4D-Kähler metrics admitting essential h-projective vector fields

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(based on joint works with D. Calderbank, V. S. Matveev, T. Mettler)

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Trivial examples: Affine transformations (such as isometries, homotheties) are h-projective.

An h-projective transformation is called essential if it is not an affine transformation.

Example: For every $A \in \text{Gl}(n+1, \mathbb{C})$ the induced mapping $f_A : \mathbb{C}P(n) \rightarrow \mathbb{C}P(n)$ is h-projective for $g_{\text{Fubini-Study}}$. It is not an isometry unless $A$ is proportional to unitary matrix.

That's similar to Beltrami's construction: For $A \in \text{Gl}(n+1, \mathbb{R})$, the mapping $f_A : S^n \rightarrow S^n, f_A(x) = \frac{Ax}{||Ax||}$, is projective transformation for $g_{\text{round}}$. 

An h-projective vector field is a vector field whose local flow consists of h-projective transformations.

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In fact, the analogous problem in 2\(D\)-projective geometry has become known as “Lie problem”: it was posed by Sophus Lie and solved by Matveev in 2012 (in the case that the metric admits exactly one projective vector field but no infinitesimal homothety) and Bryant, Matveev, Manno in 2008 (where the assumption was that there are at least two linearly independent projective vector fields).
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- The “\(h\)-projective Lie-Problem” was mentioned explicitly in our application for the joint project Canberra-Jena.
Matveev, R~, 2012:

*The only closed connected Kähler 2n-manifold with essential h-projective vector field is* $(\mathbb{C}P(n), \text{const} \cdot g_{\text{Fubini–Study}}, J_{\text{standard}})$. 
Special cases: The case of degree of mobility $\geq 3$

- Classical (Mikes, Domashev 1978): Let $(M, g, J)$ be Kähler $2n$-manifold. Then, the metrics $\tilde{g}$, h-projectively equivalent to $g$, correspond to non-degenerate symmetric hermitian $(2, 0)$-tensors $A$ satisfying

$$(*) \quad \nabla_k A^{ij} = \delta_k^i \Lambda^j + J_k^{(i} J_l^{j)} \Lambda^l,$$

where $\Lambda^i = \frac{1}{2n} \nabla_j A^{jj}$. 

- The degree of mobility of $g$ is the dimension of the space of solutions of $(*)$.

- The correspondence is given by $A = (\det \tilde{g} / \det g)^{1/2} (n + 1) \tilde{g}^{-1}$.

- Of course, $(*)$ is the non-invariant version of the h-projectively invariant PDE governing metrizability of h-projective structures.

Matveev, R., 2012: Let $(M, g, J)$ be Kähler $2n$-manifold, $n > 1$ and let $D(g) \geq 3$. Then, the h-projective vector fields of $g$ correspond to the affine vector fields on the "conification" $\hat{M} = \mathbb{R}^+ \times \mathbb{R} \times M$, $\hat{g} = dr^2 + r^2 ((dt - 2\tau)^2 + g)$, $\hat{J}$ over $(M, g, J)$, where $\tau$ is a one form on $M$ such that $d\tau = g(J, J)$. 

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  Let $(M, g, J)$ be a connected 4D-Kähler manifold (of any signature). If $D(g) \geq 3$, the metric $g$ has constant holomorphic sectional curvature.

⇒ Having an essential h-projective vector field in 4D and throwing away the trivial flat case, we are working with $D(g) = 2$.

However, it remains to be true for arbitrary dimension (in any signature) if $M$ is assumed to be closed (FKMR 2011).
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There are several equivalent formulations of “trivial” \(h\)-projective equivalence:

\(g, \bar{g}\) affinely equivalent \(\iff\) \(A\) parallel \(\iff\) \(\Lambda \equiv 0 \iff\) all eigenvalues of \(A\) are constant.
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In \( 4D \), a (non-trivial=non-parallel) solution \( A \) of \((*)\), can have either

- Case 1: two non-constant eigenvalues \( \rho_1, \rho_2 \),
- Case 2: a non-constant eigenvalue \( \rho \) and a constant eigenvalue \( c \).
David and his coworkers classified the Kähler structures \((g, J)\) admitting solutions of \((\ast)\). In 4\(D\), a (non-trivial) solution \(A\) of \((\ast)\), can have either

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In a neighborhood of almost every point, the corresponding normal forms of \((g, J)\) are:

**Case 1:** There are coordinates \(\rho_1, \rho_2, t_1, t_2\) and functions \(F_1, F_2\) of one variable such that

\[
g = \rho_1 - \rho_2 F_1(\rho_1) d\rho_1 + \rho_2 - \rho_1 F_2(\rho_2) d\rho_2 + F_1(\rho_1) \rho_1 - \rho_2 (dt_1 + \rho_2 dt_2)^2 + F_2(\rho_2) \rho_2 - \rho_1 (dt_1 + \rho_1 dt_2)^2,
\]

\[
J\frac{\partial}{\partial \rho_1} = F_1(\rho_1) \rho_1 - \rho_2 \theta, 
\]

\[
J\frac{\partial}{\partial \rho_2} = F_2(\rho_2) \rho_2 - \rho_1 \theta,
\]

where \((h, i, \Omega = h(i, .), .)\) is a 2\(D\) Kähler structure and \(\theta\) is a 1-form on \(M\) satisfying \(d\theta = -\Omega\). **Case 1** is parameterized by arbitrary functions \(F_1, F_2\). **Case 2** is parameterized by an arbitrary function \(F\) and a 2\(D\) metric \(h\).
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g &= \rho_1 - \rho_2 F_1(\rho_1) d\rho_2^1 + \rho_2 - \rho_1 F_2(\rho_2) d\rho_2^2 + F_1(\rho_1) \rho_1 - \rho_2 (dt_1 + \rho_2 dt_2)^2 + F_2(\rho_2) \rho_2 - \rho_1 (dt_1 + \rho_1 dt_2)^2,
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\]

\[
\begin{align*}
Jd\rho_1 &= F_1(\rho_1) \rho_1 - \rho_2 (dt_1 + \rho_2 dt_2),
Jd\rho_2 &= F_2(\rho_2) \rho_2 - \rho_1 (dt_1 + \rho_1 dt_2).
\end{align*}
\]

**Case 2:** There is a function \(F\) of one variable such that
\[
\begin{align*}
g &= (c - \rho) h + \rho - c F(\rho) d\rho^2 + F(\rho) \rho - c \theta^2,
\end{align*}
\]

\[
\begin{align*}
Jd\rho &= F(\rho) \rho - c \theta,
J\theta &= -\rho - c F(\rho) d\rho.
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The 4D-normal forms of Calderbank et al

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  - Case 1: There are coordinates \(\rho_1, \rho_2, t_1, t_2\) and functions \(F_1, F_2\) of one variable such that \((g, J)\) is given by
    
    \[
    g = \frac{\rho_1 - \rho_2}{F_1(\rho_1)} \, d\rho_1^2 + \frac{\rho_2 - \rho_1}{F_2(\rho_2)} \, d\rho_2^2 + \frac{F_1(\rho_1)}{\rho_1 - \rho_2} (dt_1 + \rho_2 dt_2)^2 + \frac{F_2(\rho_2)}{\rho_2 - \rho_1} (dt_1 + \rho_1 dt_2)^2, \\
    Jd\rho_1 = \frac{F_1(\rho_1)}{\rho_1 - \rho_2} (dt_1 + \rho_2 dt_2), \quad Jd\rho_2 = \frac{F_2(\rho_2)}{\rho_2 - \rho_1} (dt_1 + \rho_1 dt_2).
    \]
The 4D-normal forms of Calderbank et al

David and his coworkers classified the Kähler structures \((g, J)\) admitting solutions of (\(*\)).

In 4D, a (non-trivial) solution \(A\) of (\(*\)), can have either

- Case 1: two non-constant eigenvalues \(\rho_1, \rho_2\),
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g = \frac{\rho_1 - \rho_2}{F_1(\rho_1)} d\rho_1^2 + \frac{\rho_2 - \rho_1}{F_2(\rho_2)} d\rho_2^2 + \frac{F_1(\rho_1)}{\rho_1 - \rho_2} (dt_1 + \rho_2 dt_2)^2 + \frac{F_2(\rho_2)}{\rho_2 - \rho_1} (dt_1 + \rho_1 dt_2)^2,
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Jd\rho_2 = \frac{F_2(\rho_2)}{\rho_2 - \rho_1} (dt_1 + \rho_1 dt_2).
\]

### Case 2:
There is a function \(F\) of one variable such that \((g, J)\) is given by

\[
g = (c - \rho) h + \frac{\rho - c}{F(\rho)} d\rho^2 + \frac{F(\rho)}{\rho - c} \theta^2,
\]

\[
Jd\rho = \frac{F(\rho)}{\rho - c} \theta, \quad J\theta = -\frac{\rho - c}{F(\rho)} d\rho,
\]

where \((h, i, \Omega = h(i, .))\) is a 2D Kähler structure and \(\theta\) is a 1-form on \(M\) satisfying \(d\theta = -\Omega\).
David and his coworkers classified the Kähler structures \((g, J)\) admitting solutions of \((\ast)\).

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- Case 1: There are coordinates \(\rho_1, \rho_2, t_1, t_2\) and functions \(F_1, F_2\) of one variable such that \((g, J)\) is given by
  
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Case 1 is parameterized by arbitrary functions \(F_1, F_2\).

Case 2 is parameterized by an arbitrary function \(F\) and a 2D metric \(h\).
Theorem:
Let \((g, J)\) be 4D-Kähler structure of non-constant holomorphic section curvature with essential h-projective vector field \(v\). Then, locally \((g, J)\) and \(v\) are given by Case 1 or Case 2 below:
Theorem:
Let \((g, J)\) be 4D-Kähler structure of non-constant holomorphic section curvature with essential \(h\)-projective vector field \(v\). Then, locally \((g, J)\) and \(v\) are given by Case 1 or Case 2 below:

**Case 1:** In local coordinates \(\rho_1, \rho_2, t_1, t_2\), the Kähler structure is given by

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g = \frac{\rho_1 - \rho_2}{F_1(\rho_1)} \, d\rho_1^2 + \frac{\rho_2 - \rho_1}{F_2(\rho_2)} \, d\rho_2^2 + \frac{F_1(\rho_1)}{\rho_1 - \rho_2} (dt_1 + \rho_2 \, dt_2)^2 + \frac{F_2(\rho_2)}{\rho_2 - \rho_1} (dt_1 + \rho_1 \, dt_2)^2,
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\]

where the functions \(F_i\) are given by one of the following subcases depending on the sign of \(\frac{\beta^2}{4} - \alpha\) for certain constants \(\alpha, \beta\):

Subcase 1.1: If \(\alpha - \frac{\beta^2}{4} = 0\) we have

\[
F_i(\rho_i) = (-1)^i c_i |\rho_i + \frac{\beta}{2}|^3 e^{-\frac{3}{4} \beta^2 \rho_i + \frac{\beta}{2}}, \quad c_i > 0
\]

Subcase 1.2: If \(\frac{\beta^2}{4} - \alpha > 0\) we have

\[
F_i(\rho_i) = (-1)^i c_i \left(\frac{1}{d^2}(\rho_i + \frac{\beta}{2}) + \frac{1}{2}\right)^3 e^{\frac{3}{4} \beta^2 \rho_i - \frac{\beta}{2}}, \quad c_i > 0
\]

Subcase 1.3: If \(\frac{\beta^2}{4} - \alpha > 0\) we have

\[
F_i(\rho_i) = (-1)^i c_i \left|\frac{1}{d^2}(\rho_i + \frac{\beta}{2}) - \frac{1}{2}\right| \left|\frac{1}{d}(\rho_i + \frac{\beta}{2}) - \frac{1}{2}\right|^3 e^{-\frac{3}{4} \beta^2 \rho_i + \frac{\beta}{2}}, \quad c_i > 0
\]

Moreover, \(v\) takes the form (up to adding constant linear combinations of \(\partial / \partial t_1, \partial / \partial t_2\))

\[
v = \left(\rho_1^{\frac{1}{2}} + \beta \rho_1 + \alpha\right) \frac{\partial}{\partial \rho_1} + \left(\rho_2^{\frac{1}{2}} + \beta \rho_2 + \alpha\right) \frac{\partial}{\partial \rho_2} + \left(-\beta t_1 - \alpha t_2\right) \frac{\partial}{\partial t_1} + \left(t_1 - 2\beta t_2\right) \frac{\partial}{\partial t_2}.
\]
Theorem:
Let \((g, J)\) be 4D-Kähler structure of non-constant holomorphic section curvature with essential h-projective vector field \(v\). Then, locally \((g, J)\) and \(v\) are given by Case 1 or Case 2 below:

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\[
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Jd\rho_1 = \frac{F_1(\rho_1)}{\rho_1 - \rho_2} (dt_1 + \rho_2 dt_2),
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F_i(\rho_i) = (-1)^i c_i |\rho_i + \beta/2|^3 e^{-\frac{3\beta}{2} \rho_i + \beta/2} , \text{ where } c_i > 0 \text{ is a constant.}
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Theorem:
Let \((g, J)\) be 4D-Kähler structure of non-constant holomorphic section curvature with essential h-projective vector field \(v\). Then, locally \((g, J)\) and \(v\) are given by Case 1 or Case 2 below:

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  g = \frac{\rho_1 - \rho_2}{F_1(\rho_1)} d\rho_1^2 + \frac{\rho_2 - \rho_1}{F_2(\rho_2)} d\rho_2^2 + \frac{F_1(\rho_1)}{\rho_1 - \rho_2} (dt_1 + \rho_2 dt_2)^2 + \frac{F_2(\rho_2)}{\rho_2 - \rho_1} (dt_1 + \rho_1 dt_2)^2,
  \]

  \[
  J d\rho_1 = \frac{F_1(\rho_1)}{\rho_1 - \rho_2} (dt_1 + \rho_2 dt_2),
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    \[
    F_i(\rho_i) = (-1)^i c_i |\rho_i + \beta/2|^3 e^{-\frac{3\beta}{2} \frac{1}{|\rho_i + \beta|^2}}, \text{ where } c_i > 0 \text{ is a constant.}
    \]

  - **Subcase 1.2:** If \(d^2 = \alpha - \beta^2/4 > 0\) we have

    \[
    F_i(\rho_i) = (-1)^i c_i (\frac{1}{d^2} (\rho_i + \beta/2)^2 + 1)^{\frac{3\beta}{2d^2}} \cdot \text{arctan}(\frac{1}{d}(\rho_i + \beta/2)), \text{ where } c_i > 0 \text{ is a constant.}
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Let \((g, J)\) be 4D-Kähler structure of non-constant holomorphic section curvature with essential h-projective vector field \(v\). Then, locally \((g, J)\) and \(v\) are given by Case 1 or Case 2 below:

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  \[
  F_i(\rho_i) = (-1)^i c_i |\rho_i + \beta/2|^{3/2} e^{-\frac{3\beta}{2} \frac{1}{\rho_i + \beta/2}}, \text{ where } c_i > 0 \text{ is a constant.}
  \]

- **Subcase 1.2:** If \(d^2 = \alpha - \beta^2/4 > 0\) we have
  
  \[
  F_i(\rho_i) = (-1)^i c_i \left( \frac{1}{d^2} (\rho_i + \beta/2)^2 + 1 \right) \frac{3\beta}{2d} \arctan \left( \frac{1}{d} (\rho_i + \beta/2) \right), \text{ where } c_i > 0 \text{ is a constant.}
  \]

- **Subcase 1.3:** If \(d^2 = \beta^2/4 - \alpha > 0\) we have
  
  \[
  F_i(\rho_i) = (-1)^i c_i \left( \frac{1}{d^2} (\rho_i + \beta/2)^2 - 1 \right) \frac{3\beta}{2d} \frac{1}{\frac{1}{d} (\rho_i + \beta/2) + 1} \left( \frac{3\beta}{4d} \right), \text{ where } c_i > 0 \text{ is a constant.}
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  \]

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    F_i(\rho_i) = (-1)^i c_i |\rho_i + \beta/2|^{3/2} e^{-3\beta \rho_i/2} |\rho_i + \beta/2|^{\frac{1}{2}}, \text{ where } c_i > 0 \text{ is a constant.}
    \]

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    \[
    F_i(\rho_i) = (-1)^i c_i \left( \frac{1}{d^2} (\rho_i + \beta/2)^2 + 1 \right)^{\frac{3}{2}} e^{\frac{3\beta}{2d} \arctan(\frac{1}{d}(\rho_i + \beta/2))}, \text{ where } c_i > 0 \text{ is a constant.}
    \]

  - **Subcase 1.3:** If \(d^2 = \frac{\beta^2}{4} - \alpha > 0\) we have

    \[
    F_i(\rho_i) = (-1)^i c_i \left( \frac{1}{d^2} (\rho_i + \beta/2)^2 - 1 \right)^{\frac{3}{2}} \left| \frac{1}{d}(\rho_i + \beta/2) - 1 \right|^{\frac{3\beta}{4d}} \left| \frac{1}{d}(\rho_i + \beta/2) + 1 \right|^{\frac{3\beta}{4d}}, \text{ where } c_i > 0 \text{ is a constant.}
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Moreover, \(v\) takes the form (up to adding constant linear combinations of \(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\))

\[
\nu = (\rho_1^2 + \beta \rho_1 + \alpha) \frac{\partial}{\partial \rho_1} + (\rho_2^2 + \beta \rho_2 + \alpha) \frac{\partial}{\partial \rho_2} + (-\beta t_1 - \alpha t_2) \frac{\partial}{\partial t_1} + (t_1 - 2\beta t_2) \frac{\partial}{\partial t_2}.
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Theorem:
Let \((g, J)\) be 4D-Kähler structure of non-constant holomorphic section curvature with essential h-projective vector field \(v\). Then, locally \((g, J)\) and \(v\) are given by Case 1 or Case 2 below:

**Case 2:** There are functions \(\rho, F, a\) 2D-Kähler structure \((h, i)\) and a 1-form \(\theta\) on \(M\) with \(d\theta = -h(i, .)\) such that \((g, J)\) is given by

\[
\begin{align*}
g &= -\rho h + \frac{\rho}{F(\rho)} d\rho^2 + \frac{F(\rho)}{\rho} \theta^2, \\
Jd\rho &= \frac{F(\rho)}{\rho} \theta, \\
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\end{align*}
\]

where \(F\) is given by one of the following subcases for certain constants \(\beta, C\):

Subcase 2.1: If \(\beta \neq 0\), we obtain
\[
F(\rho) = D(\rho + \beta) C + \beta \beta \rho^2 - C \beta.
\]

Subcase 2.2: If \(\beta = 0\), we obtain
\[
F(\rho) = D \rho^3 e C + \beta \rho.
\]

Moreover, \(v\) takes the form
\[
v = \rho (\rho + \beta) \partial/\partial \rho + v_h \partial/\partial \theta + v_h.
\]
Here \(v_h\) is a homothety for \(h\), \(L v_h h = Ch\), which is lifted to the distribution \(\ker \theta \cap \ker d \rho\) in the above formula.

The vertical component \(v_\theta : M \to \mathbb{R}\) satisfies the PDE
\[
d v_\theta = C_\theta + i(v_h) b
\]
for the closed 1-form \(C_\theta + i(v_h) b\), where \((v_h)_b = h(v_h, .).

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  \[
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    \[F(\rho) = D \rho^3 e^{\frac{C + \beta}{\rho}}.\]

Moreover, \(v\) takes the form

\[v = \rho(\rho + \beta) \frac{\partial}{\partial \rho} + \nu^\theta \frac{\partial}{\partial \theta} + \nu_h.\]

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\[
v = \rho(\rho + \beta) \frac{\partial}{\partial \rho} + v^\theta \frac{\partial}{\partial \theta} + v_h.
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\[
dv^\theta = C\theta + i(v_h)^b
\]

for the closed 1-form \(C\theta + i(v_h)^b\), where \((v_h)^b = h(v_h, .)\).
Summary:

- Case 1: There are only three cases for a 4D-Kähler structure having an essential h-projective vector.
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- Case 1: There are only three cases for a 4D-Kähler structure having an essential h-projective vector.
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Summary:

- Case 1: There are only three cases for a 4D-Kähler structure having an essential h-projective vector.
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Plan for the remaining talk:

- H-projectively invariant version of the main equation.
Summary:

- Case 1: There are only three cases for a $4D$-Kähler structure having an essential $h$-projective vector.
- Case 2: The $4D$-Kähler structures admitting an essential $h$-projective vector field are parametrized by $2D$-Riemannian metrics $h$ with homothety $v_h$. For every choice of such data, there remain two cases for the Kähler structures $(g, J)$.

Plan for the remaining talk:

- H-projectively invariant version of the main equation.
- PDE system for the Kähler structure and the $h$-projective vector field.
Summary:

- Case 1: There are only three cases for a 4D-Kähler structure having an essential h-projective vector.
- Case 2: The 4D-Kähler structures admitting an essential h-projective vector field are parametrized by 2D-Riemannian metrics $h$ with homothety $\nu_h$. For every choice of such data, there remain two cases for the Kähler structures $(g, J)$.

Plan for the remaining talk:

- H-projectively invariant version of the main equation.
- PDE system for the Kähler structure and the h-projective vector field.
- H-projectively invariant distributions and splitting of the PDE system.
Let $\nabla$ be a complex torsion-free connection on $(M, J)$ and denote by $S_J^2 TM$ the bundle whose sections are symmetric hermitian $(2, 0)$-tensors.
H-projectively invariant point of view

Let $\nabla$ be a complex torsion-free connection on $(M, J)$ and denote by $S^2_j TM$ the bundle whose sections are symmetric hermitian $(2, 0)$-tensors. We consider the PDE

$$(*') \quad \nabla_k \sigma^{ij} = \frac{1}{2n} (\delta_{k}^{(i} \nabla_l \sigma^{j)l} + J_{k}^{(i} J_{j}^{j)} \nabla_l \sigma^{lm})$$

on sections $\sigma$ of $S^2_j TM \otimes (\wedge^{2n} T^* M)^{\frac{1}{n+1}}$. 

Fixing a "background metric $g$", we can identify solutions $\sigma$ of $(*)'$ with that of $(*)$ via $\sigma \mapsto \sigma = \sigma - 1 \frac{g}{\det g} \sigma^{ij} = \frac{1}{2n} (\delta_{k}^{(i} \nabla_l \sigma^{j)l} + J_{k}^{(i} J_{j}^{j)} \nabla_l \sigma^{lm})$. 

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H-projectively invariant point of view

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on sections $\sigma$ of $S^2_J TM \otimes (\wedge^{2n} T^* M)^{\frac{1}{n+1}}$.

The equation above is independent of the choice of the connection in the h-projective class $[\nabla]$.
Let $\nabla$ be a complex torsion-free connection on $(M, J)$ and denote by $S^2_J TM$ the bundle whose sections are symmetric hermitian $(2, 0)$-tensors. We consider the PDE

$$\nabla_k \sigma^{ij} = \frac{1}{2n} (\delta^i_k \nabla_l \sigma^{jl} + J^i_k J^j_l \nabla_l \sigma^{lm})$$

on sections $\sigma$ of $S^2_J TM \otimes (\wedge^{2n} T^* M)^{1/2n+1}$.

The equation above is independent of the choice of the connection in the h-projective class $[\nabla]$.

The non-degenerate solutions $\sigma$ correspond to Kähler metrics $g$ contained in $[\nabla]$. The correspondence is given by

$$\sigma_g = g^{-1} \left( \det g \right)^{1/(2(n+1))}.$$

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H-projectively invariant point of view

Let $\nabla$ be a complex torsion-free connection on $(M, J)$ and denote by $S^2_J TM$ the bundle whose sections are symmetric hermitian $(2, 0)$-tensors. We consider the PDE

\[
(\star)' \quad \nabla_k \sigma^{ij} = \frac{1}{2n} (\delta_k^i \nabla_l \sigma^{jl} + J_k^i J_l^j \nabla_l \sigma^{im})
\]

on sections $\sigma$ of $S^2_J TM \otimes (\wedge^{2n} T^* M)^{1 \over n+1}$.

The equation above is independent of the choice of the connection in the h-projective class $[\nabla]$.

The non-degenerate solutions $\sigma$ correspond to Kähler metrics $g$ contained in $[\nabla]$. The correspondence is given by

\[
\sigma_g = g^{-1} (\det g)^{1 \over 2(n+1)}.
\]

$(\star)'$ is the h-projectively invariant version of

\[
(\star) \quad \nabla_k A^{ij} = \delta_k^i \Lambda^j + J_k^i J_l^j \Lambda^l \text{ (contraction yields } \Lambda^i = \frac{1}{2n} \nabla_k A^{ki} \text{).}
\]
H-projectively invariant point of view

Let $\nabla$ be a complex torsion-free connection on $(M, J)$ and denote by $S^2_J TM$ the bundle whose sections are symmetric hermitian $(2, 0)$-tensors. We consider the PDE

$$(\ast)' \quad \nabla_k \sigma^{ij} = \frac{1}{2n} \left( \delta^{(i}_k \nabla_l \sigma^j)^l + J^{(i}_k J^{j)}_m \nabla_l \sigma^{lm} \right)$$

on sections $\sigma$ of $S^2_J TM \otimes (\wedge^{2n} T^* M)^{1 \over n+1}$.

The equation above is independent of the choice of the connection in the h-projective class $[\nabla]$.

The non-degenerate solutions $\sigma$ correspond to Kähler metrics $g$ contained in $[\nabla]$. The correspondence is given by

$$\sigma_g = g^{-1} (\det g)^{\frac{1}{2(n+1)}}.$$

$$(\ast)'$$ is the h-projectively invariant version of

$$(\ast) \quad \nabla_k A^{ij} = \delta^{(i}_k \Lambda^j) + J^{(i}_k J^{j)}_l \Lambda^l \quad \text{(contraction yields } \Lambda^i = \frac{1}{2n} \nabla_k A^{ki}).$$

Fixing a “background metric $g$”, we can identify solutions $\sigma$ of $(\ast)'$ with that of $(\ast)$ via

$$\sigma \longmapsto A = \sigma \sigma_g^{-1}.$$
1st order PDE system for \((g, J)\) and the h-projective vector field \(\nu\)

An h-projective vector field \(\nu\) preserves the space of solutions \(\text{Sol}(\nabla)\) of

\[
(\ast)' \quad \nabla_k \sigma^{ij} = \frac{1}{2n} (\delta_k^i \nabla_l \sigma^{jl} + J_k^j J_m^l \nabla_l \sigma^{lm}),
\]

i.e. the Lie derivative \(\mathcal{L}_\nu\) is an endomorphism

\[
\mathcal{L}_\nu : \text{Sol}(\nabla) \rightarrow \text{Sol}(\nabla).
\]
An h-projective vector field $v$ preserves the space of solutions $\text{Sol}([\nabla])$ of

$$(*)' \quad \nabla_k \sigma^{ij} = \frac{1}{2n} (\delta_k^i \nabla_l \sigma^{jl} + J_k^l \nabla_l \sigma^{jm}),$$

i.e. the Lie derivative $\mathcal{L}_v$ is an endomorphism

$$\mathcal{L}_v : \text{Sol}([\nabla]) \rightarrow \text{Sol}([\nabla]).$$

(this is because when $v = \partial_x$, we can find a connection $\hat{\nabla} \in [\nabla]$ whose Christoffel symbols $\hat{\Gamma}^i_{jk}$ do not depend on $x$)
1st order PDE system for \((g, J)\) and the h-projective vector field \(v\)

- An h-projective vector field \(v\) preserves the space of solutions \(\text{Sol}([\nabla])\) of
  \[
  (\ast)' \quad \nabla_k \sigma^{ij} = \frac{1}{2n} (\delta_k^i \nabla_l \sigma^{jl} + J_k^j J_m^l \nabla_l \sigma^{lm}),
  \]
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- By definition, if the degree of mobility is two, \(\dim \text{Sol}([\nabla]) = 2\).
1st order PDE system for \((g, J)\) and the h-projective vector field \(v\)

- An h-projective vector field \(v\) preserves the space of solutions \(\text{Sol}([\nabla])\) of
  \[
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  \]
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- By definition, if the degree of mobility is two, \(\dim \text{Sol}([\nabla]) = 2\).

- Choosing a basis \(\sigma, \bar{\sigma}\), we find certain constants \(\alpha, \beta, \gamma, \delta\) such that
  \[
  \mathcal{L}_v \sigma = \gamma \sigma + \delta \bar{\sigma}, \quad \mathcal{L}_v \bar{\sigma} = \alpha \sigma + \beta \bar{\sigma}.
  \]
1st order PDE system for \((g, J)\) and the h-projective vector field \(v\)

- An h-projective vector field \(v\) preserves the space of solutions \(\text{Sol}(\nabla)\) of
  \[
  (*)' \quad \nabla_k \sigma^{ij} = \frac{1}{2n} (\delta_k^i \nabla_l \sigma^{jl} + J^i_k J^j_m \nabla_l \sigma^{lm}),
  \]
i.e. the Lie derivative \(\mathcal{L}_v\) is an endomorphism
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  \mathcal{L}_v : \text{Sol}(\nabla) \to \text{Sol}(\nabla).
  \]
  (this is because when \(v = \partial_x\), we can find a connection \(\hat{\nabla} \in [\nabla]\) whose Christoffel symbols \(\hat{\Gamma}^i_{jk}\) do not depend on \(x\))

- By definition, if the degree of mobility is two, \(\dim \text{Sol}(\nabla) = 2\).

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  \[
  \mathcal{L}_v \sigma = \gamma \sigma + \delta \bar{\sigma}, \quad \mathcal{L}_v \bar{\sigma} = \alpha \sigma + \beta \bar{\sigma}.
  \]
  This is a non-linear PDE system of 1st order on \(\sigma, \bar{\sigma}\) and \(v\).
1st order PDE system for \((g, J)\) and the h-projective vector field \(\nu\)

- An h-projective vector field \(\nu\) preserves the space of solutions \(\text{Sol}([\nabla])\) of
  \[
  \nabla_k \sigma^{ij} = \frac{1}{2n} (\delta_k^i \nabla_l \sigma^{lj} + J_k^j J_m^i \nabla_l \sigma^{lm}),
  \]
  i.e. the Lie derivative \(\mathcal{L}_\nu\) is an endomorphism
  \[
  \mathcal{L}_\nu : \text{Sol}([\nabla]) \rightarrow \text{Sol}([\nabla]).
  \]
  (this is because when \(\nu = \partial_x\), we can find a connection \(\hat{\nabla} \in [\nabla]\) whose Christoffel symbols \(\hat{\Gamma}^i_{jk}\) do not depend on \(x\))

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  \[
  \mathcal{L}_\nu \sigma = \gamma \sigma + \delta \bar{\sigma}, \quad \mathcal{L}_\nu \bar{\sigma} = \alpha \sigma + \beta \bar{\sigma}.
  \]

- This is a non-linear PDE system of 1st order on \(\sigma, \bar{\sigma}\) and \(\nu\).

- Now let \(\sigma = g^{-1}(\det g)^{\frac{1}{2(n+1)}}\) correspond to a metric and let \(\nu\) be essential for \(g\).
1st order PDE system for \((g, J)\) and the h-projective vector field \(v\)

- An h-projective vector field \(v\) preserves the space of solutions \(\text{Sol}([\nabla])\) of

\[
(*)' \quad \nabla_k \sigma^{ij} = \frac{1}{2n} (\delta_{k}^{(i} \nabla_l \sigma^{j)}l + J_{k}^{(i} J_{m}^{j)} \nabla_l \sigma^{lm}),
\]

i.e. the Lie derivative \(\mathcal{L}_v\) is an endomorphism

\[
\mathcal{L}_v : \text{Sol}([\nabla]) \to \text{Sol}([\nabla]).
\]

(this is because when \(v = \partial_x\), we can find a connection \(\hat{\nabla} \in [\nabla]\) whose Christoffel symbols \(\hat{\Gamma}_jk^i\) do not depend on \(x\))

- By definition, if the degree of mobility is two, \(\dim \text{Sol}([\nabla]) = 2\).

- Choosing a basis \(\sigma, \bar{\sigma}\), we find certain constants \(\alpha, \beta, \gamma, \delta\) such that

\[
\mathcal{L}_v \sigma = \gamma \sigma + \delta \bar{\sigma}, \quad \mathcal{L}_v \bar{\sigma} = \alpha \sigma + \beta \bar{\sigma}.
\]

This is a non-linear PDE system of 1st order on \(\sigma, \bar{\sigma}\) and \(v\).

- Now let \(\sigma = g^{-1}(\det g)\frac{1}{2(n+1)}\) correspond to a metric and let \(v\) be essential for \(g\).

\[\Rightarrow \text{We can choose } \bar{\sigma} = -\mathcal{L}_v \sigma \text{ as the second basis vector such that the matrix of } \mathcal{L}_v \text{ becomes} \]

\[
\mathcal{L}_v \sigma = -\bar{\sigma}, \quad \mathcal{L}_v \bar{\sigma} = \alpha \sigma + \beta \bar{\sigma}.
\]
Let $\sigma = g^{-1}(\det g)^{\frac{1}{2(n+1)}} \in \text{Sol}([\nabla])$ correspond to the metric $g$ and let $v$ be essential for $g$.

Choose $\bar{\sigma} = -\mathcal{L}_v \sigma$ as the second basis vector such that the matrix of $\mathcal{L}_v$ becomes

$$\mathcal{L}_v \sigma = -\bar{\sigma},$$

$$\mathcal{L}_v \bar{\sigma} = \alpha \sigma + \beta \bar{\sigma}.$$
The corresponding PDE on $g$, $A$ and $\nu$

- Let $\sigma = g^{-1}(\det g)^{\frac{1}{2(n+1)}} \in \text{Sol}([\nabla])$ correspond to the metric $g$ and let $\nu$ be essential for $g$.
- Choose $\bar{\sigma} = -\mathcal{L}_\nu \sigma$ as the second basis vector such that the matrix of $\mathcal{L}_\nu$ becomes
  \[
  \mathcal{L}_\nu \sigma = -\bar{\sigma},
  \]
  \[
  \mathcal{L}_\nu \bar{\sigma} = \alpha \sigma + \beta \bar{\sigma}.
  \]
- Express this 1st order PDE in terms of the metric $g$ and the $(1,1)$-tensor $A = \bar{\sigma} \sigma^{-1}$ (that solves $\nabla_k A^{ij} = \delta_k^{(i} \wedge^j) + J_k^{(i} J_l^{j)} \wedge^l$):
The corresponding PDE on $g$, $A$ and $\nu$

- Let $\sigma = g^{-1}(\det g)^{-\frac{1}{2(n+1)}} \in \text{Sol}([\nabla])$ correspond to the metric $g$ and let $\nu$ be essential for $g$.

- Choose $\bar{\sigma} = -\mathcal{L}_\nu \sigma$ as the second basis vector such that the matrix of $\mathcal{L}_\nu$ becomes
  \[
  \mathcal{L}_\nu \sigma = -\bar{\sigma},
  \]
  \[
  \mathcal{L}_\nu \bar{\sigma} = \alpha \sigma + \beta \bar{\sigma}.
  \]

- Express this 1st order PDE in terms of the metric $g$ and the $(1, 1)$-tensor $A = \bar{\sigma} \sigma^{-1}$ (that solves $\nabla_k A^{ij} = \delta_k^{(i} \Lambda^{j)} + J_k^{(i} J^{j)} \Lambda^{l)}$): The PDE becomes equivalent to
  \[
  g^{-1} \mathcal{L}_\nu g = A + \frac{1}{2} \text{trace}(A) \text{Id},
  \]
  \[
  \mathcal{L}_\nu A = A^2 + \beta A + \alpha \text{Id}.
  \]
The corresponding PDE on $g$, $A$ and $\nu$

- Let $\sigma = g^{-1} (\det g) \frac{1}{2(n+1)} \in \text{Sol}([\nabla])$ correspond to the metric $g$ and let $\nu$ be essential for $g$.

- Choose $\bar{\sigma} = -\mathcal{L}_\nu \sigma$ as the second basis vector such that the matrix of $\mathcal{L}_\nu$ becomes

$$\mathcal{L}_\nu \sigma = -\bar{\sigma},$$

$$\mathcal{L}_\nu \bar{\sigma} = \alpha \sigma + \beta \bar{\sigma}.$$ 

- Express this 1st order PDE in terms of the metric $g$ and the $(1,1)$-tensor $A = \bar{\sigma} \sigma^{-1}$ (that solves $\nabla_k A^{ij} = \delta_k^i \Lambda^j + J_k^i J^j \Lambda^l$) : The PDE becomes equivalent to

$$g^{-1} \mathcal{L}_\nu g = A + \frac{1}{2} \text{trace}(A) \text{Id},$$

$$\mathcal{L}_\nu A = A^2 + \beta A + \alpha \text{Id}.$$ 

- Now, we can insert the normal forms for $g$, $A$ from Case 1 and Case 2 respectively and obtain a 1st order PDE on
  - Case 1: the functions $F_1$, $F_2$ and the components of $\nu$. 

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The corresponding PDE on $g$, $A$ and $v$

- Let $\sigma = g^{-1}(\det g)^{\frac{1}{2(n+1)}} \in \text{Sol}([\nabla])$ correspond to the metric $g$ and let $v$ be essential for $g$.
- Choose $\bar{\sigma} = -L_v \sigma$ as the second basis vector such that the matrix of $L_v$ becomes
  \[ L_v \sigma = -\bar{\sigma}, \]
  \[ L_v \bar{\sigma} = \alpha \sigma + \beta \bar{\sigma}. \]
- Express this 1st order PDE in terms of the metric $g$ and the $(1, 1)$-tensor $A = \bar{\sigma} \sigma^{-1}$ (that solves $\nabla_k A^i_j = \delta_k^i \Lambda^j_l + J_k^i J^j_l \Lambda^l_l$): The PDE becomes equivalent to
  \[ g^{-1}L_v g = A + \frac{1}{2} \text{trace}(A) \text{Id}, \]
  \[ L_v A = A^2 + \beta A + \alpha \text{Id}. \]

- Now, we can insert the normal forms for $g$, $A$ from Case 1 and Case 2 respectively and obtain a 1st order PDE on
  - Case 1: the functions $F_1$, $F_2$ and the components of $v$.
  - Case 2: the function $F$, the 2D-Kähler metric $h$ and the components of $v$. 
The corresponding PDE on $g$, $A$ and $v$ in Case 1

The normal forms of $g$ and $A$ in Case 1: In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

$$g = \begin{pmatrix}
\frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\
0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
\end{pmatrix}, \quad A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}$$
The corresponding PDE on $g$, $A$ and $\nu$ in Case 1

- The normal forms of $g$ and $A$ in Case 1: In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

\[
g = \begin{pmatrix}
\frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\
0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
\end{pmatrix}, \\
A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

- The PDE system we want to solve is

\[
(1) \quad g^{-1} \mathcal{L}_\nu g = A + \frac{1}{2} \text{trace}(A) \text{Id}, \\
(2) \quad \mathcal{L}_\nu A = A^2 + \beta A + \alpha \text{Id}.
\]
The corresponding PDE on $g$, $A$ and $v$ in Case 1

The normal forms of $g$ and $A$ in Case 1: In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

$$g = \begin{pmatrix}
\frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\
0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
\end{pmatrix}, \quad
A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}$$

The PDE system we want to solve is

1. $g^{-1} \mathcal{L}_v g = A + \frac{1}{2} \text{trace}(A)\text{Id},$

2. $\mathcal{L}_v A = A^2 + \beta A + \alpha \text{Id}.$

Implications of equation (2):

- $v(\rho_1) = \rho_1^2 + \beta \rho_1 + \alpha$, $v(\rho_2) = \rho_2^2 + \beta \rho_2 + \alpha.$
The corresponding PDE on $g$, $A$ and $v$ in Case 1

1. **The normal forms of $g$ and $A$ in Case 1:** In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

$$ g = \begin{pmatrix} \frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\ 0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\ 0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\ 0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2} \end{pmatrix}, \quad A = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\ 0 & 0 & -1 & 0 \end{pmatrix} $$

2. **The PDE system we want to solve is**

   $$(1) \quad g^{-1} L_v g = A + \frac{1}{2} \text{trace}(A) \text{Id},$$

   $$(2) \quad L_v A = A^2 + \beta A + \alpha \text{Id}. $$

   **Implications of equation (2):**

   - $v(\rho_1) = \rho_1^2 + \beta \rho_1 + \alpha$, $v(\rho_2) = \rho_2^2 + \beta \rho_2 + \alpha$.

   $\Rightarrow$ in the coordinates from above, $v$ looks like

   $$ v = (\rho_1^2 + \beta \rho_1 + \alpha) \frac{\partial}{\partial \rho_1} + (\rho_2^2 + \beta \rho_2 + \alpha) \frac{\partial}{\partial \rho_2} + v^3(\rho_1, \rho_2, t_1, t_2) \frac{\partial}{\partial t_1} + v^4(\rho_1, \rho_2, t_1, t_2) \frac{\partial}{\partial t_2}. $$
The corresponding PDE on \( g, A \) and \( v \) in Case 1

- The normal forms of \( g \) and \( A \) in Case 1: In certain coordinates \( \rho_1, \rho_2, t_1, t_2 \), we have

\[
g = \begin{pmatrix}
\frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\
0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
\end{pmatrix}, \quad A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

- The PDE system we want to solve is

\[
(1) \quad g^{-1} \mathcal{L}_v g = A + \frac{1}{2} \text{trace}(A) \text{Id},
\]

\[
(2) \quad \mathcal{L}_v A = A^2 + \beta A + \alpha \text{Id}.
\]

Implications of equation (2):

- \( v(\rho_1) = \rho_1^2 + \beta \rho_1 + \alpha, \quad v(\rho_2) = \rho_2^2 + \beta \rho_2 + \alpha. \)

\( \Rightarrow \) in the coordinates from above, \( v \) looks like

\[
v = (\rho_1^2 + \beta \rho_1 + \alpha) \frac{\partial}{\partial \rho_1} + (\rho_2^2 + \beta \rho_2 + \alpha) \frac{\partial}{\partial \rho_2} + v^3(\rho_1, \rho_2, t_1, t_2) \frac{\partial}{\partial t_1} + v^4(\rho_1, \rho_2, t_1, t_2) \frac{\partial}{\partial t_2}.
\]
Invariant distributions and the splitting of the PDE system

- Normal forms of $g$ and $A$ in Case 1: In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

$$g = \begin{pmatrix}
\frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\
0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
\end{pmatrix}, \quad A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}$$
Invariant distributions and the splitting of the PDE system

- Normal forms of \( g \) and \( A \) in Case 1: In certain coordinates \( \rho_1, \rho_2, t_1, t_2 \), we have

\[
g = \begin{pmatrix}
\frac{\rho_1-\rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2-\rho_1}{F_2(\rho_2)} & 0 & 0 \\
0 & 0 & \frac{F_1(\rho_1)-F_2(\rho_2)}{\rho_1-\rho_2} & \frac{\rho_2F_1(\rho_1)-\rho_1F_2(\rho_2)}{\rho_1-\rho_2} \\
0 & 0 & \frac{\rho_2F_1(\rho_1)-\rho_1F_2(\rho_2)}{\rho_1-\rho_2} & \frac{\rho_2^2F_1(\rho_1)-\rho_1^2F_2(\rho_2)}{\rho_1-\rho_2}
\end{pmatrix}, \quad A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1\rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

- Consider the distributions

\[
D = \text{span}\{\frac{\partial}{\partial \rho_1}, \frac{\partial}{\partial \rho_2}\}
\]

\[
D^\perp = \text{span}\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\}
\]

\[
L_v g \text{ splits into } L_v g = L_v D g + L_v D^\perp g + L_v D^\perp g.
\]

\[
\text{The PDE } L_v g = gA + \frac{1}{2} \text{trace}(A) g \text{ splits into an upper-left block (containing } \rho_1, \rho_2, F_1, F_2, F_1', F_2') \text{ and a lower-right block.}
\]

\[
\text{Upper-left block gives ODE's on } F_1 \text{ and } F_2 \text{ respectively. Inserting solutions for } F_1, F_2 \text{ into lower-right block gives equations on } v_3, v_4 \text{ which can be solved.}
\]
Invariant distributions and the splitting of the PDE system

- Normal forms of $g$ and $A$ in Case 1: In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

$$g = \begin{pmatrix}
\frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\
0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
\end{pmatrix}, \quad A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}$$

- Consider the distributions

$$D = \text{span}\{\frac{\partial}{\partial \rho_1}, \frac{\partial}{\partial \rho_2}\}, \quad \text{invariantly} \quad = \text{span}\{\text{grad} \ \rho_1, \text{grad} \ \rho_2\},$$

$$D^\perp = \text{span}\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\}, \quad \text{invariantly} \quad = \text{span}\{J\text{grad} \ \rho_1, J\text{grad} \ \rho_2\}.$$
Invariant distributions and the splitting of the PDE system

- Normal forms of $g$ and $A$ in Case 1: In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

$$g = \begin{pmatrix} \frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\ 0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\ 0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\ 0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2} \end{pmatrix}, \quad A = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- Consider the distributions

$$D = \text{span}\{\frac{\partial}{\partial \rho_1}, \frac{\partial}{\partial \rho_2}\} \quad \text{invariantly} \quad \text{span}\{\text{grad} \rho_1, \text{grad} \rho_2\},$$

$$D^\perp = \text{span}\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\} \quad \text{invariantly} \quad \text{span}\{J \text{grad} \rho_1, J \text{grad} \rho_2\}.$$

- $D$ is an h-projectively invariant distribution
Invariant distributions and the splitting of the PDE system

- Normal forms of $g$ and $A$ in Case 1: In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

$$g = \begin{pmatrix}
\frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
\end{pmatrix}, \\
A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}$$

- Consider the distributions

$$D = \text{span}\left\{\frac{\partial}{\partial \rho_1}, \frac{\partial}{\partial \rho_2}\right\} \quad \text{invariantly} \quad \text{span}\{\text{grad } \rho_1, \text{grad } \rho_2\},$$

$$D^\perp = \text{span}\left\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\right\} \quad \text{invariantly} \quad \text{span}\{J\text{grad } \rho_1, J\text{grad } \rho_2\}.$$ 

- $D$ is an h-projectively invariant distribution $\Rightarrow D^\perp = JD$ is h-projectively invariant.
Invariant distributions and the splitting of the PDE system

- Normal forms of $g$ and $A$ in Case 1: In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

$$g = \begin{pmatrix}
\frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\
0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
\end{pmatrix}, \quad A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}$$

- Consider the distributions

$$D = \text{span}\left\{\frac{\partial}{\partial \rho_1}, \frac{\partial}{\partial \rho_2}\right\}, \quad D^\perp = \text{span}\left\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\right\},$$

invariantly

$$D = \text{span}\left\{\text{grad } \rho_1, \text{grad } \rho_2\right\}, \quad D^\perp = \text{span}\left\{\text{Jgrad } \rho_1, \text{Jgrad } \rho_2\right\}.$$ 

- $D$ is an h-projectively invariant distribution $\implies D^\perp = JD$ is h-projectively invariant.

$\implies$ If $f$ is h-projective transformation we have $f_* D = D$, $f_* D^\perp = D^\perp$. 

S. Rosemann (FSU Jena)

Essential h-projective vector fields
Invariant distributions and the splitting of the PDE system

- Normal forms of $g$ and $A$ in Case 1: In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

$$g = \begin{pmatrix}
\frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\
0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
\end{pmatrix}, \quad
A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}$$

- Consider the distributions

$$D = \text{span}\{\frac{\partial}{\partial \rho_1}, \frac{\partial}{\partial \rho_2}\} \quad \text{invariantly} = \text{span}\{\text{grad} \rho_1, \text{grad} \rho_2\},$$

$$D^\perp = \text{span}\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\} \quad \text{invariantly} = \text{span}\{J\text{grad} \rho_1, J\text{grad} \rho_2\}.$$ 

- $D$ is an h-projectively invariant distribution $\Rightarrow D^\perp = JD$ is h-projectively invariant.

  $\Rightarrow$ If $f$ is h-projective transformation we have $f_* D = D$, $f_* D^\perp = D^\perp$.

  $\Rightarrow$ The h-projective vector field $\nu$ looks like

$$\nu = (\rho_1^2 + \beta \rho_1 + \alpha) \frac{\partial}{\partial \rho_1} + (\rho_2^2 + \beta \rho_2 + \alpha) \frac{\partial}{\partial \rho_2} + \nu_D(t_1, t_2) \frac{\partial}{\partial t_1} + \nu_D^\perp(t_1, t_2) \frac{\partial}{\partial t_2}.$$
Invariant distributions and the splitting of the PDE system

- **Normal forms of** $g$ **and** $A$ **in Case 1:** In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

  \[ g = \begin{pmatrix}
  \frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
  0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\
  \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
  \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
  \end{pmatrix},
  \quad
  A = \begin{pmatrix}
  \rho_1 & 0 & 0 & 0 \\
  0 & \rho_2 & 0 & 0 \\
  0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
  0 & 0 & -1 & 0
  \end{pmatrix} \]

- **Consider the distributions**

  \[ D = \text{span}\{\frac{\partial}{\partial \rho_1}, \frac{\partial}{\partial \rho_2}\}, \quad D^\perp = \text{span}\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\}. \]

- $D$ **is an** h-projectively invariant distribution $\Rightarrow D^\perp = JD$ **is** h-projectively invariant.

  $\Rightarrow$ If $f$ **is** h-projective transformation we have $f_\ast D = D$, $f_\ast D^\perp = D^\perp$.

  $\Rightarrow$ The h-projective vector field $v$ looks like

  \[ v = \left(\rho_1^2 + \beta \rho_1 + \alpha\right)\frac{\partial}{\partial \rho_1} + \left(\rho_2^2 + \beta \rho_2 + \alpha\right)\frac{\partial}{\partial \rho_2} + v^3(t_1, t_2)\frac{\partial}{\partial t_1} + v^4(t_1, t_2)\frac{\partial}{\partial t_2}. \]

- **The Lie derivative** $\mathcal{L}_v g$ **splits into**

  \[ \mathcal{L}_v g = \mathcal{L}_{v_D} g_D + \mathcal{L}_{v_D^\perp} g_D^\perp. \]
Normal forms of $g$ and $A$ in Case 1: In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

$$g = \begin{pmatrix} \frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\ 0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\ 0 & 0 & \frac{F_1(\rho_1)-F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_1 F_1(\rho_1) - \rho_2 F_2(\rho_2)}{\rho_1 - \rho_2} \\ 0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2} \end{pmatrix}, \quad A = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Consider the distributions

$$D = \text{span}\left\{\frac{\partial}{\partial \rho_1}, \frac{\partial}{\partial \rho_2}\right\} \quad \text{invariantly} = \text{span}\left\{\text{grad} \rho_1, \text{grad} \rho_2\right\},$$

$$D^\perp = \text{span}\left\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\right\} \quad \text{invariantly} = \text{span}\left\{J\text{grad} \rho_1, J\text{grad} \rho_2\right\}.$$

$D$ is an $h$-projectively invariant distribution $\Rightarrow D^\perp = JD$ is $h$-projectively invariant.

$\Rightarrow$ If $f$ is $h$-projective transformation we have $f_* D = D$, $f_* D^\perp = D^\perp$.

$\Rightarrow$ The $h$-projective vector field $v$ looks like

$$v = (\rho_1^2 + \beta \rho_1 + \alpha) \frac{\partial}{\partial \rho_1} + (\rho_2^2 + \beta \rho_2 + \alpha) \frac{\partial}{\partial \rho_2} + v_3(t_1, t_2) \frac{\partial}{\partial t_1} + v_4(t_1, t_2) \frac{\partial}{\partial t_2}.$$

The Lie derivative $\mathcal{L}_v g$ splits into $\mathcal{L}_v g = \mathcal{L}_{v_D} g_D + \mathcal{L}_{v_D^\perp} g_{D^\perp}$.

$\Rightarrow$ The PDE $\mathcal{L}_v g = gA + \frac{1}{2} \text{trace}(A) g$ splits into an upper-left block (containing $\rho_1, \rho_2, F_1, F_2, F_1', F_2'$) and a lower-right block.
Invariant distributions and the splitting of the PDE system

- Normal forms of $g$ and $A$ in Case 1: In certain coordinates $\rho_1, \rho_2, t_1, t_2$, we have

$$
g = \begin{pmatrix}
\frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\
0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
\end{pmatrix}, \quad A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}$$

Consider the distributions

$$
\begin{align*}
D &= \text{span}\{\partial/\partial \rho_1, \partial/\partial \rho_2\} \quad \text{invariantly} = \text{span}\{\text{grad} \rho_1, \text{grad} \rho_2\}, \\
D^\perp &= \text{span}\{\partial/\partial t_1, \partial/\partial t_2\} \quad \text{invariantly} = \text{span}\{J\text{grad} \rho_1, J\text{grad} \rho_2\}.
\end{align*}
$$

- $D$ is an h-projectively invariant distribution $\Rightarrow D^\perp = JD$ is h-projectively invariant.
  - If $f$ is h-projective transformation we have $f_\ast D = D, f_\ast D^\perp = D^\perp$.
  - The h-projective vector field $v$ looks like

$$
v = (\rho_1^2 + \beta \rho_1 + \alpha) \frac{\partial}{\partial \rho_1} + (\rho_2^2 + \beta \rho_2 + \alpha) \frac{\partial}{\partial \rho_2} + v_3(t_1, t_2) \frac{\partial}{\partial t_1} + v_4(t_1, t_2) \frac{\partial}{\partial t_2}.$$

- The Lie derivative $\mathcal{L}_v g$ splits into $\mathcal{L}_v g = \mathcal{L}_{v_D} g_D + \mathcal{L}_{v_D} g_{D^\perp} + \mathcal{L}_{v_D^\perp} g_{D^\perp}$.
  - The PDE $\mathcal{L}_v g = gA + \frac{1}{2}\text{trace}(A)g$ splits into an upper-left block (containing $\rho_1, \rho_2, F_1, F_2, F'_1, F'_2$) and a lower-right block.
  - Upper-left block gives ODE’s on $F_1$ and $F_2$ respectively.
Invariant distributions and the splitting of the PDE system

Normal forms of \( g \) and \( A \) in Case 1: In certain coordinates \( \rho_1, \rho_2, t_1, t_2 \), we have

\[
g = \begin{pmatrix}
\frac{\rho_1 - \rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\
0 & \frac{\rho_2 - \rho_1}{F_2(\rho_2)} & 0 & 0 \\
0 & 0 & \frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2} & 0 \\
0 & 0 & 0 & \frac{\rho_2 F_1(\rho_1) - \rho_1 F_2(\rho_2)}{\rho_1 - \rho_2} \\
0 & 0 & 0 & \frac{\rho_2^2 F_1(\rho_1) - \rho_1^2 F_2(\rho_2)}{\rho_1 - \rho_2}
\end{pmatrix},
\]

\[
A = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_1 + \rho_2 & \rho_1 \rho_2 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

Consider the distributions

\[ D = \text{span}\{\frac{\partial}{\partial \rho_1}, \frac{\partial}{\partial \rho_2}\} \quad \text{invariantly} \quad = \text{span}\{\text{grad} \ \rho_1, \text{grad} \ \rho_2\}, \]

\[ D^\perp = \text{span}\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\} \quad \text{invariantly} \quad = \text{span}\{J\text{grad} \ \rho_1, J\text{grad} \ \rho_2\}. \]

\( D \) is an h-projectively invariant distribution \( \Rightarrow D^\perp = JD \) is h-projectively invariant.

\( \Rightarrow \) If \( f \) is h-projective transformation we have \( f_* D = D, f_* D^\perp = D^\perp. \)

\( \Rightarrow \) The h-projective vector field \( v \) looks like

\[
v = \left( \rho_1^2 + \beta \rho_1 + \alpha \right) \frac{\partial}{\partial \rho_1} + \left( \rho_2^2 + \beta \rho_2 + \alpha \right) \frac{\partial}{\partial \rho_2} + v^3(t_1, t_2) \frac{\partial}{\partial t_1} + v^4(t_1, t_2) \frac{\partial}{\partial t_2}.
\]

\( \Rightarrow \) The Lie derivative \( \mathcal{L}_v g \) splits into \( \mathcal{L}_v g = \mathcal{L}_{v_D} g_D + \mathcal{L}_{v_D^\perp} g_D^\perp + \mathcal{L}_{v_D^\perp} g_D^\perp. \)

\( \Rightarrow \) The PDE \( \mathcal{L}_v g = gA + \frac{1}{2} \text{trace}(A)g \) splits into an upper-left block (containing \( \rho_1, \rho_2, F_1, F_2, F_1', F_2' \)) and a lower-right block.

Upper-left block gives ODE’s on \( F_1 \) and \( F_2 \) respectively. Inserting solutions for \( F_1, F_2 \) into lower-right block gives equations on \( v^3(t_1, t_2), v^4(t_1, t_2) \) which can be solved.
Thanks for listening!