Kähler-Ricci solitons on toric manifolds with positive first Chern class

Xu-Jia Wang and Xiaohua Zhu
Centre for Mathematics and Its Applications
The Australian National University
Canberra, ACT 0200
Australia

Abstract. In this paper we prove there exists a Kähler-Ricci soliton, unique up to holomorphic automorphisms, on any toric Kähler manifold with positive first Chern class, and the Kähler-Ricci soliton is a Kähler-Einstein metric if and only if the Futaki invariant vanishes.

1. Introduction

Since the celebrated work of Yau [Y] on the existence of Kähler-Einstein metrics on Kähler manifolds with negative or vanishing first Chern class, and that of Aubin [A] for Kähler manifolds with negative first Chern class, significant progress has been made in the study of Kähler-Einstein metrics on Fano manifolds, namely Kähler manifolds with positive first Chern class. Notable works include a necessary condition, namely the vanishing Futaki invariant, by Futaki [Fu], the uniqueness of Kähler-Einstein metrics by Bando and Mabuchi [BM], the existence of Kähler-Einstein metrics on compact complex surfaces [T1], and in particular a striking non-existence example [T2]. However the existence of Kähler-Einstein metrics, or more generally the existence of Kähler-Ricci solitons, remains a challenging problem in general. In this paper we give a complete resolution for the existence of Kähler-Einstein metrics or Kähler-Ricci solitons on toric Fano manifolds. That is

Theorem 1.1. There exists a Kähler-Ricci soliton, which is unique up to holomorphic automorphisms, on a toric Kähler manifold with positive first Chern class.

By definition an n-dimensional Kähler manifold is toric if there is a maximal torus $T \subset Aut(M)$ such that $T$ is isomorphic to $(\mathbb{C}^*)^n = R^n \times (S^1)^n$ and $M$ is a compactification of an open orbit of $T$ [O], where $Aut(M)$ denotes the group of bi-holomorphic automorphisms on $M$. 

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Toric manifolds form an important class of complex manifolds and have been a focus of attention in the research area. There have been a number of works on the existence of Kähler-Einstein metrics on toric Fano manifolds with vanishing Futaki invariant. The first notable one is the existence on $\mathbb{C}P^2\#k\mathbb{C}P^2$ for $k = 3$ in [S,TY], which was later extended to $\mathbb{C}P^n\#k\mathbb{C}P^m$ for $n > 2$ and $k = n + 1$ [R], where $\mathbb{C}P^n\#k\mathbb{C}P^m$ denotes the manifold obtained from the complex projective space $\mathbb{C}P^n$ by blowing-up at $k$ generic points. The existence on toric Fano 3 and 4-folds with vanishing Futaki invariant was proved respectively in [M] and in [N1, N2, BS], using the classification of toric Fano manifolds in [B1, B2]. For symmetric toric Fano manifolds, the existence of Kähler-Einstein metrics was recently proved in [BS].

For Fano manifolds with non-vanishing Futaki invariant, the notion of Kähler-Ricci soliton, which describes the limiting behaviour of Kähler-Ricci flow, was introduced in [H]. Kähler-Ricci soliton was indeed introduced much earlier, and was called quasi-Einstein Kähler metric, in physics literature, see [Fr, PTV]. The notion of Kähler-Ricci soliton is an extension of that of Kähler-Einstein metric, it includes Kähler-Einstein metric as a special case, see (1.1) below for definition.

The uniqueness of Kähler-Ricci solitons on Fano manifolds has recently been proved in [TZ1, TZ2]. The existence of Kähler-Ricci solitons on (toric) Fano manifolds with non-vanishing Futaki invariant is desirable. Indeed most toric Fano manifolds have non-vanishing Futaki invariant [B1,B2,M,N1]. For the special manifolds $\mathbb{C}P^n\#k\mathbb{C}P^n$, where $1 \leq k \leq n + 1$, the Futaki invariant vanishes only when $k = n + 1$. However our knowledge on the existence of Kähler-Ricci solitons is limited. For example it is still unknown whether there is a Kähler-Ricci soliton on the manifold $\mathbb{C}P^2\#2\mathbb{C}P^2$. There are some examples of Kähler-Einstein solitons constructed by the ODE method, see [K] for compact case and [C,PTV] for complete non-compact case. In particular in [K] it is proved that the manifold $\mathbb{C}P^n\#\mathbb{C}P^n$ admits a Kähler-Ricci soliton.

Our Theorem 1.1 solved both the existence of Kähler-Einstein metrics and the existence of Kähler-Ricci solitons on toric Fano manifolds. For the manifold $\mathbb{C}P^2\#2\mathbb{C}P^2$, our existence result answered the above mentioned question. Theorem 1.1, together with [T1], completely solved the existence problem for compact complex surfaces with positive first Chern class.

**Corollary 1.2.** There exists a Kähler-Ricci soliton on a compact complex surface with positive first Chern class, and the soliton is a Kähler-Einstein metric if and only if the Futaki invariant vanishes.

A modified Calabi conjecture asserts that the vanishing Futaki invariant is also a sufficient condition for the existence of Kähler-Einstein metrics on Fano manifolds. Our Theorem 1.1 shows this is true for toric manifolds. Another open problem is whether the
holomorphic automorphism group is reductive if the Futaki invariant vanishes. Theorem 1.1 shows this is also true for toric Fano manifolds. Note that the holomorphic automorphism group is reductive if the manifold admits a Kähler-Einstein metric with positive scalar curvature [Ma].

**Corollary 1.3.** For a toric Fano manifold there exists a Kähler-Einstein metric if and only if the Futaki invariant vanishes, and the vanishing Futaki invariant implies that the holomorphic automorphism group of the manifold is reductive.

Let $M$ be a compact Kähler manifold. A Kähler-Ricci soliton on $M$ is a pair $(X, g)$, where $X$ is a holomorphic vector field on $M$ and $g$ is a Kähler metric on $M$, such that

$$
\text{Ric}(\omega_g) - \omega_g = L_X(\omega_g),
$$

where $\text{Ric}(\omega_g)$ is the Ricci form and $\omega_g$ is the Kähler form of the metric $g$, and $L_X$ is the Lie derivative along $X$. If $X = 0$, the Kähler-Ricci soliton reduces to a Kähler-Einstein metric with positive scalar curvature. The uniqueness result in [TZ1, TZ2] shows that a Kähler-Ricci soliton, if it exists, is also unique. Namely the vector field $X$ is uniquely determined and the metric $g$ is unique up to holomorphic automorphisms.

To prove Theorem 1.1 we first need to determine the vector field $X$. This can be done by using a holomorphic invariant, which is an extension of the Futaki invariant, introduced in [TZ2]. That is $X$ is a solution if and only if the holomorphic invariant vanishes, see section 2 below. If $X = 0$, the holomorphic invariant reduces to the Futaki invariant, and the Kähler-Ricci soliton becomes Kähler-Einstein metric. Hence Corollaries 1.2 and 1.3 follows. The existence of the metric $g$ is reduced to solving a complex equation of Monge-Ampère type on the manifold $M$. This equation can be further reduced to a real Monge-Ampère equation on the entire space $\mathbb{R}^n$ for toric Fano manifolds. This property enables us to establish a uniform estimate, and also high order derivative estimates, for the solutions. Therefore the continuity method applies.

This paper is organized as follows. In Section 2 we show that the vector field $X$ can be determined by the Futaki type integral introduced in [TZ2], and that the existence of the metric $g$ can be reduced to studying a real Monge-Ampère equation in $\mathbb{R}^n$. In Section 3 we establish the a priori estimates for the real Monge-Ampère equation. The authors would like to thank Gang Tian for stimulating discussions.

## 2. A holomorphic invariant on toric Fano manifolds

Let $(M, g)$ be an $n$-dimensional compact Kähler manifold with positive first Chern class $c_1(M) > 0$. Since the Ricci form of $\omega_g$,

$$
\text{Ric}(\omega_g) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\det(g_{k\bar{l}}))
$$


represents $c_1(M)$, there is a unique smooth function $h = h_g$ on $M$ such that
\begin{equation}
\begin{cases}
\text{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} h, \\
\int_M e^h \omega^n_g = \int_M \omega^n_g,
\end{cases}
\end{equation}
where $\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^j \in c_1(M)$
denotes the Kähler form of $g$, and $\omega^n_g = \omega_g \wedge \cdots \wedge \omega_g$ is the volume form of $\omega_g$.

Since the first Chern class is positive, there is no harmonic (0,1)-form on $M$. If there
is a nontrivial holomorphic vector field $X$ on $M$, by the Hodge Theorem, there exists a
unique smooth, complex-valued function $\theta_X = \theta_X(g)$ of $M$ such that
\begin{equation}
\begin{cases}
i_X \omega_g = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta_X, \\
\int_M e^{\theta_X} \omega^n_g = \int_M \omega^n_g,
\end{cases}
\end{equation}
where $i_X \omega_g$ is a (0,1)-form, defined by $i_X \omega_g = \omega_g(X, Y)$ for any vector $Y$. The first
relation above implies
\begin{equation}
L_X \omega_g = d[i_X(\omega_g)] = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta_X.
\end{equation}
Hence if $(X, g)$ is a Kähler-Ricci soliton, $h = \theta_X$.

The following holomorphic invariant was introduced in [TZ2],
\begin{equation}
F_X(v) = \int_M v(h - \theta_X)e^{\theta_X} \omega^n_g, \quad v \in \eta(M),
\end{equation}
where $\eta(M)$ denotes the Lie algebra of holomorphic vector fields on $M$. When $X = 0$, the
functional $F_X$ is just the Futaki invariant [Fu]. It was showed in [TZ2] that $F_X$ is
independent of the choice of Kähler metric $g$ and $F_X \equiv 0$ if $(X, g)$ is a Kähler-Ricci
soliton on $M$.

Let $M$ be a toric Fano manifold. We show that there always exists a holomorphic
vector field $X$ on $M$ such that $F_X \equiv 0$. Let $T$ denote the maximal torus of Aut$(M)$
and $\eta_0(M)$ the associated Lie algebra. Then $T \cong (\mathbb{C}^*)^n = \mathbb{R}^n \times (S^1)^n$ and $\eta_0(M)$ is the
maximal Abelian Lie subalgebra of $\eta(M)$. The Cartan decomposition allows us to write
\begin{equation}
\eta(M) = \eta_0(M) + \sum_i \mathbb{C}v_i,
\end{equation}
where $\{v_i\}$ are the common eigenvectors of the adjoint actions $\text{ad}_v$ for all $v \in \eta_0(M)$, namely
\begin{equation}
\text{ad}_v[v_i] = [v, v_i] = \lambda_{v,i} v_i \quad \forall \ v \in \eta_0(M).
\end{equation}
For toric Fano manifolds, the dimension of $\eta_0(M)$ is equal to $n$. Hence for each $v_i$, there
exists $v \in \eta_0(M)$ such that $\lambda_{v,i} \neq 0.$
Lemma 2.1. Let $M$ be an $n$-dimensional toric Fano manifold. Then there exists a unique holomorphic vector field $X \in \eta_0(M)$, of which the imaginary part $\text{Im}(X)$ generates a one-parameter compact subgroup in $\text{Aut}(M)$, such that

$$F_X(v) \equiv 0 \quad \forall \ v \in \eta(M).$$

Proof. It is shown in [TZ2] that there exists a unique holomorphic vector field $X \in \eta_r(M)$, of which the imaginary part $\text{Im}(X)$ generates a one-parameter compact subgroup in $\text{Aut}(M)$, such that $F_X(v) \equiv 0$ for any $v \in \eta_r(M)$, where $\eta_r(M)$ is the reductive Lie subalgebra of $\eta(M)$ containing $\eta_0(M)$. Moreover, $X$ is either the zero vector field or an element of the center of $\eta_r(M)$, and

$$F_X([v, v']) \equiv 0 \quad \forall \ v \in \eta_r(M), \ v' \in \eta(M).$$

(2.7)

Since $X$ is an element of the center of $\eta_r(M)$, we have $X \in \eta_0(M)$. Hence by (2.5) we need only to show that $F_X(v_i) = 0$ for all the $v_i$'s. By (2.6) there exists a $v \in \eta_0(M)$ such that

$$\lambda_{v, i} \neq 0.$$

Hence $F_X(v_i) = \lambda_{v, i}^{-1} F_X([v, v_i]) = 0$. □

Since $M$ is a toric manifold, the Lie algebra $\eta_0(M)$ is spanned by the basis \{\(X_1 = \frac{\partial}{\partial w_1}, \ldots, X_n = \frac{\partial}{\partial w_n}\)\}, where \((w_1, \ldots, w_n) = (x_1 + \sqrt{-1} \theta_1, \ldots, x_n + \sqrt{-1} \theta_n)\) is the affine logarithm coordinates on $T \cong \mathbb{R}^n \times (S^1)^n$. Therefore the vector field $X$ determined in Lemma 2.1 can be expressed in the form

$$X = \sum_i c_i X_i.$$  \hfill (2.8)

In the following we show how the constants $c_i$ are determined by the manifold $M$.

Let $K_0 \cong (S^1)^n$ be the maximal compact Abelian subgroup of $T$. Then there is a $K_0$-invariant Kähler metric $g^0$ on $M$, such that under the affine logarithm coordinates $(w_1, \ldots, w_n)$, the Kähler form $\omega_{g^0}$ is determined by a convex function $u^0$ on $\mathbb{R}^n$ [BS], namely

$$\omega_{g^0} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u^0 \quad \text{on} \ T.$$  

Hence

$$\omega_{g^0}^n = \left(\frac{1}{\pi}\right)^n \det(u_{ij}^0) dx_1 \wedge \ldots \wedge dx_n \wedge d\Theta,$$  \hfill (2.9)

where $d\Theta$ is the standard volume form of $K_0$. The function $u^0$ is uniquely determined by a convex polyhedron $\Omega^* \subset \mathbb{R}^n$ associated to the toric Fano manifold $M$, and can be given explicitly [BS]. That is if $p^{(1)}, \ldots, p^{(m)}$ denote the vertices of $\Omega^*$, then

$$u^0(x) = \log \left(\sum_{i=1}^m e^{(p^{(i)}, x)}\right).$$
This expression implies that the gradient (moment) mapping \(Du^0\) is a diffeomorphism from \(\mathbb{R}^n\) to \(\Omega^*\) and
\[
|\log \det(u^0_{ij}) + u^0| < \infty.
\]
Denote
\[
\tau(x) = \max\{\langle x, p(k) \rangle \mid k = 1, \ldots, m\}.
\]
The graph of \(\tau\) is a convex cone with vertex at the origin. Then \(u^0\) also satisfies
\[
|\tau - u^0| \leq C,
\]
 naming the graph of \(\tau\) is an asymptotical cone of the graph \(u^0\).

From (2.1) we have,
\[
\partial \overline{\mathcal{J}}[e^{u^0 + h\det(u^0_{ij})}] = 0 \text{ on } \mathbb{R}^n.
\]
Hence by (2.10) we have, after normalization,
\[
\det(u^0_{ij}) = e^{-h - u^0}.
\]

From (2.2) we have
\[
\theta_{X_i} = X_i(u^0) + b_i = \frac{\partial u^0}{\partial x_i} + b_i, \quad i = 1, \ldots, n
\]
for some constants \(b_i\). Direct computation shows that \(\mathcal{J}[\Delta \theta_v + \theta_v + v(h)] = 0\) for any vector \(v \in \eta(M)\), where \(\Delta\) is the Laplacian operator with respect to the metric \(\omega^0\). Hence \(v(h) = c_v - \Delta \theta_v - \theta_v\), where the constant \(c_v\) is uniquely determined by the normalization condition (the second formula) in (2.2). Inserting it into (2.4) and integrating by parts, we have
\[
F_X(v) = -\int_M (\theta_v - c_v)e^{\theta_X} \omega^0_g.
\]
Next we show \(b_i = c_{X_i}\). First observe that
\[
\int_M (\theta_v - c_v)e^{h} \omega^0_g = \int_M (-v(h) - \Delta \theta_v)e^{h} \omega^0_g = 0.
\]
Let \(v = X_i\). By (2.9) and (2.13) we see that the left hand side is equal to
\[
\int_{\mathbb{R}^n} \left(\frac{\partial u^0}{\partial x_i} + b_i - c_{X_i}\right)e^{-u^0} dx = (b_i - c_{X_i}) \int_{\mathbb{R}^n} e^{-u^0} dx.
\]
Hence \(b_i = c_{X_i}\) and
\[
\theta_{X_i} = \frac{\partial u^0}{\partial x_i} + c_{X_i},
\]
**Lemma 2.2.** Let $X$ be the holomorphic vector field determined in Lemma 2.1. Then the constants $c_1, \ldots, c_n$ in (2.8) satisfy the equations

$$
\int_{\Omega^*} y_i \exp\{\sum_{l=1}^{n} c_l y_l\} dy = 0, \ i = 1, \ldots, n. \tag{2.16}
$$

**Proof.** By (2.14) and (2.15) we see that $F_X(X_i) = 0$ is equivalent to

$$
\int_{\mathbb{R}^n} \frac{\partial u^0}{\partial x_i} \exp\{\sum_{l=1}^{n} c_l \frac{\partial u^0}{\partial x_l}\} \det(u^0_{pq}) dx = 0,
$$

which is exactly (2.16) by the transformation $y_i = \frac{\partial u^0}{\partial x_i}$. □

**Remark.** Mabuchi [M] proved that if a toric Fano manifold admits a Kähler-Einstein metric, then (2.16) holds with $c_i = 0, \ i = 1, \ldots, n$. Namely the barycenter of $\Omega^*$ is the origin. We note that one can also prove directly the existence and uniqueness of $c_1, \ldots, c_n$ satisfying (2.16). See, e.g., [MTW].

To solve equation (1.1) we also need to determine the metric, namely to find a metric $g$ such that $(X, g)$ solves equation (1.1). Let $\omega_g := \omega_{g^0} + \sqrt{-1} \partial \bar{\partial} \varphi$. By (2.1) and (2.3) one sees that (1.1) is equivalent to the complex Monge-Ampère equation

$$
\begin{cases}
\det(g^0_{ij} + \varphi_{ij}) = \det(g^0_{ij}) \exp\{h - \theta_X - X(\varphi) - \varphi\} \\
(g^0_{ij} + \varphi_{ij}) > 0.
\end{cases} \tag{2.17}
$$

Since the metric $\omega_{g^0}$ is $K_0$-invariant, we want to find a $K_0$ invariant solution $\varphi$. Let $u = u^0 + \varphi$. Note that

$$
\theta_X + X(\varphi) = \sum_{l=1}^{n} c_l \frac{\partial u}{\partial x_l} + c_X.
$$

By (2.13) we arrive at the real Monge-Ampère equation

$$
\det(u_{ij}) = \exp\{-c_X - u - \sum_{l=1}^{n} c_l \frac{\partial u}{\partial x_l}\} \text{ in } \mathbb{R}^n. \tag{2.18}
$$

Note that if $u$ is a solution of (2.18), by the second formula in (2.17), $u$ is convex. It follows that $Du(\mathbb{R}^n) = Du^0(\mathbb{R}^n) = \Omega^*$ by the boundedness of $\varphi = u - u^0$. In the next section will use the continuity method to solve equation (2.17).
3. The a priori estimates

To prove the solvability of equation (2.17), we use the continuity method. Consider the equation

\[
\begin{align*}
\det(g^{0}_{ij} + \varphi_{ij}) &= \det(g^{0}_{ij}) \exp\{h - \theta X - X(\varphi) - t\varphi\}, \\
(g^{0}_{ij} + \varphi_{ij}) &> 0,
\end{align*}
\]

(3.1)

where \(t\) is a parameter in \([0, 1]\). When \(t = 0\), the solvability of (3.1) was proved in [Z], and the solution is unique up to a constant. Suppose \(\varphi_t\) is a \(K_0\)-invariant solution of (3.1). It is known that for any \(t \in [0, 1]\), the linearized operator

\[
L_{\varphi_t} \psi = \Delta \psi + X(\psi) + t\psi,
\]

(3.2)

is invertible [TZ1], where \(\Delta\) is the Laplacian operator with respect to the metric \(\omega_{\varphi_t} = \omega_{g^0} + \sqrt{-1} \frac{\partial X(\varphi)}{2\pi} \partial\bar{\partial} \varphi_t\). Thus by the implicit function theorem, there is a positive constant \(\varepsilon_0 > 0\) such that (3.1) admits a smooth \(K_0\)-invariant solution for any \(t \in [0, \varepsilon_0]\), which is unique by [TZ1].

We will apply the continuity method for \(t \in [\varepsilon_0, 1]\). For this purpose we need to establish the a priori estimates for equation (3.1) for \(t \in [\varepsilon_0, 1]\). By the a priori estimates in [Y], which was extended to cover equation (3.1) [TZ1], it suffices to establish the uniform estimate for (3.1) for \(t \in [\varepsilon_0, 1]\).

Let \(\varphi\) be a solution of (3.1). Let \(u = u^0 + \varphi\). Then similarly to (2.18), \(u\) satisfies

\[
\det(u_{ij}) = \exp\{-cX - w - \sum_{l=1}^{n} c_l \frac{\partial u}{\partial x_l}\} \quad \text{in} \quad \mathbb{R}^n,
\]

(3.3)

where

\[
w = w_t := tu + (1 - t)u^0.
\]

(3.4)

To obtain a uniform estimate for \(\varphi\), it suffices to establish a uniform estimate for \(u - u^0\). We will use the following well known result for convex domains [Mi].

**Lemma 3.1.** Let \(\Omega\) be a bounded convex domain in \(\mathbb{R}^n\). Then there is a unique ellipsoid \(E\), called the minimum ellipsoid of \(\Omega\), which attains minimum volume among all ellipsoids containing \(\Omega\), such that

\[
\frac{1}{n} E \subset \Omega \subset E,
\]

(3.5)

where \(\alpha E\) denotes the \(\alpha\)-dilation of \(E\) with concentrated center.

Let \(T\) be a linear transformation with \(|T| = 1\), which leaves the center \(x_0\) of \(E\) invariant, namely \(T(x) = A(x - x_0) + x_0\) for some matrix \(A\), such that \(T(E)\) is a ball \(B_R\). Then we have \(B_{R/n} \subset T(\Omega) \subset B_R\) for two balls with concentrated center.
Lemma 3.2. We have
\[ m_t = \inf_{\mathbb{R}^n} w_t(x) \leq C \] (3.6)
for some \( C > 0 \) independent of \( t \in [\varepsilon_0, 1] \).

Proof. For any nonnegative integer \( k \), we denote
\[ A_k = \{ x \in \mathbb{R}^n : m_t + k \leq w(x) \leq m_t + k + 1 \}. \]

Then for any \( k \geq 0 \), \( \bigcup_{i=0}^{k} A_i = \{ w < m_t + k + 1 \} \) is convex. Observe that \( Dw(\mathbb{R}^n) = \Omega^* \) and by (2.16), the origin is contained in \( \Omega^* \). Hence for any \( k \geq 0 \), \( A_k \) is a bounded set and the minimum \( m_t \) is attained at some point in \( A_0 \).

By equation (3.3) we have
\[ \det(w_{ij}) \geq t^n \det(u_{ij}) \geq t^n e^{-cX_d} e^{-w}, \]
where \( d = \sup \{ c_l y_l : y \in \Omega^* \} \). Recall that \( t \geq \varepsilon_0 \), we obtain
\[ \det(w_{ij}) \geq C_0 e^{-m_t} \text{ in } A_0, \]
where \( C_0 = \varepsilon_0^n e^{-cX_d-1} \). By Lemma 3.1 there exists a linear transformation \( y = T(x) \) with \( |T| = 1 \), which leaves the center of the minimum ellipsoid of \( A_0 \) invariant, such that \( B_{R/n} \subset T(A_0) \subset B_R \). The above equation is unchanged under the linear transformation. We claim
\[ R \leq \sqrt{2} n C_0^{-1/2n} e^{m_t/2n}. \] (3.7)
Indeed, let
\[ v(y) = \frac{1}{2} C_0^{1/n} e^{-m_t/n} \left[ |y - y_t|^2 - \left( \frac{R}{n} \right)^2 \right] + m_t + 1, \]
where \( y_t \) is the center of the minimum ellipsoid of \( A_0 \). Then
\[ \det(v_{ij}) = C_0 e^{-m_t} \text{ in } T(A_0), \]
and \( v \geq w \) on \( \partial T(A_0) \). Hence by the comparison principle we have \( v \geq w \) in \( T(A_0) \). In particular we have
\[ m_t \leq w(y_t) \leq v(y_t) = -\frac{1}{2} C_0^{1/n} e^{-m_t/n} \left( \frac{R}{n} \right)^2 + m_t + 1. \]
Hence (3.7) follows.
By the convexity of $w$, we have

$$T(A_k) \subset B_{2(k+1)R}.$$  

We obtain

$$\int_{\mathbb{R}^n} e^{-w} = \sum_k \int_{T(A_k)} e^{-w}$$

$$\leq \sum_k e^{-m_t-k}|T(A_k)|$$

$$\leq \omega_n \sum_k e^{-m_t-k}2(k + 1)R^n$$

$$= \omega_n (2R)^n \sum (k + 1)^n \frac{e^m}{e^k}$$

$$\leq Ce^{-m_t/2},$$

where $\omega_n$ is the area of the sphere $S^{n-1}$. We note that the above integration is invariant under any linear transformation $T$ with $|T| = 1$. Returning to the coordinates $x$, by equation (3.3) we have

$$e^{-m_t/2} \geq \frac{1}{C} \int_{\mathbb{R}^n} e^{-w} dx$$

$$= \frac{1}{C} e^c \int_{\Omega^*} \exp\left\{\sum_{i=1}^n c_i y_i \right\} dy = C_1,$$

where we have used the transformation $y = Du(x)$. Hence $m_t \leq C$. $\square$

Lemma 3.3. Let $x^t = (x^t_1, ..., x^t_n) \in \mathbb{R}^n$ be the minimal point of $w = w_t$. Then

$$|x^t| \leq C$$

for some uniform constant $C$.

Proof. By equation (3.3),

$$\int_{\mathbb{R}^n} e^{-w} dx = e^c \int_{\Omega^*} \exp\left\{\sum_{i=1}^n c_i y_i \right\} dy = \beta$$

for some constant $\beta$. Recall that $|Dw| \leq d_0 := \sup\{|x| : x \in \Omega^*\}$. Hence by (3.6) there exists $R > 0$ such that $\inf_{\partial B_R(x^t)} w \geq m_t + 1$. By convexity we have

$$|Dw(x)| \geq 1/R \quad \text{in} \quad \mathbb{R}^n \setminus B_R(x^t).$$

Hence for any $\varepsilon > 0$ small, there exists $R_\varepsilon$ sufficiently large such that

$$\int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(x^t)} e^{-w} dx \leq C \int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(x^t)} e^{|x-x^t|/R} \leq \varepsilon,$$
where both $R$ and $R_\varepsilon$ are independent of $t$.

Observe that $u^0$ is a convex function defined on $\mathbb{R}^n$ satisfying $Du^0(\mathbb{R}^n) = \Omega^*$, and by (2.16) the origin $0 \in \Omega^*$. Hence for any small $\varepsilon > 0$, there exists a large constant $C > 0$ such that if $|x^t| > C$,

$$\frac{\partial u^0}{\partial \xi} > \frac{1}{2} a_0 \text{ in } B_{R_\varepsilon}(x^t),$$

where $\xi = \frac{x^t}{|x^t|}$ and $a_0 = \inf \{|x| : x \in \partial \Omega^*\}$. To see the above inequality it suffices to consider the restriction of $u^0$ on the ray $\overrightarrow{ox^t}$. Hence

$$\int_{B_{R_\varepsilon}(x^t)} \frac{\partial u^0}{\partial \xi} e^{-w} dx \geq \frac{1}{4} a_0 \beta,$$

and

$$\int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(x^t)} \frac{\partial u^0}{\partial \xi} e^{-w} dx \leq d_0 \int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(x^t)} e^{-w} dx \leq \varepsilon d_0.$$

If $\varepsilon > 0$ is sufficiently small, we obtain

$$\int_{\mathbb{R}^n} \frac{\partial u^0}{\partial \xi} e^{-w} dx > 0.$$

On the other hand, by (2.16) and (3.3),

$$0 = \int_{\Omega^*} y_i \exp\{\sum_{l=1}^{n} c_i y_l\} dy$$

$$= \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i} \exp\{\sum_{l=1}^{n} c_i \frac{\partial u}{\partial x_l}\} \det(u_{pq}) dx$$

$$= e^{-cx} \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i} e^{-w} dx$$

$$= \frac{1 - t}{t} e^{-cx} \int_{\mathbb{R}^n} \frac{\partial u^0}{\partial x_i} e^{-w} dx.$$

We obtain

$$\int_{\mathbb{R}^n} \frac{\partial u^0}{\partial \xi} e^{-w} dx = 0$$

for any unit vector $\xi \in \mathbb{R}^n$. We reach a contradiction. This completes the proof. $\square$

**Lemma 3.4.** Let $\varphi = \varphi_t$, where $t \in [\varepsilon_0, 1]$, be a solution of (3.1). Then

$$\sup_M \varphi < C$$

(3.11)

for some $C > 0$ independent of $t \in [\varepsilon_0, 1]$.

**Proof.** Let $\overline{v}$ be the function defined in (2.11). Since the graph of $\overline{v}$ is the asymptotical cone of the graph of $u^0$ and $Du(\mathbb{R}^n) = Du^0(\mathbb{R}^n) = \Omega^*$, by convexity we have

$$\overline{v}(x) + a \geq u(x) \quad \forall x \in \mathbb{R}^n,$$
where \( a = u(0) - \overline{v}(0) \). Hence by (2.12) we have
\[
\varphi = u - u^0 \leq \overline{v} - u^0 + a \leq C + a.
\]
Hence to prove (3.11) it suffices to show \( u(0) \) is upper bounded.

Let \( \chi \) be the minimal point of \( w = w_t \). Then by Lemma 3.3, \( |\chi| \leq C \). Since \( |Dw| \leq d_0 \), by (3.6) we have \( w(0) \leq C \). From (3.9) we also have \( w(0) \geq -C \). Hence we have \( |u(0)| \leq C \) by (3.4).

It remains to prove that \( \varphi \) is lower bounded. That is

**Lemma 3.5.** Let \( \varphi \) be as in Lemma 3.4. We have
\[
\inf_M \varphi \geq -C. \tag{3.12}
\]

We have two different proofs for Lemma 3.5. We include both proofs here.

**Proof (i).** By (2.12) it suffices to prove
\[
\sup_{|x| = r} (\overline{v} - u)(x) \leq C. \tag{3.13}
\]
From the proof of Lemma 3.4, we have \( |u(0)| \leq C \). By subtracting a constant we may suppose \( u(0) = 0 \). Note that if \( u \) is a solution of (3.3), so is \( u + C \) for any constant \( C \), with the right hand side of (3.3) multiplied by \( e^{tC} \).

For \( r > 0 \) we denote
\[
z(r) = \sup_{|x| = r} (\overline{v} - u)(x). \tag{3.14}
\]
Suppose the supremum is attained at \( p = p_r \). Then we have
\[
z'(r) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( z(r) - z(r - \varepsilon) \right)
\leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ (\overline{v} - u)(p) - (\overline{v} - u)(p - \varepsilon \xi) \right]
= \partial_\xi (\overline{v} - u)(p), \tag{3.15}
\]
where \( \xi = p/|p| \) is a unit vector. We claim that for \( r > 0 \) large (say \( r > r_0 \) for some \( r_0 > 0 \) independent of \( t \)),
\[
z'(r) \leq r^{-2}. \tag{3.16}
\]
Since the right hand side is integrable from 1 to \( \infty \), Lemma 3.5 follows immediately.

Let \( F^{(k)} \subset \{ x_{n+1} = p^{(k)} \cdot x \}, k = 1, \cdots, m, \) be the faces of the convex cone \( G_{\overline{v}} \) (the graph of \( \overline{v} \)). Suppose \( (p, \overline{v}(p)) \) is located on the face \( F^{(1)} \). Replacing \( u \) by \( u - p^{(1)} \cdot x \) and \( \overline{v} \) by \( \overline{v} - p^{(1)} \cdot x \) we may suppose \( F^{(1)} \) is contained in \( \{ x_{n+1} = 0 \} \). Choosing a proper coordinate system we suppose
\[
F^{(1)} \subset \{ x_1 > a^*|\hat{x}| \} \cap \{ x_{n+1} = 0 \}, \tag{3.17}
\]
where $a^*$ is a positive constant, and $\hat{x} = (x_2, \cdots, x_n)$.

Let $q = q_r$ be a point in $\{x_1 = \frac{1}{2}p_1\}$ (where $p = (p_1, \cdots, p_n)$) such that

$$
u(q) = \inf\{u(x) : x \in F^{(1)} \cap \{x_1 = \frac{1}{2}p_1\}\}.$$  

If (3.16) does not hold at some large $r$, by (3.15) we have $\partial \zeta u(p) \leq -Cr^{-2}$ since $v = 0$ on $F^{(1)}$, where $\zeta = (p - q)/|p - q|$. Hence by convexity we have $\partial \zeta u(x) \leq -Cr^{-2}$ for any $x$ on the line segment $pq$. It follows

$$u(q) - u(p) \geq Cr^{-1}.$$ (3.18)

Next we use a technique from the study of real Monge-Ampère equation [P] to prove (3.16). Denote $S = \{x \in \mathbb{R}^n \mid u = u(q)\}$ and $S^0 = \{x \in \mathbb{R}^n \mid u < u(q)\}$. Let $\psi$ be a convex function defined on the whole space $\mathbb{R}^n$ such that the graph of $\psi$ is a convex cone with vertex at $(p, u(p))$ (i.e. $\psi(p) = u(p)$) and $\psi = u$ on $S$. Then we have

$$N_u(S^0) \supset N_\psi(S^0),$$

where $N_u$ denotes the normal mapping of $u$, defined by $N_u(E) = \bigcup_{x \in E} N_u(x)$, where

$$N_u(x) = \{p \in \mathbb{R}^n \mid u(y) \geq u(x) + p \cdot (y - x) \forall y \in \mathbb{R}^n\}.$$  

By our choice of $q$, we have $\text{dist}(0, S^0) \geq Cr$. Hence by (3.10) we have

$$w(x) \geq \delta |x| - C \geq \delta r - C \quad \text{in} \quad S^0$$

for some $\delta > 0$. By (3.3), $\det D^2 u \leq Ce^{-w}$. We therefore obtain

$$|N_u(S^0)| = \int_{S^0} \det D^2 u \leq Ce^{-\delta r}.$$  

Since the graph of $\psi$ is a convex cone, $N_\psi(S^0) = N_\psi(\mathbb{R}^n)$ is convex. By (3.17) and (3.18) we have the estimate

$$|N_\psi(S^0)| \geq Cr^{-n-1}.$$  

Therefore we obtain $r^{-n-1} \leq Ce^{-\delta r}$. When $r$ is sufficiently large, this is a contradiction.

□

Proof (ii). We prove a Harnack type inequality, namely

$$- \inf_M \varphi \leq C(1 + \sup_M \varphi).$$ (3.19)
To prove (3.19) we recall two functionals introduced in [TZ1]:

\[
I(\psi) = \frac{1}{V} \int_M \psi(e^{\theta x} \omega^n_{g^0} - e^{\theta x} X(\psi) \omega^n_\psi),
\]

\[
J(\psi) = \frac{1}{V} \int_0^1 \int_M \frac{\partial \eta_s}{ds} (e^{\theta x} \omega^n_{g^0} - e^{\theta x} X(\eta_s) \omega^n_{\eta_s}) \wedge ds,
\]

which are defined on the space

\[
\mathcal{M}_X(\omega_{g^0}) = \{ \psi \in C^\infty(M) \mid \omega_\psi > 0 \text{ and } \text{Im}(X(\psi)) = 0 \},
\]

where \( V = \int_M \omega^n_{g^0}, \omega_\psi = \omega_{g^0} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi, \) and \( \eta_s(0 \leq s \leq 1) \) is a path from 0 to \( \psi \) in \( \mathcal{M}_X(\omega_{g^0}) \). It was proved that the functional \( J(\psi) \) is independent of the choice of path \( \eta_s \) and \( I(\psi) - J(\psi) \geq 0 \). Furthermore, it was proved in [CTZ] that there is a positive number \( c_0 \) such that

\[
J(\psi) \geq c_0 I(\psi) \quad \forall \psi \in \mathcal{M}_X(\omega_{g^0}). \tag{3.20}
\]

Let

\[
F(\psi) = J(\psi) - \frac{1}{V} \int_M \psi e^{\theta x} \omega^n_{g^0}.
\]

Then one has [TZ2]

\[
F(\varphi) = -\frac{1}{t} \int_0^t (I(\varphi_s) - J(\varphi_s)) ds \leq 0,
\]

where \( \varphi_s \) is the solution of (3.1) at \( t = s \). Thus we get from (3.20),

\[
I(\varphi) \leq \frac{1}{c_0} \frac{1}{V} \int_M \varphi e^{\theta x} \omega^n_{g^0} \leq \frac{1}{c_0} \sup_M \varphi. \tag{3.21}
\]

On the other hand, by the Green formula, we have

\[
\sup_M \varphi \leq \frac{1}{V} \int_M \varphi e^{\theta x} \omega^n_{g^0} + C_1, \tag{3.22}
\]

for some uniform constant \( C_1 \) depending only on \( g^0 \) [TZ2]. Observe that

\[
V = \int_M e^{h - t\varphi} \omega^n_{g^0} = \int_M e^h \omega^n_{g^0}.
\]

We have \( \sup_M \varphi \geq 0 \). It follows

\[
\frac{1}{V} \int_M \varphi e^{\theta x} \omega^n_{g^0} \geq -C_1. \tag{3.23}
\]

Hence combining (3.21) and (3.23), we obtain

\[
-\frac{1}{V} \int_M \varphi e^{\theta x + X(\varphi)} \omega^n_\varphi \leq \frac{1}{c_0} \sup_M \varphi + C_1.
\]

Now the Harnack inequality (3.19) follows from the inequality

\[
-\inf_M \varphi \leq -\frac{1}{V} \int_M \varphi e^{\theta x + X(\varphi)} \omega^n_\varphi + C_2, \quad t \geq \varepsilon_0,
\]

for some uniform constant \( C_2 \) depending only on \( \varepsilon_0 \), which was proved in [CTZ]. □
References


[CTZ] Cao, H.D., Tian, G., and Zhu, X.H., Kähler-Ricci solitons on compact Kähler manifolds with \(c_1(M) > 0\), Preprint.


Centre for Mathematics and Its Applications, Australian National University, Canberra, ACT 0200, Australia
E-mail address: wang@maths.anu.edu.au, zhux@maths.anu.edu.au