Symmetry and geometric structures

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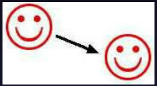
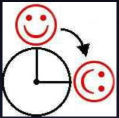
Euclidean geometry (à la Klein)

The Euclidean group $\mathbb{E}(2)$ consists of all symmetries (isometries) of $(\mathbb{R}^2, g_0)$. These are compositions of:

translations



rotations



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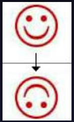

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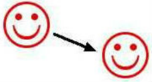
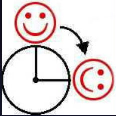
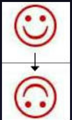

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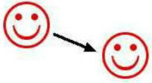

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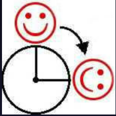

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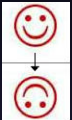

$$\mathbb{R}^2 \cong \mathbb{E}(2)/O(2)$$
$$x \mapsto \{T : T \text{ sends } 0 \text{ to } x\}$$

Riemannian geometry

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A symmetry is a diffeo. $\varphi : M \to M$ which preserves g:

$$\langle d\varphi(X), d\varphi(Y) \rangle_{\varphi(x)} = \langle X, Y \rangle_x, \qquad \forall X, Y \in T_x M.$$

N.B. Can have Riemannian manifolds with no symmetries at all. Thus, a symmetry group is not always present.

Example

$g = (x + y^2)(dx^2 + dy^2)$ has only discrete symmetries.

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$\rightsquigarrow$ bundle of orthonormal frames $\mathcal{F}_{on}(M) \to M$.

Cartan geometry

Klein
geometry
$(G \to G/H, \omega_{MC})$

$\rightsquigarrow$

Cartan
geometry
$(\mathcal{G} \to M, \omega)$

$\uparrow$ $\uparrow$

Euclidean
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$(\mathbb{R}^n, g_0)$

$\rightsquigarrow$

Riemannian
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Parabolic geometries

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In $G = SL_4$, we have for example,

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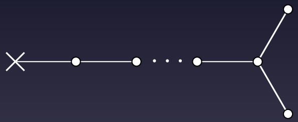
Diagram			
G/P	$\mathbb{P}^3$	$Gr_2(\mathbb{R}^4)$	$F_{1,2}(\mathbb{R}^4)$

A zoo of geometries

Example

G/P	Curved geometry
	projective structure
	2nd order ODE system
	conformal structure - odd dim
	conformal structure - even dim
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Some natural questions

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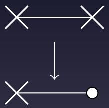

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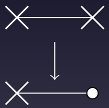

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$\boxed{\text{Curved version?}}$

Q: Which 2nd order ODE are geodesic equations for some connection?

Example 1: 2nd order ODE

Consider

$$y'' = f(x, y, y') \quad \text{up to} \quad (x, y) \mapsto (\tilde{x}(x, y), \tilde{y}(x, y)).$$

(induce $\frac{d\tilde{y}}{d\tilde{x}} = \frac{\tilde{y}_x + \tilde{y}_y y'}{\tilde{x}_x + \tilde{x}_y y'}$, etc.)

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$$\tilde{x} = \frac{a_{21} + a_{22}x + a_{23}y}{a_{11} + a_{12}x + a_{13}y}, \quad \tilde{y} = \frac{a_{31} + a_{32}x + a_{33}y}{a_{11} + a_{12}x + a_{13}y}.$$

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with $\det(a_{ij}) \neq 0$. Get $G = PGL_3$ (or SL_3 instead).

G acts transitively in (x,y,y')-space. Stabilizer at $(0,0,0)$:

$$B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \subset SL_3.$$

2nd order ODE... DG style

Use (p, q) in place of (y', y''). In (x, y, p, q)-space, have a hypersurface Σ given by $q = f(x, y, p)$.

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These properties define a geometric structure:

$$D = L_1 \oplus L_2 \subset T\Sigma$$

Hieroglyphics

$\mathfrak{sl}(3,\mathbb{R}) = 3\times 3$ trace-free matrices; bracket $[a,b] = ab - ba$.

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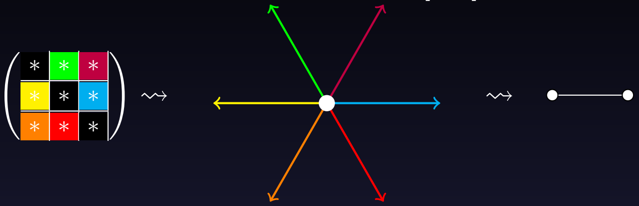

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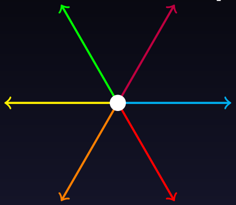

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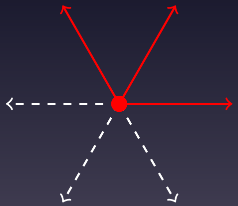

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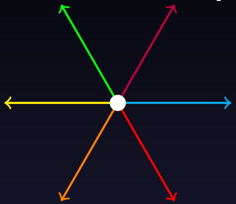
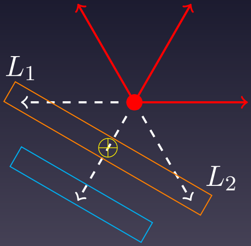
$\mathfrak{sl}(3,\mathbb{R})$ $\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$ $\rightsquigarrow$  $\rightsquigarrow$ 

$\mathfrak{b}$ $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$ $\rightsquigarrow$ 

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A flag manifold

Let ℓ be a line, Π a plane. Then

$$F_{1,2}(\mathbb{R}^3) := \{(\ell, \Pi) : \ell \subset \Pi\}.$$

Let $\ell = e_1$ and $\Pi = \{e_1, e_2\}$. Then

$$Stab_{(\ell,\Pi)} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} = B.$$

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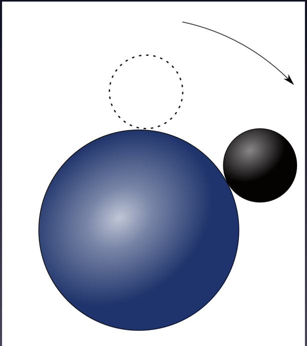
2nd order ODE geometry is a curved version of:

- $y'' = 0$, or
- $F_{1,2}(\mathbb{R}^3)$, or
- $SL(3, \mathbb{R})/B$.

Example 2: $(2,3,5)$-geometry

Example (Two balls rolling - no twisting, no slipping)

- Configuration space M is 5-dimensional.
- No twisting or slipping $\Rightarrow$ constraints on velocity space TM. Get rank 2 distribution $D \subset TM$ of allowable directions.



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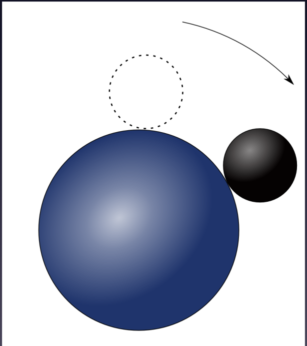

Let $\rho \geq 1$ be the ratio of the radii.
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- $\rho \neq 3$: $\mathrm{SO}(3) \times \mathrm{SO}(3)$ symmetry

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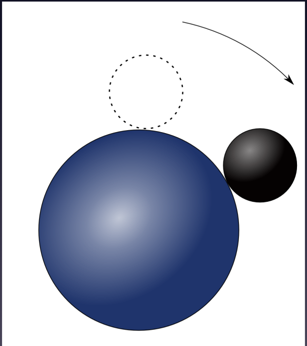

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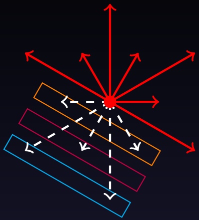
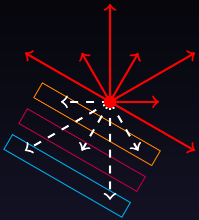
- $\rho \neq 3$: $\mathrm{SO}(3) \times \mathrm{SO}(3)$ symmetry
- $\rho = 3$: $\mathfrak{g}_2$ symmetry *(Bryant, Zelenko, Bor–Montgomery, Baez–Huerta)*
- G_2 = split real form of 14-dim exceptional simple Lie group.

G_2

$\mathfrak{p} \subset \mathfrak{g}_2$



$\mathfrak{p} \subset \mathfrak{g}_2$  

G_2 = Aut. grp of the *split*-octonions $\mathbb{O}'$. On $\mathbb{V} = \mathfrak{Im}(\mathbb{O}')$, $\exists$ G_2-inv. sig. $(3,4)$ scalar product $\langle \cdot, \cdot \rangle$. The space of *null lines* is:

$$\mathcal{Q}^5 = \{[x] \in \mathbb{P}(\mathbb{V}) : \langle x, x \rangle = 0\}.$$

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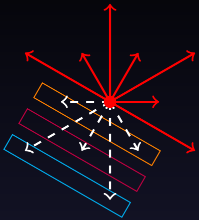

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$(2,3,5)$-geometry is a curved version of:

- two balls in 3:1 ratio rolling w/o slipping or twisting, or
- $\mathcal{Q}^5 \subset \mathbb{P}(\mathfrak{Im}(\mathbb{O}'))$, or
- G_2/P.

Symmetry gaps

Q: What is the gap between maximal and submaximal symmetry dimensions?

Structure	max	submax	Citation
2nd order ODE	8	3	Lie / Tresse (∼1890)
$(2,3,5)$-geometry	14	7	Cartan (1910)
n-dim. projective	n^2+2n	$(n-1)^2+4$	Egorov (1951)

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2013: Classified symmetry gaps for all (complex / split-real) parabolic geometries (Kruglikov & T.)

$\rightsquigarrow$ algebraic story, Dynkin diagram algorithm!

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Examples (Harmonic curvature)

- conformal geometry: Weyl $(n \geq 4)$ or Cotton $(n = 3)$;
  $\rightsquigarrow$ Penrose–Petrov classification in 4-dim Lorentzian case
- $(2, 3, 5)$-distributions: binary quartic;
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These are all instances of Lie algebra cohomology!

Why “harmonic”?

Full curvature κ of $(\mathcal{G} \to M, \omega)$ takes values in:

$$\bigwedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$$

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Kostant's BBW thm $\rightsquigarrow$ simple recipe to compute $H^2(\mathfrak{g}_-, \mathfrak{g})$.