Symmetry gaps for geometric structures

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The symmetry gap problem

For a given type of geometric structure,
what is the gap between maximal and
submaximal (infinitesimal) symmetry dimensions?

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- "Geometric structure" $\rightsquigarrow$ Cartan geometry
  (Non-examples: symplectic, contact, ...)

Example: Riemannian geometry

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n	max	submax	Citation
2	3	1	Darboux / Koenigs ($\sim$1890)
3	6	4	Wang (1947)
4	10	8	Egorov (1955)
≥ 5	$\binom{n+1}{2}$	$\binom{n}{2}+1$	Wang (1947), Egorov (1949)

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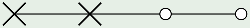
Example (Parabolic subgroups – "block upper-triangular")

G/P	P	Hieroglyphic
$\mathrm{Flag}_{1,2}(\mathbb{R}^5)$ $(\mathrm{SL}_5/P_{1,2})$	$\begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}$	

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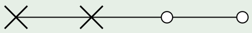
$\mathrm{Flag}_{1,2}(\mathbb{R}^5)$ $(\mathrm{SL}_5/P_{1,2})$	$\begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}$	

$(\mathfrak{g}, \mathfrak{p}) \rightsquigarrow \mathbb{Z}$-grading: $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$.

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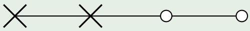
$\mathrm{Flag}_{1,2}(\mathbb{R}^5)$ $(\mathrm{SL}_5/P_{1,2})$	$\begin{pmatrix} 0 & 1 & 2 & 2 & 2 \\ -1 & 0 & 1 & 1 & 1 \\ -2 & -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 \end{pmatrix}$	

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Outline

- Background
- Results
- Proof outlines

A brief history of gaps

For parabolic geometries...

1. ≤ 2012:
   (i) 2-d projective & scalar 2nd order ODE *(Tresse, 1896)*
   (ii) $(2,3,5)$-distributions *(Cartan, 1910)*
   (iii) n-dim projective *(Egorov, 1951)*
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Moral: Can work upstairs, use representation theory.

Example: $(2,3,5)$-distributions

Let (M^5, D) with $D \subset TM$ rank 2, which is max. non-integrable.

Goursat (1896): Locally, $D = D_f$ is spanned by

$$X_1 = \partial_x + p\partial_y + q\partial_p + f(x,y,p,q,z)\partial_z, \qquad X_2 = \partial_q.$$

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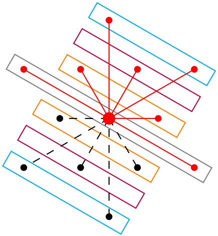
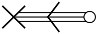
	max	submax
dim	14	7
model	D_{q^2} (G_2/P_1)	D_{q^3}





Fundamental invariant: binary quartic, i.e. $\Gamma(\bigodot^4 D^*)$.

Harmonic curvature

Theorem (Čap–Schichl, Tanaka, Morimoto)

$$\left\{\begin{array}{c} \textit{regular, normal} \\ G/P \textit{ geometries} \\ (\mathcal{G} \to M, \omega) \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \textit{underlying} \\ \textit{structures on } M \end{array}\right\}$$

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Examples (Harmonic curvature)

- conformal geometry: Weyl ($n \geq 4$) or Cotton ($n = 3$);
- $(2, 3, 5)$-distributions: binary quartic.

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The (locally) flat model is the *unique* max. sym. model. $\therefore$ Want:

$$\mathfrak{S} := \max\{\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \mid \kappa_H \neq 0\}.$$

Enter Kostant...

$$(\mathfrak{g}, \mathfrak{p}) \rightsquigarrow \mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}.$$ Have $(\mathfrak{g}_-)^* \cong \mathfrak{g}_+$.

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Curvature κ of $(\mathcal{G} \to M, \omega)$ takes values in:

$$\bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$$

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$$\rightsquigarrow \bigwedge^2 (\mathfrak{g}_-)^* \otimes \mathfrak{g} = \overbrace{\mathrm{im}(\partial^*) \oplus \underbrace{\ker(\Box)}_{}}^{\ker(\partial^*)}\!\!\!\!\!\!\!\!\underbrace{ \oplus \mathrm{im}(\partial)}_{\ker(\partial)}, \qquad \ker(\Box) \cong H^2(\mathfrak{g}_-, \mathfrak{g})$$

$\boxed{\text{cohomology!}}$

Kostant's Bott–Borel–Weil theorem

Kostant (1961), Baston–Eastwood (1989): Dynkin diagram algorithm to calculate $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ as a $\mathfrak{g}_0$-module.

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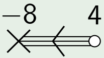
Example ((2, 3, 5)-distributions: G_2/P_1 geometry)

As a $\mathfrak{g}_0 = \mathfrak{gl}_2(\mathbb{R})$ module,

$$H^2_+(\mathfrak{g}_-, \mathfrak{g}) =$$



$$= \bigodot^4 (\mathbb{R}^2)^*,$$

i.e. binary quartic, c.f. Cartan (1910).

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$$\begin{aligned} \mathrm{pr}_{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{a}_0) &= \mathfrak{g}_- \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_+ \\ \mathfrak{a}_i &= \{X \in \mathfrak{g}_i \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{i-1}\} \quad (i \geq 1). \end{aligned}$$

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Given $0 \neq \phi \in H^2_+$, interested in $\mathfrak{a}_0 = \mathfrak{ann}(\phi)$. Let

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Let $(\mathcal{G} \to M, \omega)$ be any regular, normal G/P geometry. Then $\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \leq \dim(\mathfrak{a}^{\kappa_H(u)})$, *$\forall u$ in some open dense subset of $\mathcal{G}$.*

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K-T. (2014, in prep.): This holds $\forall u \in \mathcal{G}$.

New results (Kruglikov–T., 2013)

Fix (G, P). Define $\mathfrak{U} := \max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})\}$.

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Proposition (Extremal vectors win)

Over $\mathbb{C}$, if $\mathbb{V}$ is a $\mathfrak{g}_0$-irrep, $\phi_0 \in \mathbb{V}$ is extremal, then $\forall \phi \in \mathbb{V} \backslash \{0\}$,

$$\dim(\mathfrak{ann}(\phi)) \leq \dim(\mathfrak{ann}(\phi_0)), \qquad \dim(\mathfrak{a}^\phi) \leq \dim(\mathfrak{a}^{\phi_0}).$$

$\mathfrak{S} \leq \mathfrak{U}$: proof outline

Čap–Neusser (2009):

- Fix any $u \in \mathcal{G}$. Then $\omega_u : \mathfrak{inf}(\mathcal{G}, \omega) \hookrightarrow \mathfrak{g}$ (linearly).
- Bracket on $\mathfrak{f} = \mathrm{im}(\omega_u)$ is $[X, Y]_{\mathfrak{f}} := [X, Y]_{\mathfrak{g}} - \kappa_u(X, Y)$.
- Regularity: $\mathfrak{f}$ is filtered, so $\mathfrak{s} = gr(\mathfrak{f}) \subset \mathfrak{g}$ is a graded subalg.
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$(*)$: $\boxed{[\mathfrak{s}_{i+1}, \mathfrak{g}_{-1}] \subset \mathfrak{s}_i}$ $(i \geq -1)$ $\quad \Rightarrow \quad$ $\mathfrak{s} \subset \mathrm{pr}_{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{s}_0) \subset \mathfrak{a}^{\kappa_H(u)}$,
so $\dim(\mathfrak{s}) \leq \mathfrak{U}$ when $\kappa_H(u) \neq 0$.

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(3) Any nbd of a non-flat point contains a non-flat regular pt. $\square$

Realizability: proof outline

Define $\mathfrak{f} = \mathfrak{a} := \mathfrak{a}^{\phi_0}$ as *vector spaces*, but with deformed bracket

$$[X, Y]_{\mathfrak{f}} := [X, Y]_{\mathfrak{a}} - \phi_0(X, Y).$$

(Kostant $\rightsquigarrow$ *explicit* l.w. $\phi_0 \in \mathbb{V} \subset H^2 \cong \ker(\Box) \subset \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$.)

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Non-exceptions: $\mathfrak{f}/\mathfrak{f}^0 \rightsquigarrow$ non-flat model, $\dim(\mathfrak{f}) = \mathfrak{U}$, so $\mathfrak{S} = \mathfrak{U}$.

Have algorithm for constructing an explicit submax. sym. model.

Extremal vectors win

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By upper semi-continuity, $\dim(\mathfrak{a}_k^{\phi}) = \dim(\mathfrak{a}_k^{g_n \cdot \phi}) \leq \dim(\mathfrak{a}_k^{\phi_0})$. □

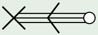
Dynkin diagram recipes - 1

1. $\mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}$, and $\mathfrak{g}_0 = \mathcal{Z}(\mathfrak{g}_0) \oplus (\mathfrak{g}_0)_{ss}$ with

$$\boxed{\begin{cases} \dim(\mathcal{Z}(\mathfrak{g}_0)) = \#\ \text{crosses;} \\ (\mathfrak{g}_0)_{ss}\ \text{D.D.} \to \text{remove crosses.} \end{cases}}$$

Since $\dim(\mathfrak{g}_-) = \dim(\mathfrak{g}_+)$, get $n = \dim(\mathfrak{g}/\mathfrak{p})$ and $\dim(\mathfrak{p})$.

Example (G_2/P_1)

 , $\dim(\mathfrak{g}_0) = 4$, $n = 5$.

Dynkin diagram recipes - 2

Let $\mathbb{V} \subset H^2_+$ be a $\mathfrak{g}_0$-irrep and $\phi_0 \in \mathbb{V}$ a l.w. vector.

2. $\dim(\mathfrak{ann}(\phi)) \leq \dim(\mathfrak{ann}(\phi_0))$, $\forall \phi \in \mathbb{V} \backslash \{0\}$, $\mathfrak{q} := \{X \in (\mathfrak{g}_0)_{ss} \mid X \cdot \phi_0 = \lambda \phi_0\}$ is parabolic, and

$$\boxed{\dim(\mathfrak{ann}(\phi_0)) = (\#\text{crosses}) - 1 + \dim(\mathfrak{q})}$$

D.D. Notation: If $\neq 0$ on uncrossed node, put $*$.

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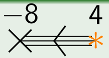
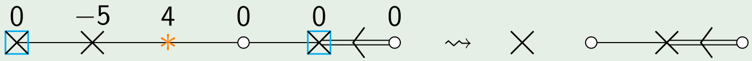
$H^2_+ =$ 

, $\quad \dim(\mathfrak{ann}(\phi_0)) = 2.$

D.D. Notation: If 0 over $\times$ $\rightsquigarrow$ put $\square$.

3. Remove all $*$ and $\times$, except $\square$ (also remove adj. edges). Then remove connected components w/o $\square$. Obtain $(\bar{\mathfrak{g}}, \bar{\mathfrak{p}})$.

Example



Dynkin diagram recipes - 3

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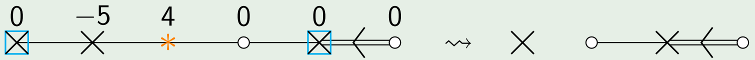

Proposition (Prolongation criterion)

No $\square \Leftrightarrow \dim(\mathfrak{a}_+^{\phi_0}) = 0$. *Otw,* $\dim(\mathfrak{a}_+^{\phi_0}) = \dim(\bar{\mathfrak{g}}/\bar{\mathfrak{p}})$.

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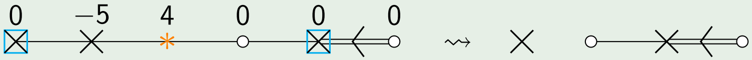

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Proposition (Maximal parabolics)

Single cross $\Rightarrow$ *no* $\square$, *so* $\mathfrak{a}_+^{\phi_0} = 0$.

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G/P	H^2_+ components	n	$\dim(\mathfrak{a}_0^{\phi_0})$	$\dim(\mathfrak{a}_+^{\phi_0})$	$\dim(\mathfrak{a}^{\phi_0})$
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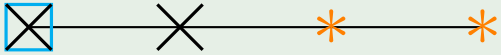
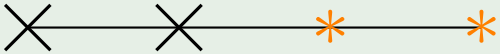
$A_4/P_{1,2}$	0 -4 3 1 	7	6	1	14
	-4 1 1 1 	7	6	0	13

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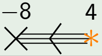
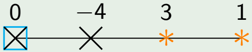
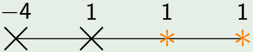
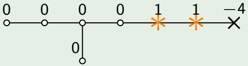
G_2/P_1		5	2	0	7
$A_4/P_{1,2}$		7	6	1	14
		7	6	0	13
E_8/P_8		57	90	0	147

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Appendix: The Tanaka property

Let $u \in \pi^{-1}(x)$, $\widetilde{\mathcal{S}} := \mathfrak{inf}(\mathcal{G}, \omega)$, $\widetilde{\mathcal{S}}^j := \{\xi \in \widetilde{\mathcal{S}} \mid \omega_u(\xi) \in \mathfrak{g}^j\}$, $\mathfrak{f}^j := \omega_u(\widetilde{\mathcal{S}}^j)$. WTS: $[\mathfrak{f}^{i+1}, \mathfrak{g}^{-1}] \subset \mathfrak{f}^i + \mathfrak{g}^{i+1}, \quad \forall i \geq -1.$

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     $$[\xi^{(i)}, \eta^{(i)}]_{u_i} \cdot F_j = 0 \quad \Rightarrow \quad [\xi, \eta]_u = \xi'_u + \chi_u \qquad (\dagger)$$

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Conclusion: $[X, Y] \in \mathfrak{f}^i + \mathfrak{g}^{i+1}$ by $(\dagger)$.