Almost H-projective structures and their description as parabolic geometries

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Kioloa, March 2013
1. Almost complex manifolds

Suppose that \((M, J)\) is an almost complex manifold with \(\dim \mathbb{R}(M) = 2n\).

We denote the Nijenhuis tensor of \(J\) by
\[
N(X, Y) = [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]).
\]

\(N\) is a two-form with values in \(TM\), which is of type \((0, 2)\), i.e.
\[
N(JX, Y) = -JN(X, Y).
\]

Theorem (Newlander-Nirenberg 1957) \((M, J)\) is a complex manifold \(\iff N \equiv 0\).

A complex connection on an almost complex manifold \((M, J)\) is an affine connection \(\nabla\) that preserves the complex structure \(\nabla J = 0\).
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Suppose that $(M, J)$ is an almost complex manifold with $\dim_{\mathbb{R}}(M) = 2n$. The Nijenhuis tensor of $J$ is given by:

$$N(X, Y) = [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]).$$

$N$ is a two-form with values in $T^*M$, which is of type $(0, 2)$, i.e. $N(JX, Y) = -JN(X, Y)$.

Theorem (Newlander-Nirenberg 1957) $(M, J)$ is a complex manifold if and only if $N \equiv 0$.

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A complex connection on an almost complex manifold \((M, J)\) is an affine connection \(\nabla\) that preserves the complex structure \(\nabla J = 0.\)
For any complex connection $\nabla$ on $(M, J)$ we have:

\[-4\text{-times the } (0, 2)\text{-part of its torsion } T_{\nabla} \text{ equals } -[T_{\nabla}(X, Y) - T_{\nabla}(JX, JY)] + J(T_{\nabla}(JX, Y) + T_{\nabla}(X, JY))\]

which coincides with the Nijenhuis tensor $N$.

the curvature has values in $\text{gl}(TM, J)$:

\[R_{\nabla}(X, Y) \circ J = J \circ R_{\nabla}(X, Y)\]

Proposition (Lichnerowicz, 1955)

On any almost complex manifold $(M, J)$ there exist a complex connection such that $T_{\nabla} = -\frac{1}{4}N$. Such a complex connection is not unique. Complex connections $\nabla$ with $T_{\nabla} = -\frac{1}{4}N$ are sometimes called minimal connections.

Corollary

There exists a complex torsion-free connection on $(M, J) \iff N \equiv 0$. 
For any complex connection $\nabla$ on $(M, J)$ we have:

-4-times the $(0, 2)$-part of its torsion $T^\nabla$ equals

$$-[T^\nabla(X, Y) - T^\nabla(JX, JY)) + J(T^\nabla(JX, Y) + T^\nabla(X, JY))]$$

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\[ \text{Proposition (Lichnerowicz, 1955)} \]
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For any complex connection $\nabla$ on $(M, J)$ we have:

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Such a complex connection is not unique. Complex connections $\nabla$ with $T \nabla = -\frac{1}{4}N$ are sometimes called **minimal connections**.

**Corollary**

There exists a complex torsion-free connection on $(M, J) \iff N \equiv 0.$
2. Almost H-projective structures

Two affine connections $\nabla$ and $\hat{\nabla}$ on an almost complex manifold $\mathcal{M} = (M, J)$ are H-projectively equivalent:

$\iff$ there exists a real 1-form $\Upsilon \in \Omega^1(M)$ such that

$$\nabla_X Y = \hat{\nabla}_X Y + \Upsilon(X)Y + \Upsilon(Y)X - \Upsilon(JX)JY - \Upsilon(JY)JX.$$

for all vector fields $X, Y \in \mathfrak{X}(M)$.

$\upsilon_{\Upsilon}(X)(Y) = \upsilon_{\Upsilon}(Y)(X)$,

Hence, if $\hat{\nabla}$ is a complex connection, then any H-projectively equivalent connection $\nabla$ is complex too.

Since $\upsilon_{\Upsilon}(X)(Y) = \upsilon_{\Upsilon}(Y)(X)$, H-projectively equivalent connections have the same torsion.
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2. Almost $H$-projective structures

Two affine connections $\nabla$ and $\hat{\nabla}$ on an almost complex manifold $(M, J)$ are $H$-projectively equivalent if there exists a real 1-form $\Upsilon \in \Omega^1(M)$ such that

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**Definition**

Suppose that \((M, J)\) is an almost complex with \(\dim \mathbb{R} > 2\).
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- An **almost H-projective structure** on \((M, J)\) is an \(H\)-projective equivalence class \([\nabla]\) of complex connections whose torsion is of type \((0, 2)\).
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- If $(M, J)$ is a complex manifold, then an almost $H$-projective structure $[\nabla]$ on $(M, J)$ is torsion-free and called an **$H$-projective structure**.

Remark

A smooth curve $c : I \to M$ is $J$-planar with respect to a complex connection $\nabla$ $\iff$ $\nabla \dot{c} \dot{c} \in \text{span} \{ \dot{c}, J \dot{c} \}$.

Two complex connections are $H$-projectively equivalent $\iff$ they have the same $J$-planar curves.
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Remark

- A smooth curve \(c : I \to M\) is \(J\)-planar with respect to a complex connection \(\nabla\) if and only if \(\nabla_{\dot{c}} \dot{c} \in \text{span}\{\dot{c}, J\dot{c}\}\).
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- Two complex connections are **\(H\)-projectively equivalent** \(\iff\) they have the same \(J\)-planar curves.
3. Parabolic geometries

Suppose that $G$ is a Lie group and $P$ a closed subgroup of $G$.

Definition

A Cartan geometry of type $(G, P)$ on a manifold $M$ is given by a principal $P$-bundle $\pi: G \rightarrow M$ together with a Cartan connection, i.e. a one form $\omega \in \Omega^1(G, g)$ such that:

1. $\omega$ is $P$-equivariant: $(r_p)_* \omega = \text{Ad}(p)^{-1} \circ \omega$, $\forall p \in P$.

2. $\omega$ reproduces generators of fundamental vector fields $\omega(u) : T_u G \rightarrow g$ is a linear isomorphism for all $u \in G$.

The principal $P$-bundle $G \rightarrow G/P$ equipped with the Maurer Cartan form $\omega_{MC} \in \Omega^1(G, g)$ is called the homogeneous model of a Cartan geometry of type $(G, P)$.

If $G$ is semisimple and $P$ a parabolic subgroup, then a Cartan geometry of type $(G, P)$ is called a parabolic geometry of type $(G, P)$.
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- If $G$ is semisimple and $P$ a parabolic subgroup, then a Cartan geometry of type $(G, P)$ is called a **parabolic geometry** of type $(G, P)$. 
Curvature

The curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of a Cartan geometry $(\mathcal{G} \to M, \omega)$ is given by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

It is horizontal and $P$-equivariant.
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Natural vector bundles

Any \( P \)-module \( V \) gives rise to a vector bundle \( V = G \times P V \cong G \times V / \sim \), where \((u, v) \sim (u \cdot p, p^{-1} \cdot v) \forall p \in P \).

Any \( P \)-module homomorphism \( V \to W \) induces a vector bundle homomorphism \( V \to W \).

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- For the homogeneous model \((G \to G/P, \omega_{MC})\) of a Cartan geometry the curvature \( K \) vanishes identically.
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Natural vector bundles

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\mathcal{V} := G \times_P \mathcal{V} := G \times \mathcal{V} / \sim, \quad \text{where } (u, v) \sim (u \cdot p, p^{-1} \cdot v) \ \forall p \in P.
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- Any \( P \)-module homomorphism \( \mathcal{V} \to \mathcal{W} \) induces a vector bundle homomorphism \( \mathcal{V} \to \mathcal{W} \).
The Cartan connection induces an isomorphism as follows:

\[ G \times_P g/p \cong TM \]

\[ [u, X + p] \mapsto T_u p \omega^{-1}(X). \]

Consequently, all tensor bundles over \( M \) are associated vector bundles.
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Consequently, all tensor bundles over $M$ are associated vector bundles.

Since the curvature $K \in \Omega^2(G, g)$ is $P$-equivariant and horizontal, it can be equivalently viewed as section of

$$\Lambda^2 T^* M \otimes \mathcal{A}M \cong \mathcal{G} \times_P \Lambda^2 (g/p)^* \otimes g,$$

where $\mathcal{A}M = \mathcal{G} \times_P g$. 
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where \( \mathcal{A}M = G \times_P g \).

In this picture \( K \) corresponds to the following \( P \)-equivariant function

\[ \kappa : G \rightarrow \Lambda^2(g/p)^* \otimes g \]

\[ \kappa(u)(X + p, Y + p) = K(\omega^{-1}(X)(u), \omega^{-1}(X)(u)). \]
Prolongation procedures of Tanaka (1979), Morimoto (1993), and Čap-Schichl (2000)

Normalising the curvature of a regular parabolic geometry induces an equivalence of categories between regular normal parabolic geometries and certain underlying geometric structures, which admit descriptions in more conventional geometric terms.
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Consider the complex for computing the homology $H_*(p_+, g)$ of the nilradical $p_+$ of the parabolic subalgebra $p$ with values in $g$:

$$0 \leftarrow g \leftarrow \partial^* p_+ \otimes g \leftarrow \partial^* \Lambda^2 p_+ \otimes g \leftarrow ...$$
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- Since $\partial^*$ is $P$-equivariant, its induces bundle maps

  \[
  \partial^* : \Lambda^i T^* M \otimes AM \rightarrow \Lambda^{i-1} T^* M \otimes AM.
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- Since $\partial^*$ is $P$-equivariant, its induces bundle maps

  $\partial^* : \Lambda^i T^* M \otimes \mathcal{A}M \rightarrow \Lambda^{i-1} T^* M \otimes \mathcal{A}M$.

- A parabolic geometry is normal : $\iff \partial^* \kappa = 0.$
The curvature $\kappa$ of a normal parabolic geometry therefore gives rise to a $P$-equivariant function, called the **harmonic curvature**, $\kappa_h : \mathcal{G} \rightarrow H_2(p_+, g)$. 

$H_2(p_+, g)$ is a completely reducible $P$-module, which can be explicitly computed via Kostant's version of the Bott-Borel-Weil Theorem (1961).

For a regular normal parabolic geometry, it can be shown that $\kappa = 0 \iff \kappa_h = 0$. 

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• The curvature $\kappa$ of a normal parabolic geometry therefore gives rise to a $P$-equivariant function, called the **harmonic curvature**, $\kappa_h : \mathcal{G} \to H_2(p_+, g)$.

• $H_2(p_+, g)$ is a completely reducible $P$-module, which can be explicitly computed via Kostant’s version of the Bott-Borel-Weil Theorem (1961).
The curvature $\kappa$ of a normal parabolic geometry therefore gives rise to a $P$-equivariant function, called the harmonic curvature,

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For a regular normal parabolic geometry, it can be shown that

$$\kappa = 0 \quad \iff \quad \kappa_h = 0.$$
4. Almost H-projective structures as parabolic geometries

Consider $\mathbb{R}^{2(n+1)}$ endowed with the complex structure $J = \begin{pmatrix} J_2 & \cdots & J_2 \end{pmatrix}$ where $J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. 

$\text{gl}(n+1, \mathbb{C}) \sim = \left\{ A \in \text{gl}(2(n+1), \mathbb{R}) : A J = J A \right\} = \left\{ \begin{pmatrix} A_1, 1 & \cdots & A_1, n+1 \\ \vdots & \ddots & \vdots \\ A_n, 1 & \cdots & A_n, n+1 \end{pmatrix} : A_{i,j} = (a_{i,j} - b_{i,j}) \right\}$. 

Then $\text{sl}(n+1, \mathbb{C}) = \left\{ ( -\text{tr} \ C (A)) Z X A : A \in \text{gl}(n, \mathbb{C}), X \in \mathbb{C}^n, Z \in \mathbb{C}^n^\ast \right\}$. 

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4. Almost H-projective structures as parabolic geometries

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\[ J = \begin{pmatrix} J_2 & & \\ & \ddots & \\ & & J_2 \end{pmatrix} \]

where $J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. 
4. Almost H-projective structures as parabolic geometries

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$$\mathcal{J} = \begin{pmatrix} J_2 & \cdots & \cdots \\ \cdots & \ddots & \cdots \\ \cdots & \cdots & J_2 \end{pmatrix}$$

where $J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$$\text{gl}(n+1, \mathbb{C}) \cong \{ A \in \text{gl}(2(n+1), \mathbb{R}) : AJ = J A \} = \left\{ \begin{pmatrix} A_{1,1} & \cdots & A_{1,n+1} \\ \vdots & \ddots & \vdots \\ A_{n+1,1} & \cdots & A_{n+1,n+1} \end{pmatrix} : A_{i,j} = \begin{pmatrix} a_{i,j} & -b_{i,j} \\ b_{i,j} & a_{i,j} \end{pmatrix} \right\}.$$
4. Almost $H$-projective structures as parabolic geometries

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$$\text{gl}(n+1, \mathbb{C}) \cong \{ A \in \text{gl}(2(n+1), \mathbb{R}) : AJ = JA \} = \left\{ \begin{pmatrix} A_{1,1} & \cdots & A_{1,n+1} \\ \vdots & \ddots & \vdots \\ A_{n+1,1} & \cdots & A_{n+1,n+1} \end{pmatrix} : A_{i,j} = \begin{pmatrix} a_{i,j} & -b_{i,j} \\ b_{i,j} & a_{i,j} \end{pmatrix} \right\}.$$ Then

$$\text{sl}(n+1, \mathbb{C}) = \left\{ \begin{pmatrix} -\text{tr}_\mathbb{C}(A) & Z \\ X & A \end{pmatrix} : A \in \text{gl}(n, \mathbb{C}), X \in \mathbb{C}^n, Z \in \mathbb{C}^{n^*} \right\}.$$
Hence, $\mathfrak{sl}(n + 1, \mathbb{C})$ admits a $|1|$-grading as follows:

$$\mathfrak{sl}(n + 1, \mathbb{C}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_0 \cong \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{g}_{-1} \cong \mathbb{C}^n$ resp. $\mathfrak{g}_1 \cong \mathbb{C}^{n^*}$ as $\mathfrak{g}_0$-modules.
Hence, \( \mathfrak{sl}(n + 1, \mathbb{C}) \) admits a \(|1|\)-grading as follows:

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The subalgebra

\[
p := \mathfrak{g}_0 \oplus \mathfrak{g}_1
\]

is a parabolic subalgebra of \( \mathfrak{g} \) with abelian nilradical \( p_+ = \mathfrak{g}_1 \).
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The subalgebra

$$\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

is a parabolic subalgebra of $\mathfrak{g}$ with abelian nilradical $\mathfrak{p}_+ = \mathfrak{g}_1$.

Set $G := PSL(n + 1, \mathbb{C})$ and let $P$ be the stabiliser in $G$ of the complex line generated by the first standard basis vector of $\mathbb{R}^{2(n+1)}$. 

It follows that the adjoint action of $G_0$ on $\mathfrak{g}$ induces an isomorphism $G_0 \cong GL(\mathfrak{g}_{-1}, J) \cong GL(n, \mathbb{C})$. 

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Therefore, the Levi subgroup $G_0$ of $P$ consists of equivalence classes of matrices of the form

$$\begin{pmatrix} \det_C(C)^{-1} & 0 \\ 0 & C \end{pmatrix}$$

where $C \in GL(n, \mathbb{C})$.

It follows that the adjoint action of $G_0$ on $\mathfrak{g}$ induces an isomorphism $G_0 \cong GL(\mathfrak{g}_{-1}, J) \cong GL(n, \mathbb{C})$. 
Theorem (Yoshimatsu (1978), Hrdina (2009))

Suppose that $M$ is a manifold with $\dim_{\mathbb{R}}(M) = 2n > 2$. Then there is an equivalence of categories between

$$\{\text{Almost } H\text{-projective structures } (J, [\nabla]) \text{ on } M\}$$

$$\updownarrow \ 1:1$$

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Given an almost $H$-projective manifold $(M, J, [\nabla])$, then $J$ defines reduction of structure group

$$G_0 \xrightarrow{\phi} \mathcal{F}M$$

$$p_0 \downarrow \quad \downarrow q$$

$$M \xrightarrow{id} M$$

corresponding to the inclusion $G_0 \cong GL(g_{-1}, J) \hookrightarrow GL(g_{-1}) \cong GL(2n, \mathbb{R})$.
The bundle map $\phi$ can be encoded by a strictly horizontal $G_0$-equivariant 1-form $\theta \in \Omega^1(G_0, \mathfrak{g}_{-1})$. Any connection $\nabla$ in the $H$-projective class can be viewed as a principal connection $\gamma \nabla \in \Omega^1(G_0, g_0)$. For $u \in G_0$ set $G_u := \{ \gamma \nabla(u) : \nabla \in [\nabla] \}$ and $G := \bigsqcup u \in G_0 G_u$. The projection $p: G \to M$ is a principal $P$-bundle, where the right action of an element $g_0 \exp(Z) \in P$ on an element $\gamma \nabla(u) \in G_u$ is given by the following connection form at $u \cdot g_0$:

$$\xi \mapsto \gamma \nabla(u \cdot g_0)(\xi) + [Z, \theta(\xi)].$$

Let $\pi: G \to G_0$ be the natural projection. The tautological 1-form $\tau \in \Omega^1(G, g_{-1} \oplus g_0)$ given by $\tau(\gamma \nabla(u))(\eta) = (\theta + \gamma \nabla(u))(T \pi \eta)$ can be extended to a normal Cartan connection $\omega \in \Omega^1(G, g)$ (which is unique up to isomorphism).
The bundle map $\phi$ can be encoded by a strictly horizontal $G_0$-equivariant 1-form $\theta \in \Omega^1(G_0, g_{-1})$.

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$$G_u := \{ \gamma^\nabla(u) : \nabla \in [\nabla] \}$$

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5. The harmonic curvature

Recall that the harmonic curvature is a $P$-equivariant function $\kappa_h : G \to H^2(p, g)$. Since $H^2(p, g)$ is completely reducible, the harmonic curvature can be viewed as $G_0$-equivariant function $\kappa_h : G_0 \to H^2(p, g)$.

Consider the complex for computing the cohomology $H^*(g - , g)$:

$$0 \to g \to g^* \to \Lambda^2 g^* \to \ldots$$

The map $\partial$ is $G_0$-equivariant and so $H^*(g - , g)$ is naturally a $G_0$-module. Kostant showed that $\partial$ and $\partial^*$ are adjoint operators for some inner product on $\Lambda^i g^* \otimes g \cong G_0 \Lambda^i p + \otimes g$. Hence, one has an algebraic Hodge structure $\Lambda^i g^* \otimes g = \ker(\partial^*) \oplus \ker(\Box) \oplus \im(\partial^*)$, where $\Box := \partial \partial^* + \partial^* \partial$. 

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- Hence, one has a algebraic Hodge structure

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In particular, as $G_0$-module $H^i(g_-, g) \cong H_i(p_+, g)$ is isomorphic to $G_0$-submodule $\ker(\Box)$ in $\Lambda^i g^* \otimes g$.

Harmonic curvature of almost $H$-projective manifold
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Harmonic curvature of almost $H$-projective manifold

- $H^2_R(\mathfrak{g}_{-1}, \mathfrak{g}) = ?$
- We have $H^2_C(\mathfrak{g}_{-1}^\mathbb{C}, \mathfrak{g}^\mathbb{C}) \cong H^2_R(\mathfrak{g}_{-1}, \mathfrak{g}) \otimes_R \mathbb{C}$
In particular, as $G_0$-module $H^i(g_-, g) \cong H_i(p_+, g)$ is isomorphic to $G_0$-submodule $\ker(\square)$ in $\Lambda^i g^* \otimes g$.

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- $H^2_{\mathbb{R}}(g_{-1}, g) = ?$
- We have $H^2_{\mathbb{C}}(g^{\mathbb{C}}_{-1}, g^{\mathbb{C}}) \cong H^2_{\mathbb{R}}(g_{-1}, g) \otimes_{\mathbb{R}} \mathbb{C}$
- The Lie algebra $g = \mathfrak{sl}(n + 1, \mathbb{C})$ can be viewed as real form of the complex Lie algebra

\[ g \oplus g = \mathfrak{sl}(n + 1, \mathbb{C}) \oplus \mathfrak{sl}(n + 1, \mathbb{C}). \]
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- Hence, $H^2_{\mathbb{C}}(g_{-1}^{\mathbb{C}}, g^{\mathbb{C}}) \cong H^2_{\mathbb{C}}(g_{-1} \oplus g_{-1}, g \oplus g)$ as $g_0^\mathbb{C} \cong g_0 \oplus g_0$-module.
\[ H^2 (\text{for } n = 5) \]

\[ H^2_C(\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1}, \mathfrak{g} \oplus \mathfrak{g}) = \]

\[
\begin{array}{cccc}
-4 & 1 & 1 & 0 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0
\end{array}
\oplus
\begin{array}{cccc}
-4 & 1 & 1 & 0 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

\[
\begin{array}{cccc}
-3 & 2 & 0 & 0 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0
\end{array}
\oplus
\begin{array}{cccc}
-2 & 1 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
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\]

Hence, \( \kappa_h \) has values in three irreducible \( G_0 \)-modules. Correspondingly, we shall write \( \kappa_h = \mathcal{W}^{2,0} + \mathcal{W}^{1,1} + \mathcal{T} \).
The theory of parabolic geometries implies then the following:

1. \((M, J, \nabla)\) is an almost \(H\)-projective manifold if and only if \(T = 0\).

2. \((M, J, \nabla)\) is locally isomorphic to \(\mathbb{CP}^n\) with its canonical \(H\)-projective structure if and only if \(\kappa_h = 0\).

3. If \(T = 0\), then \(W(1, 1) = 0\) if and only if \((M, J, \nabla)\) is complex projective structure.

4. If \(T = 0\) and \(\nabla\) is \(K\)-\(\ddot{a}\)hlerisable, then \(W(2, 0) = 0\).

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The theory of parabolic geometries implies then the following:

**Interpretation of harmonic curvature; cf. David Calderbank’s unpublished notes on Hamiltonian 2-vectors**

Suppose that \((M, J, \nabla)\) is an almost \(H\)-projective manifold with \(n > 1\).

\[\begin{align*} 
1. & \quad (M, J, [\nabla]) \text{ is an } H\text{-projective manifold } \iff T = 0 \\
2. & \quad (M, J, [\nabla]) \text{ is locally isomorphic to } \mathbb{C}P^n \text{ with its canonical } H\text{-projective structure } \iff \kappa_h = 0. \\
3. & \quad \text{If } T = 0, \text{ then } W_{(1, 1)} = 0 \iff (M, J, [\nabla]) \text{ is complex projective structure.} \\
4. & \quad \text{If } T = 0 \text{ and } [\nabla] \text{ is Kählerisable, then } W_{(2, 0)} = 0.
\end{align*}\]
The theory of parabolic geometries implies then the following:

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### Interpretation of harmonic curvature; cf. David Calderbank’s unpublished notes on Hamiltonian 2-vectors

Suppose that $(M, J, [\nabla])$ is an almost $H$-projective manifold with $n > 1$.

1. $(M, J, [\nabla])$ is an $H$-projective manifold $\iff$ $T=0$

2. $(M, J, [\nabla])$ is locally isomorphic to $\mathbb{C}P^n$ with its canonical $H$-projective structure $\iff$ $\kappa_h=0$. (torsion-free case: Tashiro 1957)

3. If $T = 0$, then

$$W^{(1,1)} = 0 \iff (M, J, [\nabla]) \text{ is complex projective structure}.$$

In this case $W^{(2,0)}$ is Weyl curvature of complex projective manifold.
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4. If \(T = 0\) and \([\nabla]\) is \(K\ddot{a}hlerisable\), then \(W^{(2,0)} = 0\).
The Hodge decomposition

\[ \Lambda^i T^* M \otimes \gr(AM) = G_0 \times G_0 \Lambda^i g_{-1}^* \otimes g = \text{im}(\partial) \oplus \ker(\square) \oplus \text{im}(\partial^*) \]

implies that for any \( \nabla \in [\nabla] \):

\[ \exists! \quad P^\nabla \in \Omega^1(M, T^* M) \quad \text{s.t.} \quad \partial^*(R^\nabla - \partial P^\nabla) = 0. \]
The Hodge decomposition

\[ \Lambda^i T^* M \otimes \text{gr}(\mathcal{A}M) = \mathcal{G}_0 \times \mathcal{G}_0 \Lambda^i \mathfrak{g}^{-1} \otimes \mathfrak{g} = \text{im}(\partial) \oplus \ker(\Box) \oplus \text{im}(\partial^*) \]

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\( W^\nabla = R^\nabla - \partial P^\nabla \) is called the Weyl curvature and \( P^\nabla \) the Rho tensor of \( \nabla \).
• The Hodge decomposition

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• \( W^\nabla = R^\nabla - \partial P^\nabla \) is called the Weyl curvature and \( P^\nabla \) the Rho tensor of \( \nabla \).

• \( (\partial P^\nabla)_{abc}^d = \delta_{[a}{}^c P_{b]}^d - J_{[a}{}^c P_{b]}^e J^d_e - P_{[ab]}^c \delta^d_c - J_{[a}{}^e P_{b]}^c J^d_e \)
The Hodge decomposition

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- \( W^\nabla = R^\nabla - \partial P^\nabla \) is called the Weyl curvature and \( P^\nabla \) the Rho tensor of \( \nabla \).

- \( (\partial P^\nabla)_{ab}^c_d = \delta_{[a}^c P^\nabla_{b]d} - J_{[a}^c P^\nabla_{b]e} J_d^e - P^\nabla_{[ab]} \delta^c_d - J_{[a}^e P^\nabla_{b]e} J^c_d \)

- \( R^\nabla_{ab}^c_d = \)

\[ = W_{ab}^c_d + \delta_{[a}^c P^\nabla_{b]d} - J_{[a}^c P^\nabla_{b]e} J_d^e - P^\nabla_{[ab]} \delta^c_d - J_{[a}^e P^\nabla_{b]e} J^c_d \]
• The Hodge decomposition

\[ \Lambda^i T^* M \otimes \text{gr}(\mathcal{AM}) = \mathcal{G}_0 \times \mathcal{G}_0 \Lambda^i g_{-1} \otimes g = \text{im}(\partial) \oplus \ker(\Box) \oplus \text{im}(\partial^*) \]

implies that for any \( \nabla \in [\nabla] \):

\[ \exists! \ P^{\nabla} \in \Omega^1(M, T^* M) \quad \text{s.t.} \quad \partial^*(R^{\nabla} - \partial P^{\nabla}) = 0. \]

• \( W^{\nabla} = R^{\nabla} - \partial P^{\nabla} \) is called the Weyl curvature and \( P^{\nabla} \) the Rho tensor of \( \nabla \).

\[ (\partial P^{\nabla})_{ab}{}^c{}^d = \delta_{[a}{}^c P_{b]}{}^d - J_{[a}{}^c P_{b]}{}^e J^d{}^e - P^{\nabla}{}_{[ab]} \delta^c{}^d - J_{[a}{}^e P_{b]}{}^e J^c{}^d \]

• \( R^{\nabla}{}_{ab}{}^c{}^d = (\partial P^{\nabla})_{ab}{}^a{}^d = \frac{2n+1}{2} P^{\nabla}{}_{bd} - \frac{1}{2} P^{\nabla}{}_{db} + J_{(b}{}^e J_{d)}{}^f P^{\nabla}{}_{fe} \)
\[ R_{(bd)}^{\nabla} = nP_{(bd)}^{\nabla} + J_{(b^e J_d)^f} P_{fe}^{\nabla} \]
\[ R_{(bd)} = nP_{(bd)} + J(b^e J_d)^f P_{fe} \]
\[ R_{[bd]} = (n + 1)P_{[bd]} \]
\item $R_{(bd)}^{\nabla} = nP_{(bd)}^{\nabla} + J(b^e J_d)^f P_{fe}^{\nabla}$
\item $R_{[bd]}^{\nabla} = (n + 1)P_{[bd]}^{\nabla}$
\item $P_{bd}^{\nabla} = \frac{1}{n+1} R_{bd}^{\nabla} + \frac{1}{(n+1)(n-1)} (R_{(bd)}^{\nabla} - J(b^e J_d)^f R_{fe}^{\nabla})$
\[ R_{(bd)}^{\nabla} = n P_{(bd)}^{\nabla} + J(b^e J_d)^f P_{fe}^{\nabla} \]
\[ R_{[bd]}^{\nabla} = (n + 1) P_{[bd]}^{\nabla} \]
\[ P_{bd}^{\nabla} = \frac{1}{n+1} R_{bd}^{\nabla} + \frac{1}{(n+1)(n-1)} \left( R_{(bd)}^{\nabla} - J(b^e J_d)^f R_{fe}^{\nabla} \right) \]

How does \( P \) and \( W \) change when one changes \( H \)-projectively?
\[
R_{(bd)}^\nabla = nP_{(bd)}^\nabla + J(b^e J_d)^f P_{fe}^\nabla \\
R_{[bd]}^\nabla = (n + 1)P_{[bd]}^\nabla \\
P_{bd}^\hat{\nabla} = \frac{1}{n+1} R_{bd}^\nabla + \frac{1}{(n+1)(n-1)} (R_{(bd)}^\nabla - J(b^e J_d)^f R_{fe}^\nabla)
\]

How does $P$ and $W$ change when one changes $\nabla$ $H$-projectively?

\[
P_{ab}^\hat{\nabla} = P_{ab}^\nabla - 2\nabla_a \gamma_b + 2(\gamma_a \gamma_b - J_a^e J_b^f \gamma_e \gamma_f)
\]
\[ R_{(bd)}^\nabla = nP_{(bd)}^\nabla + J(b^e J_d)^f P_{fe} \]
\[ R_{[bd]}^\nabla = (n + 1)P_{[bd]} \]
\[ P_{bd}^\nabla = \frac{1}{n+1} R_{bd}^\nabla + \frac{1}{(n+1)(n-1)} (R_{(bd)}^\nabla - J(b^e J_d)^f R_{fe}^\nabla) \]

How does \( P \) and \( W \) change when one changes \( \nabla \) H-projectively?

- \( \hat{P}_{ab} = P_{ab}^\nabla - 2\nabla_a \gamma_b + 2(\gamma_a \gamma_b - J_a^e J_b^f \gamma_e \gamma_f) \)

- \( W^\nabla \) is a two form with values in the complex vector bundle \( \mathfrak{gl}(TM, J) \) and hence we can decompose it into types as follows:

\[ W^\nabla = W^{2,0} + W^{1,1} + W^{0,2} \]
\[ R_{(bd)} = nP_{(bd)} + J(b^e J_d)^f P_{fe} \]

\[ R_{[bd]} = (n + 1)P_{[bd]} \]

\[ P_{bd} = \frac{1}{n+1} R_{bd} + \frac{1}{(n+1)(n-1)} (R_{(bd)} - J(b^e J_d)^f R_{fe}) \]

How does \( P \) and \( W \) change when one changes \( \nabla \) \( H \)-projectively?

\[ P^\hat{\nabla}_{ab} = P_{ab} - 2\nabla_a \gamma_b + 2(\gamma_a \gamma_b - J_a{}^e J_b{}^f \gamma_e \gamma_f) \]

\( W^\nabla \) is a two form with values in the complex vector bundle \( \mathfrak{gl}(TM, J) \) and hence we can decompose it into types as follows:

\[ W^\nabla = W^{2,0} + W^{1,1} + W^{0,2}. \]

\[ W^\hat{\nabla}_{ab}^c{}^d = \]

\[ = W_{ab}^\nabla{}^c{}^d + T_{ab}{}^e \gamma_e \delta^c{}^d + T_{ab}{}^c \gamma_d - J_e{}^f T_{ab}{}^e \gamma_f J^c{}^d - J_e{}^c T_{ab}{}^e J_d{}^f \gamma_f. \]

is of type \((0,2)\).
\[
R_{(bd)}^\nabla = nP_{(bd)}^\nabla + J(b^e J_d)^f P_{fe}^\nabla \\
R_{[bd]}^\nabla = (n + 1)P_{[bd]}^\nabla \\
P_{bd}^\nabla = \frac{1}{n+1} R_{bd}^\nabla + \frac{1}{(n+1)(n-1)}(R_{(bd)}^\nabla - J(b^e J_d)^f R_{fe}^\nabla)
\]

How does \( P \) and \( W \) change when one changes \( \nabla \) \( H \)-projectively?

- \( \hat{P}_{ab} = P_{ab}^\nabla - 2\nabla_a \gamma_b + 2(\gamma_a \gamma_b - J(a^e J_b)^f \gamma_e \gamma_f) \)

- \( \hat{W}_{\nabla}^{\hat{\nabla}} = W_{\nabla}^{\nabla} \)
- \( W_{\nabla}^{\hat{\nabla}} = W_{ab}^{\hat{\nabla}} - 2\nabla_a \gamma_b + 2(\gamma_a \gamma_b - J(a^e J_b)^f \gamma_e \gamma_f) \)
- \( W_{ab}^{\hat{\nabla}} = W_{ab}^{\nabla} + T_{ab}^e \gamma_e \delta^c_d + T_{ab}^c \gamma_d - J_e^f T_{ab}^e \gamma_f J_d^c - J_e^c T_{ab}^e J_d^f \gamma_f \)
- \( W_{ab}^{\hat{\nabla}} \) is of type \((0,2)\).

- The components \( W^{2,0} \) and \( W^{1,1} \) are independent of the choice of the connection in \( \nabla \). These are the two harmonic curvature components.