

Symmetry gaps for geometric structures

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The symmetry gap problem

For a given type of geometric structure,
what is the gap between maximal and
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- “Geometric structure” \rightsquigarrow Cartan geometry
(Non-examples: symplectic, contact, ...)

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n	max	submax	Citation
2	3	1	Darboux / Koenigs (~1890)
3	6	4	Wang (1947)
4	10	8	Egorov (1955)
≥ 5	$\binom{n+1}{2}$	$\binom{n}{2} + 1$	Wang (1947), Egorov (1949)

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Example (Parabolic subgroups – “block upper-triangular”)

G/P	P	Hieroglyphic
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- Background
- Results
- Proof outlines

A brief history of gaps

For parabolic geometries...

① ≤ 2012 :

- (i) 2-d projective & scalar 2nd order ODE (*Tresse, 1896*)
- (ii) (2, 3, 5)-distributions (*Cartan, 1910*)
- (iii) n -dim projective (*Egorov, 1951*)
- (iv) scalar 3rd order ODE (*Wafo Soh et al., 2002*)
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Moral: Can work upstairs, use representation theory.

Example: $(2, 3, 5)$ -distributions

Let (M^5, D) with $D \subset TM$ rank 2, which is max. non-integrable.

Goursat (1896): Locally, $D = D_f$ is spanned by

$$X_1 = \partial_x + p\partial_y + q\partial_p + f(x, y, p, q, z)\partial_z, \quad X_2 = \partial_q.$$

This is $(2, 3, 5)$ iff $f_{qq} \neq 0$. Studied by Cartan (1910) in his famous 5-variables paper.

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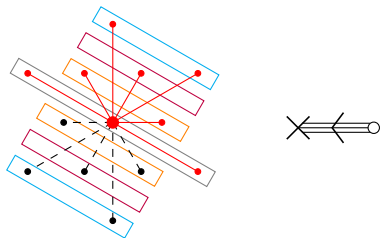
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	max	submax
dim	14	7
model	D_{q^2} (G_2/P_1)	D_{q^3}



Fundamental invariant: **binary quartic**, i.e. $\Gamma(\odot^4 D^*)$.

Theorem (Čap–Schichl, Tanaka, Morimoto)

$$\left\{ \begin{array}{l} \textit{regular, normal} \\ G/P \textit{ geometries} \\ (\mathcal{G} \rightarrow M, \omega) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \textit{underlying} \\ \textit{structures on } M \end{array} \right\}$$

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Examples (Harmonic curvature)

- conformal geometry: Weyl ($n \geq 4$) or Cotton ($n = 3$);
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The (locally) flat model is the *unique* max. sym. model. \therefore Want:

$$\mathfrak{S} := \max\{\dim(\text{inf}(\mathcal{G}, \omega)) \mid \kappa_H \neq 0\}.$$

$$(\mathfrak{g}, \mathfrak{p}) \rightsquigarrow \mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}. \quad \text{Have } (\mathfrak{g}_-)^* \cong \mathfrak{g}_+.$$

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Curvature κ of $(\mathcal{G} \rightarrow M, \omega)$ takes values in:

$$\bigwedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$$

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Kostant (1961) Laplacian:

$$\square = \partial\partial^* + \partial^*\partial$$

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$$\rightsquigarrow \bigwedge^2(\mathfrak{g}_-)^* \otimes \mathfrak{g} = \overbrace{\text{im}(\partial^*) \oplus \ker(\square)}^{\ker(\partial^*)} \oplus \underbrace{\text{im}(\partial)}_{\ker(\partial)}$$

$$\ker(\square) \cong H^2(\mathfrak{g}_-, \mathfrak{g})$$

cohomology!

Kostant's Bott–Borel–Weil theorem

Kostant (1961), Baston–Eastwood (1989): Dynkin diagram algorithm to calculate $H_+^2(\mathfrak{g}_-, \mathfrak{g})$ as a \mathfrak{g}_0 -module.

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Example $((2, 3, 5)$ -distributions: G_2/P_1 geometry)

As a $\mathfrak{g}_0 = \mathfrak{gl}_2(\mathbb{R})$ module,

$$H_+^2(\mathfrak{g}_-, \mathfrak{g}) = \begin{array}{c} -8 \qquad 4 \\ \times \leftarrow \leftarrow \leftarrow \circ \end{array} = \odot^4(\mathbb{R}^2)^*,$$

i.e. binary quartic, c.f. Cartan (1910).

Tanaka prolongation

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Given $\mathfrak{a}_0 \subset \mathfrak{g}_0$, the **Tanaka prolongation** of \mathfrak{a}_0 in \mathfrak{g} :

$$\mathrm{pr}_{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{a}_0) = \mathfrak{g}_- \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_+$$

$$\mathfrak{a}_i = \{X \in \mathfrak{g}_i \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{i-1}\} \quad (i \geq 1).$$

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Given $0 \neq \phi \in H_+^2$, interested in $\mathfrak{a}_0 = \mathrm{ann}(\phi)$. Let

$$\mathfrak{a}^\phi := \mathrm{pr}_{\mathfrak{g}}(\mathfrak{g}_-, \mathrm{ann}(\phi)).$$

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Theorem (Kruglikov, T. (2013))

Let $(\mathcal{G} \rightarrow M, \omega)$ be any regular, normal G/P geometry. Then $\dim(\mathrm{inf}(\mathcal{G}, \omega)) \leq \dim(\mathfrak{a}^{\kappa_H(u)})$, $\forall u$ in some open dense subset of \mathcal{G} .

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K-T. (2014, in prep.): This holds $\forall u \in \mathcal{G}$.

New results (Kruglikov–T., 2013)

Fix (G, P) . Define $\mathfrak{L} := \max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in H_+^2(\mathfrak{g}_-, \mathfrak{g})\}$.

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Fix (G, P) . Define $\mathfrak{U} := \max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in H_+^2(\mathfrak{g}_-, \mathfrak{g})\}$.

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If G/P is *complex or split-real*, then $\mathfrak{S} = \mathfrak{U}$ almost always.
Exception list when G is simple: A_2/P_1 , $A_2/P_{1,2}$, B_2/P_1 .

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Proposition (Extremal vectors win)

Over \mathbb{C} , if \mathbb{V} is a \mathfrak{g}_0 -irrep, $\phi_0 \in \mathbb{V}$ is extremal, then $\forall \phi \in \mathbb{V} \setminus \{0\}$,

$$\dim(\text{ann}(\phi)) \leq \dim(\text{ann}(\phi_0)), \quad \dim(\mathfrak{a}^\phi) \leq \dim(\mathfrak{a}^{\phi_0}).$$

Čap–Neusser (2009):

- Fix **any** $u \in \mathfrak{G}$. Then $\omega_u : \text{inf}(\mathfrak{G}, \omega) \hookrightarrow \mathfrak{g}$ (linearly).
- Bracket on $\mathfrak{f} = \text{im}(\omega_u)$ is $[X, Y]_{\mathfrak{f}} := [X, Y]_{\mathfrak{g}} - \kappa_u(X, Y)$.
- Regularity: \mathfrak{f} is filtered, so $\mathfrak{s} = \text{gr}(\mathfrak{f}) \subset \mathfrak{g}$ is a graded subalg.
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(*): $[\mathfrak{s}_{i+1}, \mathfrak{g}_{-1}] \subset \mathfrak{s}_i$ ($i \geq -1$) \Rightarrow $\mathfrak{s} \subset \text{pr}_{\mathfrak{g}}(\mathfrak{g}_{-}, \mathfrak{s}_0) \subset \mathfrak{a}^{\kappa_H(u)}$,
so $\dim(\mathfrak{s}) \leq \mathfrak{L}$ when $\kappa_H(u) \neq 0$.

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Definition

$x \in M$ is a **regular point** iff $\forall i$, $\dim(\mathfrak{s}_i)$ is loc. constant near x .

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$x \in M$ is a **regular point** iff $\forall i$, $\dim(\mathfrak{s}_i)$ is loc. constant near x .

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Realizability: proof outline

Define $\mathfrak{f} = \mathfrak{a} := \mathfrak{a}^{\phi_0}$ as *vector spaces*, but with deformed bracket

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Non-exceptions: $\mathfrak{f}/\mathfrak{f}^0 \rightsquigarrow$ non-flat model, $\dim(\mathfrak{f}) = \mathfrak{U}$, so $\mathfrak{S} = \mathfrak{U}$.

Have algorithm for constructing an explicit submax. sym. model.

Extremal vectors win

Work over \mathbb{C} . Let \mathbb{V} be a \mathfrak{g}_0 -irrep, and $\phi_0 \in \mathbb{V}$ an extremal vector.

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By upper semi-continuity, $\dim(\mathfrak{a}_k^\phi) = \dim(\mathfrak{a}_k^{g_n \cdot \phi}) \leq \dim(\mathfrak{a}_k^{\phi_0})$. \square

① $\mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}$, and $\mathfrak{g}_0 = \mathcal{Z}(\mathfrak{g}_0) \oplus (\mathfrak{g}_0)_{ss}$ with

$$\begin{cases} \dim(\mathcal{Z}(\mathfrak{g}_0)) = \# \text{ crosses;} \\ (\mathfrak{g}_0)_{ss} \text{ D.D.} \rightarrow \text{remove crosses.} \end{cases}$$

Since $\dim(\mathfrak{g}_-) = \dim(\mathfrak{g}_+)$, get $n = \dim(\mathfrak{g}/\mathfrak{p})$ and $\dim(\mathfrak{p})$.

Example (G_2/P_1)

$$\times \Leftarrow \Leftarrow \Leftarrow \circ \quad , \quad \dim(\mathfrak{g}_0) = 4, \quad n = 5.$$

Let $\mathbb{V} \subset H_+^2$ be a \mathfrak{g}_0 -irrep and $\phi_0 \in \mathbb{V}$ a l.w. vector.

- ② $\dim(\text{ann}(\phi)) \leq \dim(\text{ann}(\phi_0))$, $\forall \phi \in \mathbb{V} \setminus \{0\}$,
 $\mathfrak{q} := \{X \in (\mathfrak{g}_0)_{ss} \mid X \cdot \phi_0 = \lambda \phi_0\}$ is parabolic, and

$$\dim(\text{ann}(\phi_0)) = (\# \text{crosses}) - 1 + \dim(\mathfrak{q})$$

D.D. Notation: If $\neq 0$ on uncrossed node, put $*$.

Example (G_2/P_1)

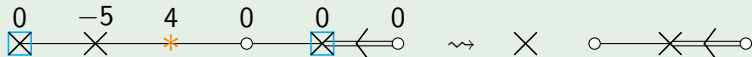
$$H_+^2 = \begin{array}{c} -8 \quad 4 \\ \times \leftarrow \leftarrow \leftarrow * \end{array}, \quad \dim(\text{ann}(\phi_0)) = 2.$$

Dynkin diagram recipes - 3

D.D. Notation: If 0 over $\times \rightsquigarrow$ put \square .

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Then remove connected components w/o \square . Obtain (\bar{g}, \bar{p}) .

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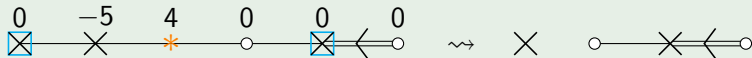


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Proposition (Prolongation criterion)

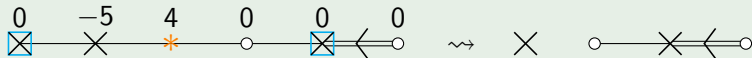
No $\square \Leftrightarrow \dim(\mathfrak{a}_+^{\phi_0}) = 0$. *Otw*, $\dim(\mathfrak{a}_+^{\phi_0}) = \dim(\bar{\mathfrak{g}}/\bar{\mathfrak{p}})$.

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Proposition (Maximal parabolics)

Single cross \Rightarrow no \square , so $\mathfrak{a}_+^{\phi_0} = 0$.

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G/P	H_+^2 components	n	$\dim(\mathfrak{a}_0^{\phi_0})$	$\dim(\mathfrak{a}_+^{\phi_0})$	$\dim(\mathfrak{a}^{\phi_0})$
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- Dim of submax space of solns of almost-Einstein scales, Killing tensors, etc. (more generally, of BGG operators)?

Appendix: The Tanaka property

Let $u \in \pi^{-1}(x)$, $\tilde{\mathcal{S}} := \inf(\mathcal{G}, \omega)$, $\tilde{\mathcal{S}}^j := \{\xi \in \tilde{\mathcal{S}} \mid \omega_u(\xi) \in \mathfrak{g}^j\}$,
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Appendix: The Tanaka property

Let $u \in \pi^{-1}(x)$, $\tilde{\mathcal{S}} := \text{inf}(\mathcal{G}, \omega)$, $\tilde{\mathcal{S}}^j := \{\xi \in \tilde{\mathcal{S}} \mid \omega_u(\xi) \in \mathfrak{g}^j\}$,
 $\mathfrak{f}^j := \omega_u(\tilde{\mathcal{S}}^j)$. WTS: $[\mathfrak{f}^{i+1}, \mathfrak{g}^{-1}] \subset \mathfrak{f}^i + \mathfrak{g}^{i+1}$, $\forall i \geq -1$.

- Let $X = \omega_u(\xi) \in \mathfrak{f}^{i+1}$ and $Y = \omega_u(\eta) \in \mathfrak{g}^{-1}$.
 - $0 = (\mathcal{L}_\xi \omega)(\eta) = d\omega(\xi, \eta) + d(\omega(\xi))(\eta) = \xi(\omega(\eta)) - \omega([\xi, \eta])$.
 - $X \in \mathfrak{p} \Rightarrow \xi_u = (\zeta_X)_u$, where ζ_X is fund. vertical v.f.
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- If $\xi \in \tilde{\mathcal{S}}^{i+1}$ and $\eta \in \Gamma(T\mathcal{G})^P$, then $\forall u_i \in \pi_i^{-1}(x)$, $\xi_{u_i}^{(i)} = 0$ and

$$[\xi^{(i)}, \eta^{(i)}]_{u_i} \cdot F_j = 0 \quad \Rightarrow \quad [\xi, \eta]_u = \xi'_u + \chi_u \quad (\dagger)$$

where $\xi' \in \tilde{\mathcal{S}}$ and $\chi_u \in T_u^{i+1}\mathcal{G}$.

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Conclusion: $[X, Y] \in \mathfrak{f}^j + \mathfrak{g}^{j+1}$ by (\dagger) .