-action and $G_m$-invariant compactification $(\hat{X};\{p_i\}_{i=1}^b)$. Let $\delta(p)$ be the $\delta$-invariant. For any subset $S \subset \{1, \ldots, b\}$ of the branches of $\hat{O}$ we define an $\hat{O}^S$-atom of genus $g$ to be any curve $C^S$ obtained by nodally attaching a smooth curve $C_0$ of genus $g - \delta(p) - |S| + 1$ to the points $\{p_i\}_{i \in S^c}$. (N.B. $S^c := \{1, \ldots, b\} \setminus S$)

In this construction, we think of $S$ as indexing the branches that ‘dangle,’ and we will say that a curve with an $\hat{O}$-singularity in which several of the branches are dangling rational components to have a dangling $\hat{O}$-singularity. We may naturally define a collection of modified $\alpha$-values associated to these dangling singularities.

**Definition 2.11.** The $\alpha$-invariant $\alpha(\hat{O}^S)$ of a Gorenstein singularity with $G_m$-action $\hat{O}$ and specified subset $S \subset [n]$ of the branches is defined to be the $\alpha$-invariant $\alpha(C, \eta)$ of any $(\hat{O}^S, \eta)$-atom $(C, \eta)$. 


Remark 2.12. An $\hat{O}^S$-atom $C^S$ of genus $g$ together with marked points $\{p_i\}_{i \in S}$ can be regarded as a point in $\mathcal{U}_{g,|S|}$, and we can define the corresponding characters $\chi_{\psi_i}(C^S; \{p_i\}_{i \in S})$; see Section 1.3 for the definition of $\psi_i$ and Corollary 3.6 for the computation of these characters.

In general, the invariants $\alpha^S(\hat{O})$ will depend on the subset $S$, which reflects the fact that curves $C^S$ may appear in the moduli stack $\mathcal{M}_g(\alpha)$ at different values of $\alpha$. The relationship between the characters of $\hat{O}$ and $\hat{O}^S$-atoms are explained in Corollaries 3.3 and 3.6. In Table 2, we list the $\alpha$-values for all dangling ADE singularities. Note that since the branches of any $A_k$ or toric planar singularity are isomorphic the only relevant feature of the subset $S \subset \{1, \ldots, n\}$ is the size. For $D_{2k+1}$ singularities, we use the labeling “1” for the singular branch and “2” for the smooth branch, and for $D_{2k+2}$-singularities, we use “1,2” for the tangent branches and “3” for the smooth branch with unique tangent direction. Similarly for the $E_7$ singularity, we use the labeling “1” for the singular branch and “2” for the smooth branch.

The final construction which we expect to play an important role in the log minimal model program is the atomic chain. While it is possible to define the atomic chain construction is much greater generality, we will focus on the simple case of $A_{2i+1/2j+1}$ and $A_{2i+1/2j}$-chains, as these are the only chains we expect to play a role for $\alpha \geq 5/9$.

Definition 2.13 ($A_{2i+1/2j+1}$, $A_{2i+1/2j}$-chains). We will say that a genus $g$ curve $C$ is an $A_{2i+1/2j+1}$-chain (resp. $A_{2i+1/2j}$-chain) if $C$ is of the form

$$C = C_0 \cup E_1 \cup E_2 \cup E_3$$

(resp. $C = C_0 \cup E_1 \cup E_2$),

where $C_0$ is a genus $g-i-j$ curve, each $E_k$ is a smooth rational curve, $E_1$ meets $C_0$ at a node, $E_2$ meets $E_1$ at an $A_{2i+1}$ singularity, and $E_3$ meets $E_2$ at an $A_{2j+1}$ singularity (resp. $E_2$ has a monomial $A_{2j}$-singularity); see Figure 7.

We define the associated $\alpha$-invariants in the standard way.

Definition 2.14. The $\alpha$-invariants $\alpha(A_{2i+1/2j+1})$ and $\alpha(A_{2i+1/2j})$ are defined to be the $\alpha$-invariants $\alpha(C, \eta)$ of an $A_{2i+1/2j+1}$-atom and $A_{2i+1/2j}$-atom respectively.
3. Character calculations

In this section, we compute the $\alpha$-invariants of all ADE, planar toric, and monomial unibranch Gorenstein singularities. We also compute the $\alpha$-invariant for the canonical ribbons with $\mathbb{G}_m$-action using results of Bayer and Eisenbud [BE95] to write down the spaces of pluricanonical differentials. These computations are summarized in Tables 1 and 2 in Section 4, where we use our results to give an outline of the log MMP for $\overline{\mathcal{M}}_g$ for $\alpha \geq 5/9$.

In Section 3.1, we explain how to compute the $\alpha$-invariant of a curve $C$ with $\mathbb{G}_m$-action using Rosenlicht differentials to write down diagonalized bases for the vector spaces $H^0(C, \omega^m_C)$. Under certain hypotheses on $C$, there is a somewhat more geometric method of computing $\alpha$-invariants. Recall from (1.2) that $K, \delta$ are formally defined by the equations

$$K := 2\lambda_2 - 13\lambda_1,$$
$$\delta := 13\lambda_1 - \lambda_2,$$

and that these classes restrict to the usual canonical class and boundary divisor on $\overline{\mathcal{M}}_g$. These functorial interpretations cannot be extended to all of $\mathcal{H}_g$, but they do extend under additional assumptions on the deformation space of $C$. In Section 3.2, we will see that if $C$ has a smooth deformation space, then the fiber $K|_{C^i}$ can be identified with $\det(T^1(C))^{\vee} \otimes \det(\text{Aut}^1(C))$, and we will use this identification to compute $\chi_K(C)$ for $A_i/A_j$-chains. Similarly, we will see that if the locus of singular deformations of $C$ is cut out formally locally by a single equation and generically parameterizes nodal curves, then $\chi_\delta(C)$ can be computed from the weight of this equation, and we use this to compute $\chi_\delta(C)$ for $A_i/A_j$-chains.

It is clear that any two of the characters $\chi_1(C, \eta), \chi_2(C, \eta), \chi_K(C, \eta), \chi_\delta(C, \eta)$ determine the others, and are therefore sufficient to determine the $\alpha$-invariant. Roughly speaking, the characters $\chi_i(C)$ are easier to compute for singularities described intrinsically, i.e. in terms of the subalgebra of functions that descend from the normalization. By contrast, $\chi_K(C, \eta), \chi_\delta(C, \eta)$ are easier to compute for singularities described by extrinsic equations, from which the deformation space and discriminant can be explicitly described.5

3.1. Computing characters $\chi_i$. In this section, we use Rosenlicht differentials to compute $\alpha$-invariants for several classes of Gorenstein singularities: We begin with the ADE singularities, which we encounter in the initial stages of the Hassett-Keel program. Next, we compute the $\alpha$-invariant for an arbitrary unibranch Gorenstein singularity, and then do an easy multi-branch example, the elliptic $m$-fold points which play a prominent role in the log MMP for $\overline{\mathcal{M}}_{1,n}$ [Smy11].

We begin by explaining how to algorithmically compute the induced characters $\chi_m(C, \eta) \in \mathbb{Z}$ in the cases when $C$ is an $\mathcal{O}^S$-atom.

3.1.1. The setup. We keep the notation of Section 2.3. Suppose $\mathcal{O}_{X, p}$ is a Gorenstein curve singularity with $\mathbb{G}_m$-action and $\mathbb{G}_m$-invariant compactification $(\tilde{X}; \{p_i\}_{i=1}^b)$. Recall that for $S \subset \{1, \ldots, b\}$, we define an $\mathcal{O}^S_{X, p}$-atom as the union $C^S = \tilde{X} \cup C_0$, where $\tilde{X}$ is nodally attached at $p_i$ to a point $q_i \in C_0$ for $i \in S^c$.

5A final point of interest: Using $\chi_K$ and $\chi_\delta$, one can define $\alpha$-invariants for non-Gorenstein curves, provided they have a smooth deformation space and the locus of worse-than-nodal deformations has codimension at least 2. This may be of interest for later stages in the log minimal model program.
Suppose \( X = \text{Spec} \, k[x_1, \ldots, x_n]/I \) and let \( \bar{X} = \text{Spec} \, k[t_1] \oplus \cdots \oplus k[t_b] \), where the normalization map \( \nu: \bar{X} \to X \) given by \( x_j \mapsto (f_{j1}(t_1), \ldots, f_{jn}(t_n)) \). In what follows, we also let \( s_i = 1/t_i \) to be the local coordinate around \( p_i \).

The \( \mathbb{G}_m \)-action on \( X \) extends to \( \bar{X} \) as \( \lambda \cdot t_i = \lambda^{-\alpha_i} t_i \) for \( \alpha_i \in \mathbb{N} \). The weight of a semi-invariant function (or differential) \( f \) with respect to this action will be called \( \eta \)-weight and denoted \( wt_\eta(f) \). Note that, \( \alpha_i \deg f_{ji} = wt_\eta(x_j) \) for all \( i, j \).

Since \( X \) is a Gorenstein curve with \( \mathbb{G}_m \)-action, we can find a \( \mathbb{G}_m \)-semi-invariant generator of \( \omega_{X,p} \), which we denote once and for all by \( \omega_0(X) \), or simply \( \omega_0 \). After rescaling \( t_i \), we have

\[
\omega_0 = \left( \frac{dt_1}{t_1^{m_1}}, \frac{dt_2}{t_2^{m_2}}, \ldots, \frac{dt_b}{t_b^{m_b}} \right),
\]

for some integers \( m_i \geq 2 \) satisfying \( (m_1 - 1)\alpha_1 = \cdots = (m_b - 1)\alpha_b \). We make the following observation:

**Lemma 3.1.** \( H^0(\bar{X}, \omega_{\bar{X}}) \) is generated by \( \mathbb{G}_m \)-semi-invariant differentials of the form \( f(x_1, \ldots, x_n)\omega_0 \), where \( f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n] \) range over all monomials satisfying \( wt_\eta(f\omega_0) > 0 \).

**Proof.** This is clear. For example, to see the last assertion, note that if \( f(x_1, \ldots, x_n)\omega_0 \) has local equation \( dt_i/t_i^a \) around \( 0 \in \mathcal{E}_i \), then its equation around \( p_i \) is \( -s_i^{a-2}ds_i \). It follows that \( f\omega_0 \) is regular at \( p_i \) if and only if \( a \geq 2 \) if and only if the \( \eta \)-weight of \( f(x_1, \ldots, x_n)\omega_0 \) is positive. \( \square \)

**Lemma 3.2.** There is a weight decomposition \( H^0(C^S, \omega_{C^S}) \cong H^0(\bar{X}, \omega_{\bar{X}}) \oplus W_0 \), where \( W_0 \) is the space of \( \eta \)-weight 0 differentials and \( H^0(\bar{X}, \omega_{\bar{X}}) \) are regular differentials on \( \bar{X} \) extended by zero to \( C_0 \).

**Proof.** By definition,

\[
H^0(C^S, \omega_{C^S}) = \{(w, u) \in H^0(\bar{X}, \omega_{\bar{X}}(\sum_{i \in S^e} p_i)) \oplus H^0(C_0, \omega_{C_0}(\sum_{i \in S^c} q_i)) : \text{Res}_{p_i} w + \text{Res}_{q_i} u = 0 \}.
\]

A global section of \( H^0(\bar{X}, \omega_{\bar{X}}(\sum_{i \in S^e} p_i)) \) is a Rosenlicht differential on \( X \) with at worst simple poles at \( \{p_i\}_{i \in S^e} \). By Lemma 3.1, \( \mathbb{G}_m \)-semi-invariant elements in \( H^0(\bar{X}, \omega_{\bar{X}}) \) have positive \( \eta \)-weight. On the other hand, any semi-invariant differential in \( H^0(\bar{X}, \omega_{\bar{X}}(\sum_{i \in S^e} p_i)) \) with a simple pole at some \( p_i \) has local equation \( ds_i/s_i = -dt_i/t_i \) and hence has \( \eta \)-weight zero. \( \square \)

**Corollary 3.3 (Computing \( \lambda_1 \)).** For any \( S \subset \{1, \ldots, b\} \), the character \( \lambda_1(C^S) \) is the sum of weights of the \( \mathbb{G}_m \)-action on the vector space \( H^0(\bar{X}, \omega_{\bar{X}}) \). In other words,

\[
\lambda_1(C^S) = \lambda_1(\bar{X}).
\]

\( \square \)

In the same vein, we can algorithmically compute \( \lambda_2(C^S) \).

**Lemma 3.4.** Suppose \( p_a(\bar{X}) \geq 2 \). Then there is a weight space decomposition

\[
H^0(C^S, \omega_{C^S}^2) \cong H^0(\bar{X}, \omega_{\bar{X}}^2) \bigoplus U_+ \bigoplus U_0,
\]

where \( U_0 \) is the space of \( \eta \)-weight 0 differentials and \( H^0(\bar{X}, \omega_{\bar{X}}^2) \) are regular quadratic differentials on \( \bar{X} \) extended by zero to \( C_0 \). Moreover, the weights of the \( \mathbb{G}_m \)-action on \( U_+ \) are precisely \( \{\alpha_i\}_{i \in S^c} \).
Remark 3.5. The Gorenstein singular curves \( \bar{X} \) with \( p_a(\bar{X}) = 1 \) are precisely elliptic \( m \)-fold points; see [Sny11]. The cases of \( m = 1 \) and \( m = 2 \) (the cusp and the tacnode) are considered in 3.1.2 and 3.1.3, and the case of \( m \geq 3 \) is considered separately in Section 3.1.8.

Proof of Lemma 3.4. We note that \( p_a(\bar{X}) \geq 2 \) implies that \( m_i \geq 3 \) for some \( i \in \{1, \ldots, b\} \).

Clearly, \( H^0(\bar{X}, \omega_{\bar{X}}^2) \) is a \( \mathbb{G}_m \)-invariant subspace of \( H^0(C^S, \omega_{C^S}^2) \), consisting of differentials whose local equation at \( 0 \in E_i \) is \( (dt_i)^2/t_i^2 \) for some \( a_i \geq 4 \). The quadratic differentials of zero weight are exactly those that have local equation \( c(dt_i)^2/t_i^2 \) around \( 0 \in E_i \) (with \( c \) possibly zero) for each \( i \in C^S \).

It follows that \( H^0(C^S, \omega_{C^S}^2)/H^0(\bar{X}, \omega_{\bar{X}}^2) \oplus U_0 \) is spanned by \( \mathbb{G}_m \)-semi-invariant differentials which have local equation \( dt_i/t_i^3 \) on some \( E_i \) with \( i \in S^e \). It remains to show that there are precisely \( |S^e| \) linearly independent quadratic differentials of this form and to compute their weight.

Consider first the case of \( b \geq 2 \). Observe that if \( E_i \) is a singular branch, then by Example 2.2, we have \( (\cdots, dt_i/t_i^3, \cdots) \in \omega_{X,p} \). We set \( u(i) := (\cdots, dt_i/t_i^2, \cdots) \omega(i, i + 1) = (\cdots, (dt_i)^2/t_i^3, \cdots) \in \omega_{x_p} \). If \( i \in S^e \), then \( u(i) \) extended by zero to \( C^S \) is a regular quadratic differential on all of \( C^S \).

Assume not all branches \( E_i \) are smooth and pairwise tangent. Consider a smooth branch \( E_i \). If \( m_i \geq 3 \), take a \( \mathbb{G}_m \)-semi-invariant function \( f \) on \( X \) whose local equation along \( E_i \) is exactly \( t_i \) and which vanishes to order \( 2 \) or higher along a branch \( E_{i'} \) for some \( i' \neq i \) (in particular, either \( \alpha_i > \alpha_{i'} \) or \( f \equiv 0 \) on \( E_{i'} \)). Then \( u(i) := f^{m_i-2} \cdot \omega dt \omega(i, i') = (\cdots, (dt_i)^2/t_i^3, \cdots) \omega(i, i') = (\cdots, (dt_i)^2/t_i^3, \cdots) \omega(i, i') \). Clearly, the constructed differentials \( \{u(i)\}_{i \in S^e} \subset H^0(C^S, \omega_{C^S}^2) \) are linearly independent and have weights \( \{\alpha_i\}_{i \in S^e} \).

Suppose all branches \( p \in X \) are smooth and pairwise tangent. Then modulo \( (t_i^2) \oplus \cdots \oplus (t_b^2) \) the ring of regular functions on \( X \) is generated by \( f = (t_1, \ldots, t_b) \). In particular, \( \alpha_1 = \cdots = \alpha_b \), and so \( m_1 = \cdots = m_b \geq 3 \). One sees at once that \( f^{m_i-2} \cdot \omega dt \omega(i, i') = (\cdots, (dt_i)^2/t_i^3, \cdots) \omega(i, i') = (\cdots, (dt_i)^2/t_i^3, \cdots) \omega(i, i') \). If \( i' \neq i \in S^e \), and \( f \cdot (\cdots, dt_i/t_i^2, \cdots) = (\cdots, (dt_i)^2/t_i^3, \cdots) \), for \( i' \neq i \in S^e \), span \( U_+ \). Therefore \( U_+ \) has dimension \( |S^e| \) and the claim follows.

Finally, if \( b = 1 \), then \( \omega_0 = dt/t^{m+1} \), where \( m \) is the largest gap of the semigroup of vanishing orders in \( \hat{O}_{X,p} \). Since \( m \geq 3 \), \( t^{2m-1} \omega_0 \in \omega_{X,p} \). We conclude that \( (dt_i)/t^3 = t^{2m-1} \omega_0 \in \omega_{X,p} \).

If \( S = \emptyset \), then \( (dt)^2/t^3 \) spans \( U_+ \) and has weight \( \alpha_1 \). The proof is finished.

Corollary 3.6 (Computing \( \lambda_2 \)). Suppose \( p_a(\bar{X}) \geq 2 \). For any \( S \subset \{1, \ldots, b\} \), the character \( \lambda_2(C^S) \) is the sum of the weights of the \( \mathbb{G}_m \)-action on the vector space \( H^0(\bar{X}, \omega_{\bar{X}}^2) \) and \( \sum_{i \in S^e} \alpha_i \).

The character of the \( |S| \)-pointed curve \( (C^S; \{p_i\}_{i \in S}) \) computed with respect to the cotangent line bundle \( \psi_i \) is

\[
\chi_{\psi_i}(C^S; \{p_i\}_{i \in S}) = \alpha_i.
\]

In particular,

\[
\lambda_2(C^S) = \lambda_2(\bar{X}) + \sum_{i \in S^e} \alpha_i = \lambda_2(C) - \sum_{i \in S} \chi_{\psi_i}(C^S; \{p_i\}_{i \in S}).
\]

Proof. The first assertion follows directly from Lemma 3.2. The second assertion follows from the observation that the fiber of \( \psi_i \) at \( (C^S; \{p_i\}_{i \in S}) \) is \( ds_i = -dt_i/t_i^2 \) and so \( \mathbb{G}_m \) acts on it with weight \( \alpha_i \).
3.1.2. $A_{2k}$-singularity: $y^2 = x^{2k+1}$. Let $C = C_0 \cup \bar{X}$, where $X \simeq \text{Spec} \, k[x, y]/(y^2 - x^{2k+1})$, be an $A_{2k+1}$-atom. The $G_m$-action on $X$ is $\eta: \lambda \cdot (x, y) = (\lambda^{-1}x, \lambda^{-(2k+1)}y)$. The normalization map is given by $(x, y) \mapsto (t^2, t^{2k+1})$ and the $G_m$-action extends as $t \mapsto t^{1-1}t$.

It is easy to see that $\omega_0 := dt/t^{2k}$ is a $G_m$-semi-invariant generator for $\omega_X$ at the cusp, of $\eta$-weight $2k - 1$. By Lemma 3.1, a basis of $H^0(\bar{X}, \omega_{\bar{X}})$ diagonalizing the $G_m$-action is $\langle x^i w_0^i \mid 0 \leq i \leq k - 1 \rangle$. The $\eta$-weights of this basis are $(2k - 1), (2k - 3), \ldots, 1$. Thus by Corollary 3.3, the character $\chi_1$ is

$$\chi_1(A_{2k}) = \sum_{i=1}^{k} (2i - 1) = k^2.$$ 

Similarly, if $k \geq 2$, a basis of $H^0(\bar{X}, \omega_{\bar{X}})^2$ diagonalizing the $G_m$-action is $\langle x^i \omega_0^i, x^j y w_0^j \mid 0 \leq i \leq 2k - 2, 0 \leq j \leq k - 3 \rangle$. Thus, by Corollary 3.6 the character $\chi_2$ is given by

$$\chi_2(A_{2k}) = \sum_{i=0}^{2k-2} (4k - 2 - 2i) + \sum_{j=0}^{k-3} (2k - 3 - 2j) + 1 = 5k^2 - 4k + 1.$$ 

If $k = 1$, then the only weight space of $H^0(C, \omega_C^2)$ with non-zero weight is spanned by $(dt)^2/t^4$, hence $\chi_2(A_2) = 2$.

3.1.3. $A_{2k+1}$-singularity: $y^2 = x^{2k+2}$. Next, we consider an $A_{2k+2}$-atom $C = C_0 \cup \bar{X}$, where $X \simeq \text{Spec} \, k[x, y]/(y^2 - x^{2k+2})$. The $G_m$-action on $X$ is $\eta: \lambda \cdot (x, y) = (\lambda^{-1}x, \lambda^{-(k+1)}y)$. The normalization map is $(x, y) \mapsto ((t_1, t_2), (t_1^{k+1}, -t_2^{k+1}))$, then the $G_m$-action extends as $\lambda \cdot (t_1, t_2) = (\lambda^{1-t_1}, \lambda^{1-t_2})$. It is easy to see that $\omega_0 := (dt_1/t_1^{k+1}, -dt_2/t_2^{k+1})$ is a $G_m$-semi-invariant generator for $\omega_X$ at the tacnode, of $\eta$-weight $k$. If $k \geq 2$, we may write down bases of $H^0(\bar{X}, \omega_{\bar{X}})$ and $H^0(\bar{X}, \omega_{\bar{X}}^2)$ diagonalizing the $G_m$-action as

$${H^0}(\bar{X}, \omega_{\bar{X}}) = \langle x^i \omega_0^i \mid 0 \leq i \leq k - 1 \rangle,$$

$${H^0}(\bar{X}, \omega_{\bar{X}}^2) = \langle x^i \omega_0^i, x^j y w_0^j \mid 0 \leq i \leq 2k - 2, 0 \leq j \leq k - 3 \rangle.$$ 

It follows by Corollaries 3.3 and 3.6 that

$$\chi_1(A_{2k+1}) = \sum_{i=0}^{k-1} (k - i) = \frac{k^2 + k}{2},$$

$$\chi_2(A_{2k+1}) = \sum_{i=0}^{2k-2} (2k - i) + \sum_{j=0}^{k-3} (k - 1 - j) + (1 + 1) = \frac{5k^2 + k}{2}.$$ 

(N.B. If $k = 1$, we need to modify the computation of $\chi_2$. Namely, we observe that the only weight spaces of $H^0(C, \omega_C^2)$ with non-zero weights are spanned by $((dt_1)^2/t_1^3, (dt_2)^2/t_2^3)$ and $((dt_1)^2/t_1^3, (dt_2)^2/t_2^3)$, hence $\chi_2(A_3) = 3$.)

3.1.4. $D_{2k+1}$-singularity: $x(y^2 - x^{2k-1}) = 0$. Here, $X \simeq \text{Spec} \, k[x, y]/x(y^2 - x^{2k-1})$ and the normalization map is given by $(x, y) \mapsto ((t_1^2, 0), (t_1^{2k-1}, t_2))$ with the $G_m$-action on the normalization being $\lambda \cdot (t_1, t_2) = (\lambda^{1-t_1}, \lambda^{-(2k-1)}t_2)$. A $G_m$-semi-invariant generator for $\omega_X$ is $\omega_0 := (dt_1/t_1^{2k-1}, -dt_2/t_2^{2k-1})$, of $\eta$-weight $2k - 1$. It follows that the bases of $H^0(\bar{X}, \omega_{\bar{X}})$ and $H^0(\bar{X}, \omega_{\bar{X}}^2)$ diagonalizing the $G_m$-action are

$${H^0}(\bar{X}, \omega_{\bar{X}}) = \langle x^i \omega_0^i \mid 0 \leq i \leq k - 1 \rangle,$$

$${H^0}(\bar{X}, \omega_{\bar{X}}^2) = \langle x^i \omega_0^i, x^j y w_0^j \mid 0 \leq i \leq 2k - 2, 1 \leq j \leq k - 2 \rangle.$$
Therefore, by Corollaries 3.3 and 3.6
\[
\chi_1(D_{2k+1}) = \sum_{i=0}^{k-1} (2k - 1 - 2i) = k^2,
\]
\[
\chi_2(D_{2k+1}) = \sum_{i=0}^{2k-2} (4k - 2 - 2i) + \sum_{j=1}^{k-2} (2k - 1 - 2j) + (1 + 2k - 1) = 5k^2 - 2k.
\]

3.1.5. \(D_{2k+2}\)-singularity: \(x(y^2 - x^{2k}) = 0\). Let \(X \simeq k[x,y]/x(y^2 - x^{2k})\). The normalization map is given by \((x,y) \mapsto (t_1, t_2, 0, (t_1^k, -t_2, t_3))\), with the \(\mathbb{G}_m\)-action being \(\lambda \cdot (t_1, t_2, t_3) = (\lambda^{-1}t_1, \lambda^{-1}t_2, \lambda^{-k}t_3)\). The generator for \(H^0(\bar{X}, \omega_{\bar{X}})\) is \(\omega_0 = (dt_1/t_1^{k+1}, -dt_2/t_2^{k+1}, -2dt_3/t_3^2)\), of \(\eta\)-weight \(k\). The bases of \(H^0(\bar{X}, \omega_{\bar{X}})\) and \(H^0(\bar{X}, \omega_{\bar{X}}^2)\) diagonalizing the \(\mathbb{G}_m\)-action are
\[
H^0(\bar{X}, \omega_{\bar{X}}) = \langle x^i\omega_0 \mid 0 \leq i \leq k - 1 \rangle,
\]
\[
H^0(\bar{X}, \omega_{\bar{X}}^2) = \langle x^i w_0^2, x^j yw_0^2 \mid 0 \leq i \leq 2k - 2, 1 \leq j \leq k - 2 \rangle.
\]

Therefore, by Corollaries 3.3 and 3.6
\[
\chi_1(D_{2k+2}) = \sum_{i=0}^{k-1} (k - i) = \frac{k^2 + k}{2},
\]
\[
\chi_2(D_{2k+2}) = \sum_{i=0}^{2k-2} (2k - 2i) + \sum_{j=1}^{k-2} (k - j) + (1 + 1 + k) = \frac{5k^2 + 3k}{2}.
\]

3.1.6. Exceptional simple singularities: \(E_6, E_7, E_8\). To complete a list of \(\chi_i\) characters of all ADE singularities, it remains to consider the cases of \(E_6, E_7, E_8\) singularities. Among these, \(E_6 : y^3 - x^4 = 0\) and \(E_8 : y^3 - x^5 = 0\) are unibranch. We compute the characters of all unibranch curve singularities with \(\mathbb{G}_m\)-action in Section 3.1.7, but we list here the characters of \(E_6\) and \(E_8\)-atoms for the reader’s convenience:

\[(3.3)\]
\[
\begin{align*}
\chi_1(E_6) &= 8, \\
\chi_2(E_6) &= 33.
\end{align*}
\]

\[(3.4)\]
\[
\begin{align*}
\chi_1(E_8) &= 14, \\
\chi_2(E_8) &= 63.
\end{align*}
\]

Let \(C = C_0 \cup \bar{X}\) be an \(E_7\)-atom, where \(X \simeq \text{Spec} \ k[x,y]/y(g^2 - x^3)\). The \(\mathbb{G}_m\)-action on \(X\) is \(\eta : \lambda \cdot (x,y) = (\lambda^{-x}x, \lambda^{-3}y)\). The normalization map is given by \((x,y) \mapsto ((t_1^2, t_2), (t_1^2, 0))\) and the \(\mathbb{G}_m\)-action extends as \(\lambda \cdot (t_1, t_2) = (\lambda^{-1}t_1, \lambda^{-2}t_2)\). The generator of \(H^0(\bar{X}, \omega_{\bar{X}})\) is \(\omega_0 = (dt_1/t_1^2, -dt_2/t_2^2)\), of \(\eta\)-weight 4. The bases of \(H^0(\bar{X}, \omega_{\bar{X}})\) and \(H^0(\bar{X}, \omega_{\bar{X}}^2)\) diagonalizing the \(\mathbb{G}_m\)-action are
\[
H^0(\bar{X}, \omega_{\bar{X}}) = \langle \omega_0, x\omega_0, y\omega_0 \rangle,
\]
\[
H^0(\bar{X}, \omega_{\bar{X}}^2) = \langle \omega_0^2, x\omega_0^2, x^2\omega_0^2, y\omega_0^2, y^2\omega_0^2 \rangle
\]

It follows that
\[
\chi_1(E_7) = 4 + 2 + 1 = 7,
\]
\[
\chi_2(E_7) = (8 + 6 + 4 + 5 + 3 + 2) + 1 + 2 = 31.
\]
3.1.7. Monomial unibranch singularities. Suppose \( N \setminus \{b_1, \ldots, b_k\} \) is a semigroup. Let \( \bar{X} \) be the \( \mathbb{G}_m \)-invariant compactification of \( X \simeq \text{Spec} \, k[t^n : n \geq 0, n \notin \{b_1, \ldots, b_k\}] \), with \( \mathbb{G}_m \)-action given by \( \eta : \lambda \cdot t = \lambda^{-1}t \). Clearly, \( p_\eta(\bar{X}) = k \), the normalization of \( \bar{X} \) is \( \mathbb{P}^1 \), and \( \bar{X} \) has an isolated monomial unibranch singularity at \( t = 0 \) with the gap sequence \( \{b_1, \ldots, b_k\} \). From now on we assume that \( \bar{X} \) is Gorenstein, which is equivalent to the gap sequence \( \{b_1 = 1, \ldots, b_k\} \) being symmetric:

\[
 n \in \{b_1, \ldots, b_k\} \iff 2k - 1 - n \notin \{b_1, \ldots, b_k\}.
\]

In particular, \( b_k = 2k - 1 \). Evidently, a \( \mathbb{G}_m \)-semi-invariant generator for \( \omega_{\bar{X}} \) in a neighborhood of zero is given by \( dt/t^{b_k + 1} \). Therefore, we can write down the bases of \( H^0(\bar{X}, \omega_{\bar{X}}) \) and \( H^0(\bar{X}, \omega_{\bar{X}}^2) \) diagonalizing the \( \mathbb{G}_m \)-actions as

\[
 H^0(\bar{X}, \omega_{\bar{X}}) = \langle \frac{dt}{t^{b_1+1}}, \frac{dt}{t^{b_2+1}}, \ldots, \frac{dt}{t^{b_k+1}} \rangle,
\]

\[
 H^0(\bar{X}, \omega_{\bar{X}}^2) = \langle \frac{(dt)^2}{t^{2b_k+2-j}} : j \in \{0, \ldots, 2b_k - 2\} \setminus \{b_1, \ldots, b_k\} \rangle.
\]

From this, we compute

\[
 \chi_1(\bar{X}) = \sum_{i=1}^{k} b_i, \tag{3.5}
\]

\[
 \chi_2(\bar{X}) = \sum_{j=0}^{2b_k-2} (2b_k - j) - \sum_{i=1}^{k} (2b_k - b_i) = (2k - 1)^2 + \sum_{i=1}^{k} b_i - 1. \tag{3.6}
\]

If \( C = C_0 \cup \bar{X} \) is an \( \hat{O}_{X,p} \)-atom, then by Corollaries 3.3 and 3.6, we have

\[
 \chi_1(C) = \sum_{i=1}^{k} b_i, \tag{3.7}
\]

\[
 \chi_2(C) = (2k - 1)^2 + \sum_{i=1}^{k} b_i.
\]

For the toric planar singularity \( x^p = y^q \) with \( p \) and \( q \) coprime, the local ring of the singularity is \( k[p^{i+j} : i, j \in \mathbb{Z}_{\geq 0}] \). The gap sequence \( \{b_1, \ldots, b_k\} \) is the set of positive integers that cannot be expressed as \( pi + jq \) with \( i, j \geq 0 \). The study of this sequence, e.g. finding its cardinality and the largest element is classically known in elementary number theory as the Frobenius problem [RA05]. It is well-known that the largest gap is \( b_k = pq - p - q \). It is also easy to see that the gap sequence is symmetric: \( n \) is a gap if and only if \( pq - p - q - n \) is not a gap. It follows that the genus of the singularity \( x^p = y^q \) is \( k = (p-1)(q-1)/2 \). By [BS93] (see also [Rôd94]), the sum of gaps is

\[
 \sum_{i=1}^{k} b_i = (p-1)(q-1)(2pq - p - q - 1)/12. \tag{3.8}
\]

It follows from Equations (3.5)-(3.6) that

\[
 \chi_1(\bar{X}) = \frac{1}{12}(p-1)(q-1)(2pq - p - q - 1), \tag{3.9}
\]

\[
 \chi_2(\bar{X}) = (pq - p - q)^2 + \frac{1}{12}(p-1)(q-1)(2pq - p - q - 1) - 1.
\]
Remarkably, intersection theory calculations of Proposition 5.3 give an independent algebro-geometric proof of the highly nontrivial combinatorial Formula (3.8); see Corollary 5.4 in Section 5 below.

3.1.8. Elliptic m-fold points. Let \( m \geq 3 \). An elliptic \( m \)-fold point \( X \) is a Gorenstein union of \( m \) general lines through a point in \( \mathbb{A}^{m-1} \) [Smy11]. Every such singularity is isomorphic to the cone over points \( p_1 = (1,0,\ldots,0), \ p_2 = (0,1,\ldots,0), \ldots, \ p_{m-1} = (0,0,\ldots,1), \) and \( p_m = (1,\ldots,1) \), with the vertex at \( p = (0,\ldots,0) \in \mathbb{A}^{m-1} \). If \( (x_1,\ldots,x_{m-1}) \) are coordinates centered at the vertex then the normalization map from \( m \) copies of \( \mathbb{A}^1 \) to \( X \) is given by

\[
\begin{pmatrix}
    x_1 \\
    \vdots \\
    x_{m-1}
\end{pmatrix} \rightarrow
\begin{pmatrix}
    t_1 & 0 & \cdots & 0 & t_m \\
    0 & t_2 & \cdots & t_m \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & \cdots & 0 & t_m \\
    0 & \cdots & \cdots & 0 & t_{m-1} - t_m
\end{pmatrix}.
\]

The \( \mathbb{G}_m \)-action \( \eta \) is defined by \( \lambda \cdot (t_1, \ldots, t_m) = (\lambda^{-1}t_1, \ldots, \lambda^{-1}t_m) \). We let \( C \) be the singular curve obtained by attaching \( \bar{X} \) to a smooth curve \( C_0 \) nodally at points \( p_1, \ldots, p_m \). We note that \( H^0(X, \omega_X) \) is one-dimensional and is generated by

\[
\omega_0 = \left( \frac{dt_1}{t_1^2}, \ldots, \frac{dt_{m-1}}{t_{m-1}^2}, \frac{-dt_m}{t_m^2} \right).
\]

Thus, \( \chi_1(C, \eta) = 1 \) by Corollary 3.6. A generator for \( \omega_X^2 \) in the neighborhood of the \( m \)-fold point is

\[
\omega_0^2 = \left( \frac{(dt_1)^2}{t_1^4}, \ldots, \frac{(dt_{m-1})^2}{t_{m-1}^4}, \frac{(dt_m)^2}{t_m^4} \right),
\]

and the only weight spaces of \( H^0(C, \omega_C^2 \eta^2) \) with non-zero \( \eta \)-weights are spanned by

\[
(\omega_0^2, 0), \ (x_1\omega_0^2, 0), \ldots, \ (x_{m-1}\omega_0^2, 0).
\]

It follows that \( \chi_2(C) = 2 + (m-1) = m+1 \). Thus, the \( \alpha \)-invariant of the elliptic \( m \)-fold atom is \( \alpha = \frac{11-2m}{12-m} \).

3.2. Computing \( \chi_K \) and \( \chi_\delta \). In this section, we explain how to compute the characters \( \chi_K(C, \eta) \) and \( \chi_\delta(C, \eta) \) for a curve \( C \) with \( \mathbb{G}_m \)-action \( \eta : \mathbb{G}_m \rightarrow \text{Aut}(C) \) by studying the induced action on the deformation space of \( C \). Throughout this section, we make the following additional assumptions:

(A1) The miniversal deformation space of \( C \) is smooth.
(A2) The locus of worse-than-nodal deformations of \( C \) has codimension at least 2.
(A3) The locus \( \Delta \subset \mathcal{U}_\eta \) of singular deformations of \( C \) is Cartier in a neighborhood of \([C] \).

First, we explain how to compute the character of \( \chi_K(C, \eta) \) by observing that the smoothness hypothesis gives a functorial interpretation to the canonical divisor. Indeed, since \( \mathcal{U}_\eta \) is a smooth stack in a neighborhood of \([C] \), it admits admits a canonical line bundle \( K_{\mathcal{U}_\eta} \), defined as det \( L_{\mathcal{U}_\eta} \), where \( L_{\mathcal{U}_\eta} \) is the cotangent complex of \( \mathcal{U}_\eta \) [Ols07]. Using this fact, it is easy to see that \( \chi_{K_{\mathcal{U}_\eta}}(C, \eta) \) can be computed as the determinant of the action of \( \eta \) on \( T^1(C)^{\vee} \), the dual of the space of first-order deformations.

**Lemma 3.7.** Let \((C, \eta)\) be a curve with \( \mathbb{G}_m \)-action satisfying (A1). Then there is a canonical identification

\[
K_{\mathcal{U}_\eta}[C] = \det(T^1(C))^{\vee} \otimes \det(\mathfrak{g}),
\]

where \( \mathfrak{g} \) is the Lie algebra of \( \mathbb{G}_m \).
where $g$ is the adjoint representation of $\text{Aut}(C)$. In particular, if $\text{Aut}(C)^0$ is abelian, then $\chi_{K_{U_g}}(C, \eta)$ is simply the weight of the action of $\eta$ on $\text{det}(T^1(C))^\vee$.

Proof. Choose a miniversal deformation space $p: M \to U_g$ around $[C]$ with $M$ smooth; that is, $M \to U_g$ is a smooth morphism and there is a point $m \in M$ above $[C]$ inducing an isomorphism $T_{M,m} \to T^1(C)$ of tangent spaces. Moreover, we may assume that there is a cartesian diagram

$$\begin{array}{ccc}
\text{Spec } k & \xrightarrow{\nu} & M \\
\downarrow & & \downarrow p \\
B \text{Aut}(C) & \xrightarrow{i} & U_g
\end{array}$$

where $B \text{Aut}(C) \to U_g$ is the residual gerbe associated to $[C]$. The cotangent complex $L_{M/U_g}$ is supported in degree 0 by a locally free sheaf which we denote by $\Omega_{M/U_g}$. There is an exact triangle

$$p^*L_{U_g} \to L_M \to L_{M/U_g} \to p^*L_{U_g}[1]$$

inducing an exact sequence

$$0 \to H^0(p^*L_{U_g}) \to \Omega_M \to \Omega_{M/U_g} \to H^1(p^*L_{U_g}) \to 0$$

of quasi-coherent sheaves on $M$. It follows that $p^*K_{U_g} = K_M \otimes (\text{det } \Omega_{M/U_g})^\vee$. The restriction $i^*\Omega_{M/U_g} = \Omega_{\text{Spec } k/B \text{Aut}(C)}$ is canonically identified with $g^\vee$. Therefore, $i^*K_{U_g}$ corresponds to the natural action of $\text{Aut}(C)$ on $\text{det}(T^1(C))^\vee \otimes \text{det}(g)$.

Corollary 3.8. Suppose that $C \in U_g$ is a curve with $\mathbb{G}_m$-action $\eta: \mathbb{G}_m \to \text{Aut}(C)$ satisfying Assumption (A1)-(A2). Then $\chi_K(C, \eta)$ is the weight of the action of $\eta$ on $\text{det}(T^1(C))^\vee$.

Proof. Since $K_{U_g}$ and $K$ are line bundles defined in a neighborhood of $[C] \in U_g$ that agree away from the codimension 2 locus by Remark 1.1, we have $K \simeq K_{U_g}$. The claim now follows from Lemma 3.7.

Next, we explain how to compute the character $\chi_{\delta}(C, \eta)$. The key observation is that the character $\chi_{\mathcal{O}(D)}(C, \eta)$ corresponding to an effective $\mathbb{G}_m$-invariant Cartier divisor $D$ in the miniversal deformation space of $C$ can be read off from the weight of the defining equation of $D$.

Lemma 3.9. Let $C$ be a curve in with $\mathbb{G}_m$-action $\eta: \mathbb{G}_m \to \text{Aut}(C)$. Take $\text{Spf}(A)$ to be a formal miniversal deformation space of $C$. Note that $\eta$ induces a $\mathbb{G}_m$-action on $\text{Spf}(A)$. Suppose $D$ is a Cartier divisor defined on an étale neighborhood of $[C] \in U_g$ and that $D\mid_{\text{Spf } A} = \{ f = 0 \}$, where $f \in A$. Then

$$\chi_{\mathcal{O}(D)}(C, \eta) = -d,$$

where $f \mapsto \lambda^d f$ under the induced action of $\mathbb{G}_m = \text{Spec } k[\lambda, \lambda^{-1}]$ on $\text{Spf } A$.

Proof. Denote by $\sigma: A \to k[\lambda, \lambda^{-1}] \otimes A$ the dual action of $\mathbb{G}_m$ on $\text{Spf } A$. The exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_{U_g} \to \mathcal{O}_D \to 0$$

restricted to $\text{Spf } A$ corresponds to the exact sequence

$$0 \to A_\eta \xrightarrow{f} A \to A/f \to 0$$

where $A_\eta$ is the $\mathbb{G}_m$-$A$-module corresponding to the character $\mathbb{G}_m \xrightarrow{d} \mathbb{G}_m$; that is, $A_\eta$ is $A$ as an $A$-module with coaction $a \mapsto \lambda^d \sigma(a)$. Therefore $\mathcal{O}(-D)|_{\text{Spec } \mathbb{G}_m}$ corresponds to the character $\mathbb{G}_m \xrightarrow{d} \mathbb{G}_m$ and $\chi_{\mathcal{O}(-D)}(C, \eta) = d$. 

□
**Corollary 3.10.** Suppose that $C \in U_g$ is a curve with $\mathbb{G}_m$-action $\eta : \mathbb{G}_m \to \text{Aut}(C)$ satisfying Assumptions (A1)-(A3). Then $\Delta$ is formally locally cut out by a $\mathbb{G}_m$-semi-invariant of weight $d$ and $\chi(C) = -d$.

**Proof.** The locus $\Delta$ of singular deformations is clearly a $\mathbb{G}_m$-invariant divisor. Since $\Delta$ is Cartier by Assumption (A3), it is cut out by a $\mathbb{G}_m$-semi-invariant of weight $d$. Finally, $\mathcal{O}_{U_g}(\Delta)|_{\mathfrak{m} \mathfrak{m}} \simeq \delta_{|\mathfrak{m} \mathfrak{m}}$ by Remark 1.1. Since the line bundles $\mathcal{O}_{U_g}(\Delta)$ and $\delta$ are defined in a neighborhood of $[C]$, this isomorphism extend to a neighborhood of $[C] \in U_g$ and the claim follows. □

In what follows, we use Corollaries 3.8 and 3.10 to present an alternative computation of the characters of $A_1$-atoms and to compute the characters of $A_i/A_j$-chains.

3.2.1. $A_{2k}$-singularity. For $k \geq 2$, let $C = C_0 \cup \bar{X}$ be an $A_{2k}$-atom as in 3.1.2, with $p \in X$ being the $A_{2k}$-singularity. Let $q = C_0 \cap \bar{X}$ be the node. The space of first-order deformations $T^1(C)$ can be decomposed as:

$$T^1(C) = T^1(C_0, q) \times T^1(\hat{O}_{C,p}) \times T^1(\hat{O}_{C,q}) \times \text{Cr}^1(\hat{O}_{C,p}),$$

where $T^1(C_0, q)$ are the first-order deformations of the pointed curve $(C_0, q)$, $T^1(\hat{O}_{C,p})$ and $T^1(\hat{O}_{C,q})$ are the first-order deformations of the $A_{2k}$-singularity and the node respectively, and $\text{Cr}^1(\hat{O}_{C,p})$ are the first-order “crimping” deformations of $C$ which preserve the $A_{2k}$-singularity at $p$ and the normalization of $C$ at $p$; see [vdW10] for a detailed treatment of crimping deformations. Choose coordinates $a_0, \ldots, a_{2k-1}$ on $T^1(\hat{O}_{C,p})$ and $n$ on $T^1(\hat{O}_{C,q})$ corresponding to the miniversal deformations

$$y^2 = x^{2k+1} + a_{2k-1}x^{2k-1} + \cdots + a_1x + a_0,$$

$$zw = n,$$

of the $A_{2k}$-singularity and the node. Here, we choose $z$ to be the uniformizer of $\bar{X}$ at $q$. The action of $\mathbb{G}_m$ is given by

$$\lambda \cdot (x, y, a_i) = (\lambda^{-2}x, \lambda^{-(2k+1)}y, \lambda^{-(4k+2-2i)}a_i),$$

$$\lambda \cdot (z, w, n) = (\lambda z, w, \lambda n).$$

Therefore, the induced $\mathbb{G}_m$-action on $\det(T^1(\hat{O}_{C,p}))$ has weight $\sum_{i=0}^{2k-1}(4k+2-2i) = (4k^2+6k)$, while the induced action on $T^1(\hat{O}_{C,q})$ has weight $-1$. (N.B. The tangent space to the miniversal deformation $\hat{O}_{C,p}$ is $T^1(\hat{O}_{C,q}) = ((a_0, \ldots, a_{2k-1})/(a_0, \ldots, a_{2k-1})^2)^{\vee}$, hence the reversal of signs).

The $\mathbb{G}_m$-action on the normalization of $X$ is $\lambda \cdot t = \lambda^{-1}t$. By [vdW10, Example 1.111], the weights of the induced $\mathbb{G}_m$-action on $\text{Cr}^1(\hat{O}_{C,p})$ are $-1, -3, \ldots, -(2k-3)$. We note that this can be seen from the fact that we may choose coordinates $c_1, \ldots, c_k$ on $\text{Cr}^1(\hat{O}_{C,p})$ corresponding to the crimping deformation of the $A_{2k}$-singularity as given by

$$(t^2 + c_1t^3 + c_2t^5 + \cdots + c_kt^{2k-1}, t^{2k+1}) \in k[t].$$

Thus, the induced character on $\det(\text{Cr}^1(\hat{O}_{C,p}))$ is $-\sum_{i=1}^{k-1}(2i-1) = -(k-1)^2$. Finally, since the $\mathbb{G}_m$-action on $T^1(C_0, q)$ is trivial, we conclude by Corollary 3.8 that

$$\chi(C_0) = -(4k^2 + 6k) - (k-1)^2 = -3k^2 - 8k + 2.$$

Let $\Delta(a_0, \ldots, a_{2k-1})$ denote the discriminant of the polynomial $x^{2k+1} + a_{2k-1}x^{2k-1} + \cdots + a_1x + a_0$. Then the locus of singular deformations of $C$ is cut out formally locally by the polynomial $n\Delta(a_0, \ldots, a_{2k-1})$. Since, $\Delta(a_0, \ldots, a_{2k-1})$ is a quasi-homogeneous polynomial of
\(\eta\)-weight \(-4k(2k + 1)\), we see that \(G_m\) acts on \(n\Delta(a_0, \ldots, a_{2k-1})\) with weight \(1 - 4k(2k + 1)\). Thus, by Corollary 3.10, we have

\[
(3.11) \quad \chi_\delta(A_{2k}) = 8k^2 + 4k - 1.
\]

Remark 3.11. Note that we have \(\chi_\delta(A_{2k}) = 13\chi_1(A_{2k}) - 2\chi_3(A_{2k})\) (Table 1), as guaranteed by Equation (1.2).

3.2.2. \(A_{2k+1}\)-singularity. Let \(C = C_0 \cup \tilde{X}\) be an \(A_{2k+1}\)-atom as in 3.1.3, with \(p \in X\) being the \(A_{2k+1}\)-singularity. Let \(\{q_1, q_2\} = C_0 \cap \tilde{X}\) be the nodes. The space of first-order deformations \(T^1(C)\) can be decomposed as:

\[
T^1(C) = T^1(C_0, q_1, q_2) \times \Cr^1(\tilde{O}_{C, p}) \times T^1(\tilde{O}_{C, p}) \times T^1(\tilde{O}_{C, q_1}) \times T^1(\tilde{O}_{C, q_2}),
\]

where \(\Cr^1(\tilde{O}_{C, p})\) are the first-order deformations crimping deformations of \(C\) which preserve the \(A_{2k+1}\)-singularity at \(p\) and the normalization of \(C\) at \(p\); see [vdW10]. Choose coordinates \(a_0, \ldots, a_{2k}\) on \(T^1(\tilde{O}_{C, p})\) and \(n_j\) on \(T^1(\tilde{O}_{C, q_j})\) corresponding to the miniversal deformations

\[
y^2 = x^{2k+2} + a_{2k}x^{2k} + \cdots + a_1x + a_0, \\
z_1w = n_1, \\
z_2w = n_2.
\]

of the \(A_{2k+1}\)-singularity and the nodes. Here, we choose \(z_j\) to be the uniformizer of \(q_j \in \tilde{X}\).

The action of \(G_m\) is given by

\[
\lambda \cdot (x, y, a_i) = (\lambda^{-2}x, \lambda^{-(k+1)}y, \lambda^{-(2k+2-i)}a_i), \\
\lambda \cdot (z, w, n_j) = (\lambda z, w, \lambda n_j).
\]

Therefore, the induced \(G_m\)-action on \(\det(T^1(\tilde{O}_{C, p}))\) has weight \(\sum_{i=0}^{2k}(2k+2-i) = (2k^2+5k+2)\), while the induced action on \(T^1(\tilde{O}_{C, q_j})\) has weights \(-1\). The \(G_m\)-action on \(T^1(C_0, q)\) is trivial.

The \(G_m\)-action on the normalization of \(X\) is \(\lambda \cdot (t_1, t_2) = (\lambda^{-1}t_1, \lambda^{-1}t_2)\). By [vdW10, Example 1.112], the weights of the induced \(G_m\)-action on \(\Cr^1(\tilde{O}_{C, p})\) are \(-1, -2, \ldots, -(k-1)\). This can be seen from the fact that we may choose coordinates \(c_1, \ldots, c_{k-1}\) on \(\Cr^1(\tilde{O}_{C, p})\) corresponding to the crimping deformation of the \(A_{2k+1}\)-singularity as given by

\[
k[(t_1, t_2 + c_1t_2^2 + c_2t_2^3 + \cdots + c_{k-1}t_2^{k-1}), (t_1^{k+1}, t_2^{k+1})] \subset k[t_1] \oplus k[t_2].
\]

Therefore, the induced character on \(\det(\Cr^1(\tilde{O}_{C, p}))\) is \(-\sum_{i=1}^{k-1} i = -(k-1)k/2\). By Corollary 3.8, we conclude

\[
(3.12) \quad \chi_\delta(A_{2k+1}) = -((2k^2 + 5k + 2) - 2 - (k-1)k/2) = -(3k^2 + 11k)/2.
\]

Let \(\Delta(a_0, \ldots, a_{2k})\) denote the discriminant of the polynomial \(x^{2k+2} + a_{2k}x^{2k} + \cdots + a_1x + a_0\). Then the locus of singular deformations of \(C\) is cut out formally locally by the polynomial \(n_1n_2\Delta(a_0, \ldots, a_{2k})\). \(G_m\) acts on this polynomial with weight \(2 - (2k+1)(2k+2)\). Thus, by Corollary 3.10, we have

\[
(3.13) \quad \chi_\delta(A_{2k+1}) = 4k^2 + 6k.
\]

In the following two examples, we build on Sections 3.2.1 and 3.2.2 to compute the characters of \(A_i/A_j\)-chains from Definition 2.13.
3.2.3. $A_{2i+1}/A_{2j}$-chains. Let $C = C_0 \cup E_1 \cup E_2$ be an $A_{2i+1}/A_{2j}$-atom, and let $q = E_1 \cap C_0$ denote the node, $p_1 = E_1 \cap E_2$ denote the $A_{2i+1}$-singularity, and $p_2 \in E_2$ denote the $A_{2j}$-singularity. The space of first-order deformations $T^1(C)$ can be decomposed as:

$$T^1(C) = T^1(C_0, q) \times T^1(\tilde{O}_{C, q}) \times T^1(\tilde{O}_{C_1, p_1}) \times Cr^1(\tilde{O}_{C, p_2}) \times T^1(\tilde{O}_{C_2, p_2}) \times Cr^1(\tilde{O}_{C, p_2}).$$

Choose coordinates $(a_0, \ldots, a_{2j-1})$, $(b_0, \ldots, b_2)$ on $T^1(\tilde{O}_{C, p_2})$, $T^1(\tilde{O}_{C_1, p_1})$, and $n$ on $T^1(\tilde{O}_{C, q})$ corresponding to the miniversal deformations

$$y^2 = x^{2j+1} + a_{2j-1}x^{2j-1} + \cdots + a_1x + a_0,$$  
$$w^2 = z^{i+j} + b_2z^{2i} + \cdots + b_1z + b_0,$$  
$$uv = n.$$

The action of $Aut(C) \simeq G_m$ is given by via $\lambda(x, y) = (\lambda^{-2}x, \lambda^{-(2j+1)}y), \lambda(z, w) = (\lambda z, \lambda^{i+1}w)$, and $\lambda \cdot u = \lambda^{-1}u$.

From 3.2.1, the characters of $\det(T^1(\tilde{O}_{C_1, p_1}))$ and $\det(Cr^1(\tilde{O}_{C, p_2}))$ are $4j^2 + 6j$ and $-(j-1)^2$, respectively. From 3.2.2, the characters of $\det(T^1(\tilde{O}_{C, p_2}))$ and $\det(Cr^1(\tilde{O}_{C, q}))$ are $-2i^2 + 5i + 2$ and $(i^2 - i)/2$ (note that the present action on the $A_{2i+1}$-atom has a sign opposite from that in 3.2.2). Finally, the characters of $T^1(\tilde{O}_{C, q})$ and $T^1(C_0, q)$ are 1 and 0, respectively.

By Lemma 3.7, we conclude

$$\chi_K(A_{2i+1}/A_{2j}) = 2i^2 + 5i + 2 - \frac{j^2 - i}{2} = (4j^2 + 6j) - (j - 1)^2 - 1 = \frac{3i^2 + 11i}{2} + 2 - (3j^2 + 8j).$$

If $\Delta(a_0, \ldots, a_{2j-1})$ is the discriminant of $x^{2j+1} + a_{2j-1}x^{2j-1} + \cdots + a_1x + a_0$ and $\Delta(b_0, \ldots, b_2)$ is the discriminant of $z^{i+j} + b_2z^{2i} + \cdots + b_1z + b_0$, then the locus of singular deformations of $C$ is cut out formally locally by the polynomial $n\Delta(a_0, \ldots, a_{2j-1})\Delta(b_0, \ldots, b_2)$. Using (3.11) and (3.13), and applying Corollary 3.10, we obtain

$$\chi_\delta(A_{2i+1}/A_{2j}) = (8j^2 + 4j - 1) - (4i^2 + 6i).$$

3.2.4. $A_{2i+1}/A_{2j+1}$-chains. The computation of the characters $\chi_K$ and $\chi_\delta$ for the $A_{2i+1}/A_{2j+1}$-chain follows straightforwardly from the computation in 3.2.2 using the strategy of Section 3.2.3. For the reader’s convenience we list these characters here:

$$\chi_K(A_{2i+1}/A_{2j+1}) = (2i^2 + 5i + 2) - \frac{j^2 - i}{2} = (2i^2 + 5j + 2) + \frac{j^2 - j}{2} + 1$$
$$\chi_\delta(A_{2i+1}/A_{2j+1}) = (4j^2 + 6j) - (4i^2 + 6i) - 1$$

3.3. Non-reduced curves: A case study of ribbons. The character theory is particularly suited to the study of non-reduced Gorenstein schemes. Here, we treat the case of canonical rational ribbons with $G_m$-action. Ribbons occur as certain flat limits (in the Hilbert scheme) of canonically embedded smooth curves degenerating to hyperelliptic curves [Fon93]. Our exposition is self-contained but we refer the reader to [BE95] for a more systematic study of ribbons.

A rational ribbon is a scheme obtained by gluing together two copies of $\mathbb{A}^1[x, \varepsilon] := Spec\ k[x, \varepsilon]/(\varepsilon^2)$. Precisely, let $U_1 = Spec\ k[x, \varepsilon]/(\varepsilon^2)$ and $U_2 = Spec\ k[y, \eta]/(\eta^2)$, and let $(U_1)_x$ and $(U_2)_y$ be the corresponding open affine subchemes. Then a rational ribbon of genus $g$ is given by a gluing isomorphism $\varphi: (U_1)_y \rightarrow (U_2)_x$ defined by

$$x \mapsto y^{-1} - y^{-2}f(y)\eta,$$
$$\varepsilon \mapsto y^{-g-1}\eta,$$
where \( f(y) = f_1 y^{-1} + \cdots + f_{g-2} y^{-(g-2)} \in \frac{k[y, y^{-1}]}{k[y] + y^{g+1} k[y^{-1}]} \); see [BE95, p. 733] for more details.

We focus here on non-split ribbons that admit \( \mathbb{G}_m \)-action. There are \( g-2 \) such ribbons, each given by \( f(y) = y^{-\ell} \), for \( \ell \in \{1, \ldots, g-2\} \). Denote the ribbon corresponding to \( f(y) = y^{-\ell} \) by \( C_{\ell} \). Then the \( \mathbb{G}_m \)-action on \( C_{\ell} \) is given by \( \lambda \cdot (x, y, \varepsilon, \eta) = (\lambda x, \lambda^{-1} y, \lambda^{g-\ell} \varepsilon, \lambda^{-\ell-1} \eta) \).

By adjunction, the sections of \( \omega_{C_{\ell}} \) over \( U_1 \) are identified with restrictions to \( U_1 \) of 2-forms \( f(x, \varepsilon) \frac{dx \wedge d\varepsilon}{\varepsilon^2} \) on \( \text{Spec} \ k[x, \varepsilon] \), and the sections of \( \omega_{C_{\ell}} \) over \( U_2 \) are identified with restrictions to \( U_2 \) of 2-forms \( f(y, \eta) \frac{dy \wedge d\eta}{\eta^2} \) on \( \text{Spec} \ k[y, \eta] \). With this in mind, we can write down \( g \) linearly independent global sections of \( \omega_{C_{\ell}} \):

For \( k = 0, \ldots, g - \ell - 2 \), take

\[
\omega_k = x^k \frac{dx \wedge d\varepsilon}{\varepsilon^2} = -(y^{g-1-k} + (g - \ell - k - 1)y^{g-\ell-k-2}\eta) \frac{dy \wedge d\eta}{\eta^2},
\]

for \( k = g - \ell - 1, \ldots, g - 1 \), take

\[
\omega_k = (x^k + (\ell + k - 1 + g)x^{\ell+k-1}\varepsilon) \frac{dx \wedge d\varepsilon}{\varepsilon^2} = -y^{g-1-k} \frac{dy \wedge d\eta}{\eta^2}.
\]

It follows that \( \{\omega_i\}_{i=0}^{g-1} \) form the basis of \( H^0(C_{\ell}, \omega_{C_{\ell}}) \) diagonalizing the \( \mathbb{G}_m \)-action. Note that we recover the second part of [BE95, Theorem 5.1], namely the identification of the sections of \( H^0(C_{\ell}, \omega_{C_{\ell}}) \) with functions

\[
1, y, y^2, \ldots, y^\ell, y^{\ell+1} + \eta, y^{\ell+2} + 2y\eta, \ldots, y^{g-1} + (g - \ell - 1)y^{g-\ell-2}\eta,
\]

under a trivialization of \( \omega_{C_{\ell}} \) on \( U_2 \).

We now proceed with character computations. Under the \( \mathbb{G}_m \)-action above, we have that

\[
\lambda \cdot \omega_k = \lambda^{k-g+\ell+1} \omega_k.
\]

Summing up the weights of the \( \mathbb{G}_m \)-action on the basis \( \{\omega_i\}_{i=0}^{g-1} \), we obtain the character of \( \chi_1 \):

\[
\chi_1(C_{\ell}) = \sum_{k=0}^{g-1} (k - g + \ell + 1) = g(\ell + 1 - g) + g(g - 1)/2 = g \left( \ell - \frac{g-1}{2} \right).
\]

It remains to compute the weights of the \( \mathbb{G}_m \)-action on \( H^0(C_{\ell}, \omega_{C_{\ell}}^2) \) and the corresponding character \( \chi_2(C_{\ell}) \). Since \( h^0(C_{\ell}, \omega_{C_{\ell}}^2) = 3g - 3 \), it suffices to exhibit \( 3g - 3 \) linearly independent \( \mathbb{G}_m \)-semi-invariant sections. One such choice is presented by

\[
1, y, y^2, \ldots, y^2, y^{2g+1} + y^{\ell} \eta, \ldots, y^{\ell+g-1} + (g - \ell - 1)y^{g-2}\eta, \\
y^{\ell+g} + (g - \ell)y^{g-1}, \ldots, y^{2g-2} + (2g - 2\ell - 2)y^{2g-\ell-3}\eta, \eta, \ldots, y^{g-3}\eta.
\]

In particular, taking into account that the weight of \( \left( \frac{dy \wedge d\eta}{\eta^2} \right)^2 \) is \( 2\ell \), we see that the weights of the \( \mathbb{G}_m \)-action on \( H^0(C_{\ell}, \omega_{C_{\ell}}^2) \) are

\[
2\ell, 2\ell - 1, \ldots, 2\ell - (2g - 2), \ell - 1, \ldots, \ell - g + 2.
\]

Summing up these weights, we obtain the character of \( \chi_2 \):

\[
\chi_2(C_{\ell}) = 2(2g - 1)\ell - (g - 1)(2g - 1) + (g - 2)\ell - (g - 2)(g - 1)/2 \\
= (5g - 4)(\ell - \frac{g-1}{2}) = (5g - 4)\chi_1(C_{\ell}).
\]
Using $\delta = 13\lambda_1 - \lambda_2$, we deduce that the character of $\delta$ is $\chi_\delta(C_\ell) = (8g + 4)(\ell - \frac{g-2}{2})$. In particular, if $\ell \neq (g - 1)/2$, then all three characters $\chi_1(C_\ell), \chi_2(C_\ell), \chi_\delta(C_\ell)$ are non-zero, and we have

$$\frac{\chi_\delta(C_\ell)}{\chi_\lambda(C_\ell)} = \frac{8g + 4}{g}.$$  

**Remark 3.12.** Generalizing the computations above, Anand Deopurkar computed characters of Gorenstein $n$-ribbons with $\mathbb{G}_m$-action and verified that always

$$\chi_\delta = \frac{12(2g + n - 1)n}{n^2 + (4g - 3)n + 2 - 2g} \chi_\lambda.$$  

This recovers the ratio $\frac{8g + 4}{g}$ for 2-ribbons, gives the ratio $\frac{36(g + 1)}{9g + 1}$ for 3-ribbons (see Corollary 5.5 and the subsequent discussion for the significance of this slope), and more generally gives the same ratio $\frac{\chi_\delta}{\chi_\lambda}$ as that of the toric singularity $y^n = x^{2g/(n-1)+1}$ (note that the arithmetic genus of an $n$-ribbon always satisfies $n - 1 \mid 2g$).

<table>
<thead>
<tr>
<th>Singularity type</th>
<th>$\lambda$</th>
<th>$\lambda_2$</th>
<th>$\delta$</th>
<th>$\alpha$-value</th>
<th>slope</th>
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<tr>
<td>$A_{2k} : y^2 - x^{2k+1}$</td>
<td>$k^2$</td>
<td>$5k^2 - 4k + 1$</td>
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<td>$\frac{3k^2 + 8k - 2}{8k^2 + 4k - 1}$</td>
<td>$\frac{8k^2 + 4k - 1}{k^2}$</td>
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<td>$4k^2 + 6k$</td>
<td>$\frac{3k + 11}{8k + 12}$</td>
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<td>$D_{2k+1} : x^2(y^2 - x^{2k-1})$</td>
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<td>$5k^2 - 2k$</td>
<td>$8k^2 + 2k$</td>
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<td>$4k^2 + 5k$</td>
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<td>152</td>
<td>251</td>
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<td>$y^3 - x^8$</td>
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<tr>
<td>$T_{p,q} : x^p - y^q$</td>
<td>See Corollary 5.4 for character values</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>monomial unibranch with gaps $(b_1, \ldots, b_k)$</th>
<th>$\sum b_i$</th>
<th>$(2k - 1)^2 + \sum b_i$</th>
<th>$12\sum b_i - (2k - 1)^2$</th>
<th>$\frac{11\sum b_i - 2(2k - 1)^2}{12\sum b_i - (2k - 1)^2}$</th>
<th>$12 - \frac{(2k-1)^2}{\sum b_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ribbon $C_\ell$</td>
<td>$g(\ell - \frac{2g-1}{2})$</td>
<td>$(5g - 4)(\ell - \frac{2g-1}{2})$</td>
<td>$(8g + 4)(\ell - \frac{2g-1}{2})$</td>
<td>$\frac{3g+8}{8g+4}$</td>
<td>$8 + \frac{4}{g}$</td>
</tr>
</tbody>
</table>

**Table 1.** Character values of some Gorenstein singular curves

---

An $n$-ribbon is a non-reduced scheme supported on $\mathbb{P}^1$ and locally isomorphic to $U \times \text{Spec} \ k[e]/(e^n)$, where $U \subset \mathbb{P}^1$ is affine.
4. OUTLINE OF THE LOG MMP FOR $\overline{M}_g$

In this section, we summarize the outcome of the computations from Section 3. We begin, by collecting the character and $\alpha$-value computations for all ADE, planar toric, and Gorenstein unibranch singularities in Table 1: The characters of all ADE atoms are computed in 3.1.2 – 3.1.6, the characters of the unibranch Gorenstein singularities are computed in 3.1.7, and the characters of ribbons are computed in Section 3.3.

The characters of dangling singularities in Table 2 are obtained from the characters in Table 1 using the comparison results of Corollaries 3.3 and 3.6. Finally, the characters of $A_i/A_j$-atoms are computed in Sections 3.2.3 and 3.2.4.

<table>
<thead>
<tr>
<th>Dangling type</th>
<th>$\lambda$</th>
<th>$\lambda_2$</th>
<th>$\delta$</th>
<th>$\alpha$-value</th>
<th>Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{2k}^{(1)}: y^2 - x^{2k+1}$</td>
<td>$k^2$</td>
<td>$5k^2 - 4k$</td>
<td>$8k^2 + 4k$</td>
<td>$5k + 8$</td>
<td>$8k + 4$</td>
</tr>
<tr>
<td>$A_{2k+1}^{(1,2)}: y^2 - x^{2k+2}$</td>
<td>$k^2 + k$</td>
<td>$5k^2 + k - 4$, $2$</td>
<td>$4k^2 + 6k + 2$</td>
<td>$5k + 8$</td>
<td>$8k + 4$</td>
</tr>
<tr>
<td>$A_{2k+1}^{(1)}: y^2 - x^{2k+2}$</td>
<td>$k^2 + k$</td>
<td>$5k^2 + k - 2$, $2$</td>
<td>$4k^2 + 6k + 1$</td>
<td>$5k^2 + 11k + 4$, $8k^2 + 12k + 2$</td>
<td>$8k^2 + 12k + 2$, $k^2 + k$</td>
</tr>
<tr>
<td>$D_{2k+1}^{(1)}: x(y^2 - x^{2k-1})$</td>
<td>$k^2$</td>
<td>$5k^2 - 2k - 1$</td>
<td>$8k^2 + 2k + 1$</td>
<td>$5k^2 + 4k + 2$, $8k^2 + 2k + 1$</td>
<td>$8k^2 + 2k + 1$, $k^2 + k$</td>
</tr>
<tr>
<td>$D_{2k+2}^{(1,2)}: x(y^2 - x^{2k})$</td>
<td>$k^2 + k$</td>
<td>$5k^2 + 3k - 4$, $2$</td>
<td>$4k^2 + 5k + 2$</td>
<td>$5k^2 + 7k + 4$, $8k^2 + 10k + 4$</td>
<td>$8k^2 + 10k + 4$, $k^2 + k$</td>
</tr>
<tr>
<td>$D_{2k+2}^{(1)}: x(y^2 - x^{2k})$</td>
<td>$k^2 + k$</td>
<td>$5k^2 + 3k - 2$, $2$</td>
<td>$4k^2 + 5k + 1$</td>
<td>$5k^2 + 7k + 4$, $8k^2 + 10k + 2$</td>
<td>$8k^2 + 10k + 2$, $k^2 + k$</td>
</tr>
<tr>
<td>$E_6^{(1)}: y^3 - x^4$</td>
<td>8</td>
<td>32</td>
<td>72</td>
<td>5/9</td>
<td>9</td>
</tr>
<tr>
<td>$E_7^{(1)}: y(y^2 - x^3)$</td>
<td>7</td>
<td>30</td>
<td>61</td>
<td>31/61</td>
<td>61/7</td>
</tr>
<tr>
<td>$E_8^{(1)}: y^3 - x^5$</td>
<td>14</td>
<td>62</td>
<td>120</td>
<td>29/60</td>
<td>60/7</td>
</tr>
<tr>
<td>$A_{i}/A_{j}$-chains</td>
<td>$\lambda$</td>
<td>$\lambda_2$</td>
<td>$\delta$</td>
<td>$\alpha$-value</td>
<td>Slope</td>
</tr>
<tr>
<td>$A_{2i+1}/2j$</td>
<td>$j^2 - \left(\frac{j^2 + 1}{2}\right)$</td>
<td>$8j^2 + 4j - 4\ell^2 - 6i - 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{2i+1}/2j+1$</td>
<td>$j^2 + j^2 \ell^2 - \ell$</td>
<td>$4j^2 + 6j - 4\ell^2 - 6i + 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Character values for dangling ADE singularities

Using heuristics provided by both intersection theory and character computations, we offer predictions in Table 3 for modular interpretations of $\overline{M}_g(\alpha)$ for $\alpha \geq 5/9$. (For small $g$ these predictions have to be modified. For example, in $\overline{M}_3$ Weierstrass genus 2 tails are also elliptic tails and so $\alpha = 2/3$ is not a threshold value, as can be seen from [HL10]; similarly, in genus 4 the $A_2/A_3$-atom is dangling, so one needs to add 1 to the character of $\lambda_2$. ) We note that the first two lines of Table 3 have been verified in the work of Hassett and Hyeon [HH09, HH13], where the reader can find a precise definition of elliptic tails and bridges. For the definition and extended discussion of all other tails, bridges, and triboroughs featured in the rightmost column of Table 3 we refer the reader to [FS13a, Section 4.4].

Remark 4.1. There are in fact other singular curves with $G_m$-action whose characters allow them to appear in $\overline{M}_g(\alpha)$ for $\alpha \in [5/9,1]$. However, they can be excluded by geometric considerations. Heuristically, a necessary condition for a curve to appear in $\overline{M}_g(\alpha)$ is that it is an isotrivial specialization of curves in $\overline{M}_g(\alpha + \epsilon)$. For example, an $A_5/4$-curve has $\alpha$-value 9/11 but obviously is not an isotrivial specialization of any stable curve.
SINGULARITIES WITH $\mathbb{G}_m$-ACTION AND THE LOG MMP FOR $\mathcal{M}_g$

<table>
<thead>
<tr>
<th>$\alpha$-value</th>
<th>Singularity type added at $\alpha$</th>
<th>Locus removed at $\alpha - \epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9/11</td>
<td>$A_2$ elliptic tails attached nodally</td>
<td></td>
</tr>
<tr>
<td>7/10</td>
<td>$A_4$ elliptic bridges attached nodally</td>
<td></td>
</tr>
<tr>
<td>2/3</td>
<td>$A_4$ genus 2 tails attached nodally at a Weierstrass point</td>
<td></td>
</tr>
<tr>
<td>19/29</td>
<td>$A_5^{(1)}$ genus 2 tails attached nodally</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A_{3/4}$ genus 2 tails attached tacnodally at a Weierstrass point</td>
<td></td>
</tr>
<tr>
<td>12/19</td>
<td>$A_{3/5}$ genus 2 tails attached tacnodally</td>
<td></td>
</tr>
<tr>
<td>17/28</td>
<td>$A_5$ genus 2 bridges attached nodally at conjugate points</td>
<td></td>
</tr>
<tr>
<td>49/83</td>
<td>$A_{6}$ hyperelliptic genus 3 tails attached nodally at a Weierstrass point</td>
<td></td>
</tr>
<tr>
<td>32/55</td>
<td>$A_{6}^{(1)}$ hyperelliptic genus 3 tails attached nodally</td>
<td></td>
</tr>
<tr>
<td>42/73</td>
<td>$A_{3/56}$ hyperelliptic genus 3 tails attached tacnodally at a Weierstrass point</td>
<td></td>
</tr>
<tr>
<td>5/9</td>
<td>$D_4$ elliptic triboroughs attached nodally</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$D_5$ genus 2 bridges attached nodally at a Weierstrass and free point</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$D_6^{(1)}$ genus 2 bridges attached nodally at two free points</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A_{3/7}$ hyperelliptic genus 3 tails attached tacnodally</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A_{5/7}$ hyperelliptic genus 3 tails attached oscnodally</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A_{5/6}$ hyperelliptic genus 3 tails attached oscnodally at a Weierstrass point</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A_{7}$ hyperelliptic genus 3 bridges attached nodally at conjugate points</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Predictions for the log MMP for $\alpha \geq 5/9$

Giving a complete description of $\mathcal{M}_g(\alpha)$ is much more subtle than generally describing the singularities added and the loci removed. For instance, after removing elliptic tails (connected genus 1 subcurves which are attached nodally) at $\alpha = 9/11 - \epsilon$, in each subsequent moduli stack parameterizing curves with additional singularities, one needs to redefine what is meant by an elliptic tail by specifying the allowed attaching singularities; see [HH13] and [AFSvdW13] for examples.

5. Connections to Intersection Theory

In this section, we explore a remarkable connection between the intersection theory on $\mathcal{M}_g$ and $\alpha$-invariants. To see why such a connection should exist, suppose that the modularity principle holds and consider a worse-than-nodal curve $[C] \in \mathcal{M}_g(\alpha)$. If $T_C \subset \mathcal{M}_g$ denotes the set of all possible stable limits of the curve $[C]$, then $T_C$ should lie in the exceptional locus of the rational map $\mathcal{M}_g \dashrightarrow \mathcal{M}_g(\alpha)$. Conversely, given the definition of $\mathcal{M}_g(\alpha)$ given in Equation (1.1), one would expect the $\alpha$-value at which $[C]$ first appears in $\mathcal{M}_g(\alpha)$ to be the supremum of $\alpha$-values for which $T_C$ lies in the stable base locus of $K_{\mathcal{M}_g} + \alpha \delta$. This motivates us to make the following conjecture:

**Conjecture 5.1.** Let $\hat{O}_{X,p}$ be a Gorenstein curve singularity. If $C$ is an $\hat{O}_{X,p}$-atom and $T_C \subset \mathcal{M}_g$ is the set of all possible stable limits of $C$, then

$$\alpha(\hat{O}_{X,p}) = \sup \left\{ \alpha \mid T_C \text{ lies in the stable base locus of } K_{\mathcal{M}_g} + \alpha \delta \right\}.$$
In this section, we give some evidence for the conjecture by proving, under certain hypotheses, a relationship between the characters $\chi_L(C, \eta)$ and intersection numbers on $T_C$.

5.1. Intersection numbers and characters. To set the notation, let $C \in U_g$ be an $\hat{O}_{X,p}$-atom of a Gorenstein singularity $\hat{O}_{X,p}$ with $\mathbb{G}_m$-action, with the convention that $\mathbb{G}_m$ acts with negative weights on the maximal ideal of $\hat{O}_{X,p}$. Then $C$ admits a $\mathbb{G}_m$-action $\eta: \mathbb{G}_m \to \text{Aut}(C)$. Assume that the versal deformation space $\text{Def}(C)$ is normal and $\text{Aut}(C)$ is linearly reductive. Then $C$ admits a versal deformation space with $\mathbb{G}_m$-action, i.e., a pointed affine scheme $0 \in \text{Def}(C)$ with a $\mathbb{G}_m$-action and a smooth morphism $[\text{Def}(C)/\mathbb{G}_m] \to U_g$ sending $0$ to $[C]$ [Alp10, Theorem 3] (see also [Pin74, Proposition 2.3] for the formal case).

According to Pinkham’s theory of deformations of varieties with $\mathbb{G}_m$-action [Pin74, Proposition 2.2], the space of infinitesimal deformations of $C$ has a decomposition into weight spaces:

$$T^i(C) = \bigoplus_{\nu = -\infty}^{\infty} T^i(C)(\nu).$$

Following Pinkham [Pin74, Section (3.1)], we define $\text{Def}^+(C)$ to be the closed subscheme of $\text{Def}(C)$ corresponding to positive deformations. The tangent space to $\text{Def}^+(C)$ is $\bigoplus_{\nu > 0} T^i_C(\nu)$ and the maximal ideal of $0$ in the coordinate ring of $\text{Def}^+(C)$ is negatively graded.

Since every curve in $[(\text{Def}^+(C) - 0)/\mathbb{G}_m]$ admits an isotrivial degeneration to $C$, we have that $[(\text{Def}^+(C) - 0)/\mathbb{G}_m] \cap \overline{M}_g \subset T_C$. The precise relationship between intersection theory on $[\text{Def}^+(C)/\mathbb{G}_m]$ and characters is given by the following observation:

**Theorem 5.2.** Let $B$ be any complete curve with a non-constant map $B \to [\text{Def}^+(C)/\mathbb{G}_m]$ and let $L$ be any line bundle defined on a neighborhood of $[C] \in U_g$. Then

$$\chi_L(C, \eta) = \frac{L \cdot B}{\deg(B)},$$

where $\deg(B)$ is the degree of $B$ with respect to the natural $\mathcal{O}(-1)$ on $[\text{Def}^+(C)/\mathbb{G}_m]$. In particular,

$$\chi_i(C, \eta) = \frac{\lambda_i \cdot B}{\deg(B)}.$$  

Moreover, if the discriminant divisor $\Delta \subset \text{Def}^+(C)$ is Cartier and the locus of worse-than-nodal deformations of $C$ has codimension at least 2, then $\chi_\Delta(C, \eta) = \frac{\Delta \cdot B}{\deg(B)}$.

**Proof.** We can write $\text{Def}^+(C) = \text{Spec} \, A$ with $A$ a non-positively graded $k$-algebra. The line bundle $L$ corresponds to a graded projective $A$-module which is free of rank 1 by [Eis95, Theorem 19.2]. It follows that $L = A(d) = \mathcal{O}(d)$ for some $d$. Therefore $\chi_L(C, \eta) = -d$ and $L \cdot B = \deg_B(L) = -d \deg(B)$. \qed

5.2. Toric singularities. We apply Theorem 5.2 to compute characters of toric planar singularities. Let $X \simeq \text{Spec} \, k[x, y]/(x^p - y^q)$, where $p, q \geq 2$. We set $b := \gcd(p, q)$ and $p_0 := p/b, q_0 := q/b$. The $\mathbb{G}_m$-action is given by $\eta: \lambda \cdot (x, y) = (\lambda^{-p_0}x, \lambda^{-q_0}y)$. The miniversal deformation of $X$ is given by

$$x^p = y^q + \sum a_{ij} x^i y^j, \quad 0 \leq i \leq p - 2, \quad 0 \leq j \leq q - 2.$$

Note that our signs are opposite of Pinkham’s.
Let $C$ be an $\mathcal{O}_{X,D}$-atom of genus $g$. Then we have that $\Def^+(C) \simeq \Spec k[a_{ij} : qi + pj < pq]$. We also note that the weighted projective stack $[(\Def^+(C) - 0)/G_m]$ is identified with the moduli space of curves on $\mathbb{P}(q,p,1)$ defined by the weighted homogeneous equation
\begin{equation}
\begin{aligned}
x^p &= y^q + \sum a_{ij} x^{i} y^{j} z^{pq+qij-pj}, \quad 0 \leq i \leq p-2, \ 0 \leq j \leq q-2, \ qi + pj < pq.
\end{aligned}
\end{equation}
The arithmetic genus of these curves is $h := (pq - p - q - b + 2)/2$; e.g. [Has00, Theorem 6.3]. To apply Theorem 5.2, we need to write down a complete non-constant family $B \to [\Def^+(C)/G_m]$. We do this via a degeneration argument in Proposition 5.3.

**Proposition 5.3.** Let $Z \simeq \mathbb{P}_{[s:t]}(pq - p - q, pq) \subset [(\Def^+(C) - 0)/G_m]$ be a complete family of genus $h = (pq - p - q - b + 2)/2$ curves defined by the weighted homogeneous equation
\begin{equation}
\begin{aligned}
x^p &= y^q + sxyz^{pq-p-q} + tz^{pq} = 0.
\end{aligned}
\end{equation}
Then $Z$ is a family of at-most nodal curves and
\begin{equation}
\begin{aligned}
\lambda \cdot Z &= \frac{b}{12} (pq_0b - p_0 - q_0)^2 + pq_0(pq - p - q + 1) - 1, \\
\delta \cdot Z &= pq_0b(pq - p - q + 1).
\end{aligned}
\end{equation}

**Proof.** Suppose $F_0$ is a pencil of plane curves of degree $d > 0$ containing a curve $D$ with a unique singularity $x^p = y^q$ and such that the total family over $F_0$ is smooth. Consider a deformation $F = \{F_s\}$ of $F_0$ such that $F_1$ is a general pencil and such that the total space of the family $\{F_s\}$ has local equation $x^p = y^q + sxyz + t$. The simultaneous stable reduction of this family is obtained by the weighted blow-up of $\mathbb{A}_{s,t}^2$ with weight $(s,t) = (pq - p - q, pq)$. Denote by $Z \simeq \mathbb{P}(pq - p - q, pq)$ the exceptional divisor of this blow-up. For $[s : t] \in Z$, the stable curve over $[s : t]$ is a union of the normalization $\bar{D}$ of $D$ and the curve on $\mathbb{P}(q,p,1)$ defined by the weighted homogeneous equation
\begin{equation}
\begin{aligned}
x^p &= y^q + sxyz^{pq-p-q} + tz^{pq} = 0.
\end{aligned}
\end{equation}
It is easy to verify that every member of this family is in fact an irreducible at-most nodal curve. In particular, $Z \subset [(\Def^+(C) - 0)/G_m] \cap \mathcal{M}_h$.

We proceed to compute the intersection numbers of $Z$ with the divisor classes $\lambda$ and $\delta$ using the observation that
\begin{equation}
\begin{aligned}
\lambda \cdot Z &= (\lambda \cdot F_1 - \lambda \cdot F_0), \\
\delta \cdot Z &= (\delta \cdot F_1 - \delta \cdot F_0).
\end{aligned}
\end{equation}
Let $X_i$ be the total families of pencils $F_i$ ($i = 0, 1$). Our first goal is to compare the degrees of $\delta$ and $\kappa$ on $X_0$ and $X_1$: Since $F_1$ is a general pencil, we have $\delta(X_1) = \delta_0(X_1) = 3(d - 1)^2$. To find the number of singular fibers in $X_0 \sim D$, we observe that the topological Euler characteristic of $D$ is $2 - 2g(D) - (b - 1)$, where $g(D) = \binom{d - 1}{2} - h - b + 1$ is the geometric genus of $D$. Since the topological Euler characteristics of $X_0$ and $X_1$ are the same, we have that
\begin{equation}
\begin{aligned}
\delta_0(X_0 \sim D) &= \delta_0(X_1) = (2h + b - 1) = \delta_0(X_1) - (pq - p - q + 1).
\end{aligned}
\end{equation}
Since $X_0$ and $X_1$ are pencils of plane curves of the same degree, we have $\kappa(X_0) = \kappa(X_1)$.

Next, to compare intersection numbers of $F_0$ and $F_1$ with $\lambda$ and $\delta$, we need to write down a family of stable curves over each $F_i$. There is nothing to do in the case of $F_1$, since it is already a general pencil of plane curves of degree $d$. In particular, we have $\lambda \cdot F_1 = (\kappa(X_1) + \delta_0(X_1))/12$ by Mumford’s formula. To write down a stable family over $F_0$, we perform stable reduction of $X_0 \to F_0$ in two steps:
**Base change:** To begin, make a base change of order \( p_0 q_0 b \) to obtain the family \( \mathcal{Y} \) with local equation \( x^p - y^q = t^{p_0 q_0 b} \). The numerical invariants of \( \mathcal{Y} \) are

\[
\kappa(\mathcal{Y}) = p_0 q_0 b (\kappa_0) = p_0 q_0 b \kappa_1, \quad \text{and} \quad \delta_0(\mathcal{Y} \setminus D) = p_0 q_0 b \delta_0(\mathcal{X}_0 \setminus D) = p_0 q_0 b (\delta_0(\mathcal{X}_1)) - (pq - p - q + 1).
\]

**Weighted blow-up:** Let \( Z \) be the weighted blow-up of \( \mathcal{Y} \), centered at \( x = y = t = 0 \), with weight \((x, y, t) = (q_0, p_0, 1) \). The central fiber of \( Z \) becomes \( \overline{D} \cup T \) of the form described in [Has00, Theorem 6.3], with smooth \( T \). The self-intersection of the tail \( T \) inside \( Z \) is \((-b)\). By intersecting both sides of \( K_Z = \pi^* K_Y + a T \) with \( T \), we find that \( a = p_0 + q_0 - p_0 q_0 b \). It follows that

\[
\kappa(Z) = \kappa(Y) - b(p_0 q_0 b - p_0 - q_0)^2 = p_0 q_0 b \kappa(X_1) - (pq - p - q + 1).
\]

The number \( \delta_0(Z \setminus (\overline{D} \cup T)) \) of singular fibers in \( Z \) is the same as in \( \mathcal{Y} \setminus D \) and equals to

\[
p_0 q_0 b \delta_0(\mathcal{X}_1) = p_0 q_0 b (pq - p - q + 1).
\]

We conclude that

\[
\delta \cdot Z = p_0 q_0 b \delta_0(\mathcal{X}_1) - \delta_0(Z) = p_0 q_0 b (pq - p - q + 1),
\]

\[
\kappa \cdot Z = p_0 q_0 b \kappa(\mathcal{X}_1) - \kappa(Z) = b(p_0 q_0 b - p_0 - q_0)^2.
\]

Using Mumford’s formula \( \lambda = (\kappa + \delta)/12 \), we obtain

\[
\lambda \cdot Z = \frac{b}{12}((p_0 q_0 b - p_0 - q_0)^2 + p_0 q_0 (pq - p - q + 1) - 1).
\]

**Corollary 5.4.** Let \( X = \text{Spec } k[x, y]/(x^p - y^q) \) with \( \mathbb{G}_m \)-action \( \eta : (x, y) = (\lambda^{-q} x, \lambda^{-p} y) \). Then

\[
\chi_1(\overline{X}, \eta) = \frac{b}{12}((p_0 q_0 b - p_0 - q_0)^2 + p_0 q_0 (pq - p - q + 1) - 1),
\]

\[
\chi_2(\overline{X}, \eta) = p_0 q_0 b (pq - p - q + 1).
\]

**Proof.** This follows immediately from Theorem 5.2 and Proposition 5.3. \( \square \)

An amusing consequence of Theorem 5.2 is that it becomes possible to compute the slope of special subvarieties of \( \mathcal{M}_g \) simply by using characters. For example, by considering the monomial unibranch singularities \( y^3 = x^{3k+1} \) and \( y^3 = x^{3k+2} \), we recover the following result of Stankova [SF00] (see also Remark 3.12).

**Corollary 5.5.** For \( g \equiv 0, 1 \ (mod \, 3) \), there is a complete family \( B \) of generically smooth genus \( g \) stable trigonal curves such that \( (\delta \cdot B)/(\lambda \cdot B) = 36(g + 1)/(5g + 1) \).

**Proof.** Let \( C \) be the monomial unibranch singularity \( y^3 = x^{3k+1} \). From Equation (5.1) the restriction of its miniversal deformation to \( \text{Def}^+(C) \) is

\[
y^3 = x^{3k+1} + y(a_2 x^{2k} + \cdots + a_0) + (b_{3k-1} x^{3k-1} + \cdots + b_0).
\]

It follows that \( \left( \text{Def}^+(C) - 0 \right)/\mathbb{G}_m \simeq \mathbb{P}(2, 5, \ldots, 6k + 2, 6, 9, \ldots, 9k + 3) \) is the moduli space of curves defined by Equation (5.2) on \( \mathbb{P}(3, 3k + 1, 1) \).

Evidently, there is a complete family \( B \rightarrow \left( \text{Def}^+(C) - 0 \right)/\mathbb{G}_m \) of at-worst nodal irreducible curves. Applying Theorem 5.2 to this family and using computations of Example 3.1.7, we find
that $\lambda \cdot B = \chi_1(C) = 2g(5g + 1)/12$ and $\delta \cdot B = 13\chi_1(C) - \chi_2(C) = 6g(g + 1)$. This gives slope $36(g + 1)/(5g + 1)$.

Considering $y^3 = x^{3k+2}$, we obtain in an analogous fashion a complete family of trigonal curves of genus $g = 3k + 1$ with slope $36(g + 1)/(5g + 1)$. \hfill $\Box$

We note that in contrast to a simple construction above, an extremal family achieving slope $36(g + 1)/(5g + 1)$ is obtained by a laborious construction in [SF00]. However, our methods do not establish a stronger result, also due to Stankova, that $36(g + 1)/(5g + 1)$ is the maximal possible among slopes of trigonal families of genus $g$.

6. Connections to GIT

In this section, we show that the $\alpha$-invariant can be reinterpreted in terms of the Hilbert-Mumford index in geometric invariant theory, and explain how the character computations of Section 3 can be used to prove the instability of certain Hilbert points. Recall that for every $g$, $n$, and $m$, we have the Hilbert and Chow GIT quotients

\[ \text{Hilb}^m_{g,n} \sslash SL_N \quad \text{and} \quad \text{Chow}^m_{g,n} \sslash SL_N \]

parameterizing, respectively, semistable $m^{th}$ Hilbert points of $n$-canonically embedded curves of genus $g$, and semistable Chow points of $n$-canonically embedded curves of genus $g$ curves, up to projectivities. Here, $N = g$ if $n = 1$, and $N = (2n - 1)(g - 1)$ if $n > 1$.

**Proposition 6.1.** Let $C$ be a Gorenstein $n$-canonically embedded genus $g$ curve which admits a $\mathbb{G}_m$-action $\eta : \mathbb{G}_m \to \text{Aut}(C)$. Consider the induced one-parameter subgroup $\tilde{\eta} : \mathbb{G}_m \to SL_N$. Then the Hilbert-Mumford indices of the $m^{th}$ Hilbert point of $C$, respectively the Chow point of $C$, with respect to $\tilde{\eta}$ are

\[ \mu_{\text{Hilb}_{g,n}^m}(C, \tilde{\eta}) = \begin{cases} \chi_\lambda + (m - 1) \left[ \frac{(4g + 2)m - g + 1}{2} \chi_\delta - \frac{2m}{2} \chi_\delta \right], & \text{if } n = 1, \\
(m - 1)(g - 1) \left[ \frac{6mn^2 - 2mn - 2n + 1}{2} \chi_\lambda - \frac{mn^2}{2} \chi_\delta \right], & \text{if } n > 1, \end{cases} \]

and

\[ \mu_{\text{Chow}_{g,n}^m}(C, \tilde{\eta}) = \begin{cases} (4g + 2)\chi_\lambda - \frac{2}{2} \chi_\delta, & \text{if } n = 1, \\
(g - 1)n(6n - 2)\chi_\lambda - \frac{2}{2} \chi_\delta, & \text{if } n > 1. \end{cases} \]

**Proof.** This result follows directly by computing the divisor classes of the GIT linearizations as in [Mum77, Theorem 5.15] or [HH13, Section 5]. \hfill $\Box$

This proposition implies that if one can compute the characters of $\lambda$ and $\delta$ (or equivalently $\alpha$ and $\lambda_2$) with respect to one-parameter subgroups of the automorphism group, then one immediately knows the Hilbert-Mumford indices for all Hilbert and Chow quotients for such one-parameter subgroups. This allows us to use character computations to prove instability of certain Hilbert points (cf. [Hye10, Propositions 2 and 3]).

As an example, let us see how to use Proposition 6.1 and the computations of Example 3.3 to prove some new results on the instability of Hilbert points of canonically embedded ribbons.

**Theorem 6.2** (Hilbert stability of ribbons). Let $C_\ell$ be a ribbon defined by $f(y) = y^{-\ell}$ for some $\ell \in \{1, \ldots, g - 2\}$. Then $C_\ell$ admits a $\mathbb{G}_m$-action $\eta : \mathbb{G}_m \to \text{Aut}(C_\ell)$ and

1. If $\ell \neq (g - 1)/2$, then the $m^{th}$ Hilbert point of the $n$-canonical embedding of $C_\ell$ is unstable for all $m \geq 2$ and $n \geq 1$. 
(2) If \( \ell = (g - 1)/2 \), then the \( m \)th Hilbert point of the \( n \)-canonical embedding of \( C_\ell \) is unstable for all \( m \geq 2 \) and \( n \geq 2 \).

(3) If \( \ell = (g - 1)/2 \), then the \( m \)th Hilbert point of the canonical embedding of \( C_\ell \) is strictly semistable with respect to \( \eta \) for all \( m \geq 2 \).

Proof. The ribbon \( C_\ell \) is obtained by gluing \( \text{Spec} \, k[x, \epsilon]/(\epsilon^2) \) and \( \text{Spec} \, k[y, \eta]/(\eta^2) \) along open affines \( \text{Spec} \, k[x, x^{-1}, \epsilon]/(\epsilon^2) \) and \( \text{Spec} \, k[y, y^{-1}, \eta]/(\eta^2) \) by
\[
\begin{align*}
x & \mapsto y^{-1} - y^{-\ell-2}\eta, \\
\epsilon & \mapsto y^{-g-1}\eta,
\end{align*}
\]
We consider the \( G_m \)-action on \( C_\ell \) given by \( \lambda \cdot (x, y, \epsilon, \eta) = (\lambda x, \lambda^{-1}y, \lambda^{g-\ell}\epsilon, \lambda^{-\ell-1}\eta) \). It induces a one-parameter subgroup \( \eta \colon G_m \to \text{Aut}(C_\ell) \). By Example 3.3, the characters of \( C_\ell \) are
\[
\chi_\lambda(C_\ell, \eta) = g(\ell - \frac{g-1}{2}), \quad \chi_\delta(C_\ell, \eta) = (5g-4)(\ell - \frac{g-1}{2}).
\]
It follows by Proposition 6.1 that the Hilbert-Mumford index with respect to \( \eta \) of the \( m \)th Hilbert point of the canonical embedding of \( C_\ell \) is
\[
\mu^{\text{Hilb}_m}(C, \eta) = g(g + m - gm)(\ell - \frac{g-1}{2}).
\]
In particular, it is 0 if and only if \( \ell = (g - 1)/2 \). Similarly, we verify that the Hilbert-Mumford index \( \mu^{\text{Hilb}_{m,n}}(C, \eta) \) of the \( m \)th Hilbert point of the \( n \)-canonical embedding of \( C_\ell \) is non-zero for every \( n, m \geq 2 \). This finishes the proof.

Corollary 6.3. If \( g \) is even, then every canonically embedded genus \( g \) ribbon with \( G_m \)-action has unstable \( m \)th Hilbert point for all \( m \geq 2 \).

References


[AFSvdW13] Jarod Alper, Maksym Fedorchuk, David Smyth, and Frederick van der Wyck. Log minimal model program for \( \overline{M}_{g,n} \): The second flip. 2013. preprint.


SINGULARITIES WITH $\mathbb{G}_m$-ACTION AND THE LOG MMP FOR $\overline{\mathcal{M}}_g$

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