AMSI Numerical Solution of Hyperbolic PDE’s

Assignment 1

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Due 5pm Wednesday 23rd January

• Your solutions can either be \LaTeX up or hand written. But at the end of the day, a pdf file should be produced and sent to my email address, mailto:stephen.roberts@anu.edu.au. You should be able to get help scanning your solutions at the mathematics office.

• Fully justify your answers, quoting the appropriate theorems from the lectures, text books or course notes (no need to reference exact page numbers).

• Outside sources should be referenced fully (so as to allow us to find them easily).

• For the computational problems you should have a discussion section which provides the results of your computer experiments, and a discussion of the results, and then provide your code as an appendix. Hopefully the code will be fairly simple, just a variation on the code given in the first computer lab. The computational questions will be uploaded soon.

• You should produce well structured mathematical arguments. Indeed a component of your assignment mark will be given for the style and presentation of your solutions.
Question 1  (Riemann Problem for Fourth Power Flux.)  15 pts

Consider the conservation law
\[ u_t + \left( \frac{1}{32} u^4 \right)_x = 0 \]
with initial condition
\[ u(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x \geq 0. \end{cases} \]

For \( u_L > u_R \) we would expect a solution with a shock and for \( u_L < u_R \) a rarefaction fan.

(a) Find the Rankine-Hugoniot shock speed condition for this problem.  \hspace{1cm} (4 subpts)

(b) Hence find the solution to the problem with the initial condition \( u_L = 2, u_R = -1 \).
   Draw a picture of the \( x-t \) plane showing representative characteristic curves and the
   shock curve for this solution. \hspace{1cm} (3 subpts)

(c) Find the expression for the rarefaction fan for this problem. In particular find the
   function \( g \) such that \( u(x,t) = g(x/t) \) is a solution of this PDE. \hspace{1cm} (4 subpts)

(d) Hence find the solution to the problem with the initial condition \( u_L = -1, u_R = 2 \).
   Draw a picture of the \( x-t \) plane showing representative characteristic curves and the
   rarefaction fan for this solution. \hspace{1cm} (4 subpts)
Question 2  (Linearization of the Shallow water equation)  10 pts

Consider the ocean with a depth of $H$ metres. Suppose there is a tsunami which moves the whole vertical column of the ocean at a speed $u$ metres per second. The tsunami will produce a small disturbance of height $h$. I.e. the overall height of the ocean will be $H + h$. Here $h$ and $u$ are functions of space $x$ and time $t$. $H$ is assumed constant.

This problem can be modelled by the system of equations

$$h_t + (Hu)_x = 0 \quad (1)$$
$$u_t + (gh)_x = 0 \quad (2)$$

where $g$ is the gravitation acceleration ($g \approx 10ms^{-2}$).

(a) Derive these equations as a linearization of the shallow water wave equations. (2 sub-pts)

(b) By differentiating the equations appropriately, find a second order equation for $h$ alone. (Hint: It should be a wave equation (no pun intended)). (2 subpts)

(c) Find the general solution of this equation. (2 subpts)

(d) How fast do the components of this solution move? This should be a formula in terms of $H$ and $g$. (2 subpts)

(e) Suppose the ocean depth is 4000 metres, how fast does the disturbance created by the tsunami move? (in $ms^{-1}$ and in $km/hr$). (2 subpts)
Consider the system of conservation laws

\[ \rho_t + (u\rho)_x = 0 \]
\[ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \]

where \( \rho \) denotes density, \( u \) velocity and \( p(\rho) \) is pressure as a function of density. We suppose that \( p'(\rho) > 0 \).

(a) Write this system in terms of the primitive variables \( \rho \) and \( u \). \((3 \text{ subpts})\)

(b) What are the characteristic speeds for this system (i.e. what are the eigenvalues of the matrix \( A \) associated with the equation found in part (a)). \((3 \text{ subpts})\)

(c) Suppose we specifically assume that the pressure satisfies a gamma gas law, \( p(\rho) = c\rho^\gamma \). In this case derive the Riemann Invariants for this system of equations. \((4 \text{ subpts})\)