A Variational Theory of the Hessian Equation

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Abstract

By studying a negative gradient flow of certain Hessian functionals we establish the existence of critical points of the functionals and consequently the existence of ground states to a class of nonhomogenous Hessian equations. To achieve this we derive uniform, first- and second-order a priori estimates for the elliptic and parabolic Hessian equations. Our results generalize well-known results for semilinear elliptic equations and the Monge-Ampère equation.

1 Introduction

In this paper we study the Dirichlet problem of the k-Hessian equation ($k = 1, 2, \dots, n$)

$$\begin{cases} S_k(D^2u) = \psi(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $\psi(x,u)$ is a nonnegative function in $\overline{\Omega} \times \mathbb{R}$. Here $S_k(D^2u)$ is the k-Hessian operator of u. Recall that it is defined in the following way: Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be the eigenvalues of the Hessian matrix of u, D^2u, and let $S_k(\lambda)$ be the k^{th} elementary symmetric function of λ. Then $S_k(D^2u) = S_k(\lambda(D^2u))$. Alternatively, it can be written as the sum of the $k \times k$ principal minors of D^2u.

To work in the realm of elliptic operators, one has to restrict the class of functions and domains. Following [4], a function u in $C^2(\Omega) \cap C^0(\overline{\Omega})$ is called a k-admissible function if $\lambda(D^2u(x))$, $x \in \Omega$, belongs to the symmetric cone given by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_j(\lambda) > 0,\ j = 1, 2, \dots k\}\,.$$

Note that Γ_k always contains the positive cone $\Gamma_n = \{\lambda \in \mathbb{R}^n : \lambda_1, \dots, \lambda_n > 0\}$. The k-Hessian operator is elliptic at any k-admissible u, i.e.,

$$\left\{S_k^{ij}(D^2u)\right\} \equiv \left\{\frac{\partial}{\partial r_{ij}} S_k(D^2u)\right\}$$

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is positive definite. On the other hand, a hypersurface in $\mathbb{R}^n$ is k-convex (respectively, strictly k-convex) for $k \in \{1, 2, \dots, n-1\}$ if its principal curvatures $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$ satisfy $S_k(\kappa) \geq 0$ (respectively, $S_k(\kappa) \geq \delta_0 > 0$ for some δ_0) everywhere on the hypersurface. It is shown in [4] that whenever (1.1) admits a classical solution in $C^2(\overline{\Omega})$, it is necessary that $\partial\Omega$, regarded as a hypersurface in $\mathbb{R}^n$, be strictly $(k-1)$-convex.

From now on we shall always assume Ω is strictly $(k-1)$–convex and look for k-admissible solutions in (1.1). Notice that a k-admissible solution is subharmonic and, by the maximum principle, is negative in Ω. This, in particular, means that the value of $\psi(x, z)$ for $z > 0$ is irrelevant in solving (1.1).

The Hessian equations (1.1) constitute an important class of fully nonlinear elliptic equations. It is semilinear when $k = 1$ and of Monge-Ampère type when $k = n$. General fully nonlinear elliptic equations have been studied by many authors including [2, 4, 9, 10, 13, 18, 19, 27]. A priori global estimates for the solutions can be found in [4, 9] where ψ is nondegenerate (i.e., $\psi \geq \psi_0 > 0$) and in [13] for the degenerate case (i.e., $\psi \geq 0$). The regularity result was extended to Hessian quotient equations in [19]. With the a priori estimates at hand, one can derive existence and uniqueness results by using the method of continuity. A basic assumption for which the method works is that ψ must be monotone increasing in z.

In many situations the monotonicity condition is not satisfied and $\psi(x, 0) = 0$. In this case (1.1) always admits the trivial solution $u \equiv 0$. However, one is interested in looking for nonzero solutions that do not change sign in Ω, i.e., ground states. Let's look at the semilinear case

$$\begin{cases} -\Delta u = \psi(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.2}$$

The search for ground states was motivated by applications in physics and geometry. Nowadays there is a rich spectrum of results concerning the ground states for (1.2); see, for instance, [3, 5, 15]. Among them a fundamental and influential result is the following theorem of Ambrosetti and Rabinowitz [1]:

THEOREM 1.1 *Suppose that $\psi \in C^{0,1}(\overline{\Omega} \times \mathbb{R})$ satisfies*

$$\lim_{z\to 0} \frac{\psi(x,z)}{z} < \lambda_1\,, \quad \lim_{z\to+\infty} \frac{\psi(x,z)}{z} > \lambda_1\,, \quad \lim_{z\to+\infty} \frac{\psi(x,z)}{z^{(n+2)/(n-2)}} = 0\,,$$

uniformly in $\overline{\Omega}$, and there exists a constant $\theta \in (0, \frac{1}{2})$ such that

$$\int_0^z \psi(x,s)ds \leq \theta z \psi(x,z)$$

for large z. Then (1.2) *has a positive solution.*

Here λ_1 is the first Dirichlet eigenvalue for the Laplacian operator. When applied to the special case $\psi = |u|^{p-1}u$, it shows that a positive solution exists if

$1 < p < \frac{n+2}{n-2}$. On the other hand, the well-known Pohozaev's identity implies that no positive solution can exist when $p \geq \frac{n+1}{n-2}$ and Ω is star-shaped.

In this paper we develop a variational theory for (1.1). Our main result is a full generalization of Ambrosetti-Rabinowitz's result to other Hessian equations.

THEOREM 1.2 *Consider* (1.1) *where* $\psi \in C^{1,1}(\overline{\Omega \times \mathbb{R}^-})$ *and* Ω *is of class* $C^{3,1}$. *Suppose that* $\psi(x,z) > 0$ *for* $z < 0$ *and satisfies*

$$\lim_{z \to 0^-} \frac{\psi(x,z)}{|z|^k} < \lambda_1 \,, \tag{1.3}$$

$$\lim_{z \to -\infty} \frac{\psi(x,z)}{|z|^k} > \lambda_1 \,, \tag{1.4}$$

and

$$\begin{cases} \lim_{z \to -\infty} \frac{\psi(x,z)}{|z|^{k^*-1}} = 0 & \text{if } k < \frac{n}{2} \\ \lim_{z \to -\infty} \frac{\psi(x,z)}{|z|^{p}} = 0 & \text{if } k = \frac{n}{2} \end{cases} \tag{1.5}$$

for some large $p > 0$ *uniformly in* $\overline{\Omega}$, *and there exist constants* $\theta > 0$ *and large* M *such that when* $z < -M$,

$$\int_z^0 \psi(x,s)ds \leq \frac{1-\theta}{k+1}|z|\psi(x,z) \,. \tag{1.6}$$

Then (1.1) *has a nontrivial* k*-admissible solution in* $C^{3,\alpha}(\Omega) \cap C^{0,1}(\overline{\Omega})$ *for some* $\alpha \in (0,1)$.

Here k^* is the critical exponent for the k-Hessian operator,

$$k^* \begin{cases} = \frac{n(k+1)}{n-2k} & \text{if } 2k < n \\ < \infty & \text{if } 2k = n \\ = \infty & \text{if } 2k > n \,. \end{cases}$$

(Nevertheless, our recent studies show that one should take $k^* = n(k+1)/(n-2k)$ when $2k > n$ in some other cases.) Moreover, λ_1 is the "first eigenvalue" for the k-Hessian operator. Actually, it was proven in [28] that for each k there exists a unique $\lambda_1 > 0$ such that the problem

$$\begin{cases} S_k(D^2u) = \lambda_1 |u|^k & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits an admissible solution that is unique up to multiplication by a positive number.

The condition (1.4), which corresponds to the superlinear case in (1.2), will also be referred to as the superlinear case. Under (1.6), it is equivalent to

$$\lim_{z \to -\infty} \frac{\psi(x,z)}{|z|^k} = \infty \,.$$

We may also consider the sublinear case, that is,

$$\lim_{z\to-\infty}\frac{\psi(x,z)}{|z|^k}<\lambda_1 \tag{1.7}$$

uniformly on Ω. The following theorem, which is easier to prove than Theorem 1.2, covers the sublinear case.

THEOREM 1.3 *Consider* (1.1) *where* $\psi\in C(\overline{\Omega\times\mathbb{R}^-})\cap C^{1,1}(\overline{\Omega}\times\mathbb{R}^-)$ *and* Ω *is of class* $C^{3,1}$. *Suppose that* $\psi(x,z)>0$ *for* $z<0$ *and satisfies* (1.7) *and*

$$\lim_{z\to0^-}\frac{\psi(x,z)}{|z|^k}>\lambda_1 \tag{1.8}$$

uniformly on Ω. *Then* (1.1) *has a nontrivial k-admissible solution in* $C^{3,\alpha}(\Omega)\cap C(\overline{\Omega})$ *for some* $\alpha\in(0,1)$.

In particular, Theorems 1.3 and 1.2 apply to

$$\psi(x,z)=|z|^p\,,\quad p\in(0,k)\text{ and }(k,k^*-1),\ \text{respectively.}$$

Special cases of Theorems 1.2 and 1.3 were established in [25, 28] for $k=n$ and in [6] for general k in the radial case.

Our proof of these theorems explores the variational structure of the problem [26]. First of all, we have (see, e.g., [17])

$$\sum_j\frac{\partial}{\partial x_j}(S^{ij}(D^2u))=0\quad\text{for each } j=1,2,\dots,n, \tag{1.9}$$

(we have dropped the subscript k in $S_k^{ij}(D^2u)$). Hence

$$S_k(D^2u)=\frac{1}{k}\sum u_{ij}S^{ij}(D^2u)=\frac{1}{k}\sum\frac{\partial}{\partial x_j}(u_iS^{ij}(D^2u))\,.$$

Denote by $\Phi_0^k=\Phi_0^k(\Omega)$ the collection of all admissible functions vanishing on the boundary $\partial\Omega$. We introduce the functional

$$E_k(u)=-\int_\Omega uS_k(D^2u)dx=\frac{1}{k}\int_\Omega S^{ij}(D^2u)u_iu_j\,dx\,,\quad u\in\Phi_0^k\,.$$

Since $S^{ij}(D^2u)$ is positive definite, we have $E_k(u)\geq0$ for any $u\in\Phi_0^k$. Setting

$$\|u\|_{\Phi_0^k}=[E_k(u)]^{1/(k+1)}\,,\quad u\in\Phi_0^k\,,$$

$\|\cdot\|_{\Phi_0^k}$ is a norm in Φ_0^k [29]. Using (1.9) it is easy to see that the Euler-Lagrange equation of the functional

$$J(u)=\frac{-1}{k+1}\int_\Omega uS_k(D^2u)dx-\int_\Omega\Psi(x,u)dx$$

where

$$\Psi(x,z)=\int_z^0\psi(x,s)ds$$

is precisely (1.1).

The critical exponent k^* for the Hessian operator was first determined in [26]. In fact, similar to the semilinear elliptic equation (1.2) we have a corresponding Pohozaev's identity which shows that (1.1) does not admit admissible solutions if Ω is star-shaped and $\psi = |u|^p$, $p > k^* - 1$, or more generally, ψ satisfies

$$n\Psi(x,z) - \frac{n-2k}{k+1} z\psi(x,z) + x_i \Psi_i(x,z) > 0 \quad \text{in } \Omega \times \mathbb{R}\,.$$

It is clear that one needs to establish a Sobolev-type inequality for the Hessian operator in the study of the existence theory. The Sobolev inequality was proven in [24] in the convex category. Its full version was subsequently established in [20, 29]. Following the latter, we state the following:

THEOREM 1.4 (Hessian Sobolev Inequality) *Let Ω be a $(k-1)$-convex domain with C^2 boundary and let $u \in \Phi_0^k(\Omega)$.*

(i) *For $1 \le k < \frac{n}{2}$,*
$$\|u\|_{L^p(\Omega)} \le C\|u\|_{\Phi_0^k}\,, \quad \forall p \in [1, k^*]\,,$$
where C depends only on n, k, p, and $|\Omega|$.

(ii) *For $k = \frac{n}{2}$,*
$$\|u\|_{L_\Psi^*(\Omega)} \le C\|u\|_{\Phi_0^k}\,,$$
where C depends only on n and $\operatorname{diam}\Omega$, $\Psi(t) = e^{t^{(n+2)/n}} - 1$, *and $L_\Psi^*(\Omega)$ is the Orlicz space associated with Ψ.*

(iii) *For $\frac{n}{2} < k \le n$,*
$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{\Phi_0^k}\,,$$
where C depends on n, k, and Ω.

We note that in (i), the best constant C is attained when $\Omega = \mathbb{R}^n$ by the function

$$u(x) = (1 + |x|^2)^{(2k-n)/2k}$$

at the critical case $p = k^*$. Incidentally, we point out that further integral estimates can be found in [22, 23]. For instance, it is shown that for $u \in \Phi_0^k$,

$$\|u\|_{W^{1,p}(\Omega')} \le C\|u\|_{L^1(\Omega)} \tag{1.10}$$

for any $p < nk/(n-k)$ and $\Omega' \Subset \Omega$, where C depends only on n, k, p, and $\operatorname{dist}(\Omega', \partial\Omega)$. In particular, any admissible function is locally Hölder-continuous when $k > n/2$. In [22] we also proved a Poincaré-type inequality for admissible functions. A special case is

$$\int_\Omega |Du|^2 \le C\left|\int_\Omega uS_k(D^2u)\right|^{2/(k+1)}\,, \quad u \in \Phi_0^k(\Omega)\,. \tag{1.11}$$

With the Sobolev inequality at hand, we can, in principle, use the powerful variational methods developed for the semilinear problem (1.2) to study (1.1). A main technical difficulty is that, however, unlike the linear elliptic operator where $W^{2,p}$ regularity theory is available, the regularity theory for the Hessian operators is not so easy. We need to establish appropriate uniform estimates, gradient estimates, and in particular the interior second-order derivative estimate. For the uniform estimates we shall show that the solution is bounded when the function ψ in (1.1) lies in $L^p(\Omega)$ for some $p > \max(1, n/2k)$. The interior second-order derivative estimate is even more interesting. We state it as an independent result.

THEOREM 1.5 *Suppose* $\psi \in C^{1,1}(\overline{\Omega} \times \mathbb{R})$, $\psi \geq \psi_0 > 0$, *for some constant* ψ_0 *in* $\overline{\Omega}$. *Then for any admissible solution of* (1.1), *we have*

$$u^4(x)|D^2u(x)| \leq C\,, \tag{1.12}$$

where C *depends on* n, k, ψ_0, $\|\psi\|_{C^{1,1}}$, *and* $\|u\|_{C^1}$ *but is independent of* Ω.

The power 4 in (1.12) can be improved to any constant larger than 1; see the remark after Theorem 4.1. Theorem 1.5 extends a well-known result of Pogorelov on the Monge-Ampère equations [7, 16]. In this case, due to the special structure of the Monge-Ampère equation, the power 4 in (1.12) can be replaced by 1.

Now, one may attempt to use the mountain pass lemma to prove Theorem 1.2. However, since the relevant functional J is defined in a cone rather than a Hilbert space, we cannot apply the result directly. Instead we shall use its underlying idea. We shall introduce the parabolic Hessian equation

$$\mu(S_k(D^2u)) - u_t = \mu(\psi(x,u))\,, \quad (x,t) \in \Omega \times (0,\infty)\,,$$

to serve as the negative gradient flow for J. Here, in order to preserve admissibility, μ is a certain concave function. Given a path $\gamma : [0,1] \to \Phi_0^k$ satisfying certain conditions, we shall show that there exists some $s \in [0,1]$ such that, for $u(0,t) = \gamma(s)$, the flow has a global solution converging to a solution of (1.1). For this purpose we need to establish the corresponding a priori estimates for the parabolic equations.

This paper is arranged as follows: In Section 2 we derive a uniform estimate for solutions of (1.1). In Section 3 we derive an interior gradient estimate. A by-product of this estimate is a Liouville theorem for entire solutions of (1.1) when $\psi \equiv 0$. Interior second-order estimates will be discussed in Section 4. The main result, Theorem 4.1, contains Theorem 1.5 as its special case. We begin the study of the parabolic Hessian equation in Section 5 and apply it to prove Theorem 1.3 in Section 6. Finally, in Section 7 we prove Theorem 1.2.

A draft of this paper was completed in 1996. After that we learned that some estimates in this paper, including the uniform estimate in Section 2 and the interior gradient estimate in Section 3, were also proven by Trudinger [20, 21]. However, the proofs are different and the estimates are not completely the same. Since they are of independent interest, we decided to keep them. For further development, one may consult [23]. Finally, we would like to thank the referee for a careful reading of an earlier version of the paper. His/her comments have been very useful in improving the presentation.

2 Uniform Estimates

In this section we derive a uniform estimate for the solution of (1.1).

THEOREM 2.1 *Consider* (1.1) *for* $1 \le k < n/2$ *where* ψ *is independent of* z, $\psi \in L^p(\Omega)$ *for some* $p > n/2k$, *and* Ω *is of class* C^2. *Then there exists a positive constant* $C > 0$ *depending only* n, k, p, *and the volume* $|\Omega|$ *such that for any admissible solution* u,

$$\|u\|_{L^\infty(\Omega)} \le C\|\psi\|_{L^p(\Omega)}^{1/k} . \tag{2.1}$$

PROOF: There is no loss of generality in assuming that $|\Omega| = 1$ and $\|\psi\|_{L^p} \le 1$. Then it suffices to prove that for any solution u of

$$S_k(D^2u) = K^k\psi(x) \quad \text{in } \Omega , \tag{2.2}$$

$$\|u\|_{L^\infty(\Omega)} \le 1 \tag{2.3}$$

holds provided K is sufficiently small.

From equation (2.2) and by the Hessian Sobolev inequality, we have

$$\|u\|_{\Phi_0^k}^{k+1} = \left| \int\limits_\Omega K^k\psi(x)u(x)dx \right| \le K^k\|\psi\|_{L^p}\|u\|_{L^q} \le CK^k\|u\|_{\Phi_0^k}$$

where p and q are conjugate. By the Sobolev inequality again, we obtain

$$\|u\|_{L^1} \le C\|u\|_{\Phi_0^k} \le CK$$

where C depends only on n, k, and p. Hence

$$\left|\left\{u(x) \le -\tfrac{1}{2}\right\}\right| \le CK . \tag{2.4}$$

By Sard's theorem, for a.e. t, $0 > t > \inf_\Omega u(x)$, the level set $\{u(x) < t\}$ has a $(k-1)$–convex, smooth boundary. For simplicity we may assume that for each positive j, the boundary of $\{u(x) < -\sum_{i=1}^j 2^{-i}\}$ is smooth.

Taking (u, Ω, ψ, K) as $(u_0, \Omega_0, \psi_0, K_0)$, we are going to define a sequence $(u_j, \Omega_j, \psi_j, K_j)$, $j \ge 0$, inductively as follows: Let R_j be defined by

$$\omega_n R_j^n = \left|\left\{u_j(x) < -\tfrac{1}{2}\right\}\right| .$$

By (2.4) we have

$$R_j \le CK_j^{1/n} . \tag{2.5}$$

Define

$$\Omega_{j+1} = R_j^{-1}\left\{u_j(x) < -\tfrac{1}{2}\right\} \quad \text{and} \quad u_{j+1}(x) = 2\left(u_j(R_jx) + \tfrac{1}{2}\right) .$$

So $|\Omega_{j+1}| = 1$ and u_{j+1} satisfies

$$\begin{cases} S_k(D^2u_{j+1}) = K_{j+1}^k\psi_{j+1}(x) & \text{in } \Omega_{j+1} \\ u_{j+1} = 0 & \text{on } \partial\Omega_{j+1} , \end{cases}$$

where now ψ_{j+1} and K_{j+1} are given by

$$\psi_{j+1}(x) = R_j^{n/p}\psi(R_j x) \quad \text{and} \quad K_{j+1} = 2R_j^{2-n/kp}K_j\,, \quad \text{respectively.}$$

It is easy to see

$$\|\psi_{j+1}\|_{L^p(\Omega_{j+1})} \le \|\psi_j\|_{L^p(\Omega_j)} \le 1\,.$$

By (2.5) we have

$$K_{j+1} \le K_j \tag{2.6}$$

provided K_j is sufficiently small. Hence, similar to (2.4) we have

$$\left|\left\{u_{j+1}(x) < -\tfrac{1}{2}\right\}\right| \le CK_{j+1} \quad \text{for all } j\,.$$

Now

$$\begin{aligned}\left|\left\{u_{j+1}(x) < -\tfrac{1}{2}\right\}\right| &= R_j^{-n}\left|\left\{u_j(x) < -\tfrac{1}{2} - \tfrac{1}{4}\right\}\right| \\ &= \cdots = \left|\left\{u_0(x) < -\sum_{i=1}^{j} 2^{-i}\right\}\right| \prod_{i=1}^{j} R_i^{-n}\,.\end{aligned}$$

We obtain, in view of (2.5) and (2.6),

$$\left|\left\{u_0(x) < -\sum_{i=1}^{j} 2^{-i}\right\}\right| \le CK_{j+1}\prod_{i=1}^{j} R_i^{n} \le (CK_0)^{j+1}\,. \tag{2.7}$$

Hence (2.3) follows provided $CK_0 < 1$. This completes the proof of Theorem 2.1. □

We point out that an estimate similar to (2.1) was established by Trudinger [20]. His proof is different from ours. Here the iteration argument may be useful in other situations.

We note that when $k = n/2$ and $\psi \in L^p(\Omega)$, $p > 1$, a modification of the above argument also yields (2.1) with the constant C depending on $\mathrm{diam}(\Omega)$. Indeed, for any $q > 1$, $\delta > 0$, by the Hessian Sobolev inequality and the Hölder inequality, there exists $C > 0$ depending only on n, q, δ, and $|\Omega|$ such that

$$\|u\|_{L^\infty(\Omega)} \le C[\mathrm{diam}(\Omega)]^\delta \|u\|_{\Phi_0^k}\,.$$

The estimate (2.1) follows from the above iteration provided δ is chosen sufficiently small; see also (3.13) and (3.14) in [20].

3 Gradient Estimates

In this section we derive an interior gradient estimate for (admissible) solutions of the following Hessian equation:

$$S_k(D^2u) = \psi(x, u, \nabla u) \quad \text{in } \Omega \tag{3.1}$$

where ψ is a nonnegative Lipschitz-continuous function. The interior gradient estimate was also proven in [21]. Here we give a different proof, following [30].

First we introduce some inequalities for the polynomials $S_k(\lambda)$. More inequalities on $S_k(\lambda)$ can be found in [8, 14]. Let's denote $S_0(\lambda) = 1$ and $S_k(\lambda) = 0$ for $k < 0$ and $k > n$,

$$S_{k;i_1i_2\cdots i_j}(\lambda) = S_k(\lambda)\big|_{\lambda_{i_1}=\lambda_{i_2}=\cdots=\lambda_{i_j}=0},$$

and $S_{k;i_1i_2\cdots i_j}(\lambda) = 0$ if $i_r = i_s$ for some $1 \le r < s \le j$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \overline{\Gamma}_k$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$; then

$$S_{k-1;n}(\lambda) \ge S_{k-1;n-1} \ge \cdots \ge S_{k-1;1}(\lambda) \ge 0.$$

It is proven in [14] that

$$S_{k-1;k}(\lambda) \ge \theta\lambda_1 S_{k-2;1k}(\lambda) \quad \text{for some } \theta = \theta(n,k) > 0,$$

from which it follows that

$$S_{k-1;i}(\lambda) \ge \theta\lambda_1\lambda_2\cdots\lambda_{k-1} \quad \text{for } i \ge k \tag{3.2}$$

for some different θ. Using

$$S_{k-1}(\lambda) = \frac{1}{n-k+1}\sum S_{k-1;i}(\lambda), \tag{3.3}$$

we have

$$S_{k-1}(\lambda) \ge \theta\lambda_1\lambda_2\cdots\lambda_{k-1}.$$

The following lemma will not be used until the next section. Nevertheless, it is appropriate to place it here.

LEMMA 3.1 *Suppose* $\lambda \in \overline{\Gamma}_k$ *and* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. *Then there exists* $\theta = \theta(n,k) > 0$ *such that*

$$\lambda_1 S_{k-1;1}(\lambda) \ge \theta S_k(\lambda). \tag{3.4}$$

Moreover, for any $\delta \in (0,1)$ *there exists* $K > 0$ *such that if*

$$S_k(\lambda) \le K\lambda_1^k \quad \textit{or} \quad |\lambda_i| \le K\lambda_1 \quad \textit{for } i = k+1, 2, \dots, n,$$

we have

$$\lambda_1 S_{k-1;1}(\lambda) \ge (1-\delta)S_k(\lambda). \tag{3.5}$$

PROOF: We have

$$S_k(\lambda) = S_{k-1;1}(\lambda)\lambda_1 + S_{k;1}(\lambda). \tag{3.6}$$

By

$$S_{k;1}(\lambda) \le C_{n,k}S_{k-1;1}^{k/(k-1)}(\lambda) \le C\lambda_1 S_{k-1;1}(\lambda),$$

(3.4) follows.

To prove (3.5) we first consider the case $S_k(\lambda) \le K\lambda_1^k$. We may assume $S_k(\lambda) = 1$. If (3.5) is not true, then

$$S_{k-1;1}(\lambda) < (1-\delta)\lambda_1^{-1} \le K^{1/k};$$

hence

$$S_{k;1}(\lambda) \le CS_{k-1;1}^{k/(k-1)}(\lambda) \le CK^{1/(k-1)}.$$

In view of (3.6), (3.5) follows.

Next, we consider the case $|\lambda_i| \le K\lambda_1$ for $i = k+1, 2, \dots, n$. Observing that if $\lambda_k \ll \lambda_1$, we have $S_k(\lambda) \ll \lambda_1^k$, and so (3.5) holds. Hence we may assume $|\lambda_i| \ll \lambda_k$ for $i = k+1, 2, \dots, n$. In this case both $S_k(\lambda)$ and $\lambda_1 S_{k-1;1}(\lambda)$ are equal to $\lambda_1\lambda_2\cdots\lambda_k(1+o(1))$ with $o(1) \to 0$ as $K \to 0$. Again (3.5) holds. □

Now, we turn to the interior gradient estimate. Let $u \in C^3(\Omega)$ be a solution of (3.1) and $\Omega = B_r(0)$. Let

$$G(x,\xi) = u_\xi(x)\varphi(u)\rho(x)$$

where $\rho(x) = (1-|x|^2/r^2)^+$, $\varphi(u) = 1/(M-u)^{1/2}$, and $M = 4\sup|u|$. Suppose G attains its maximum at $x = x_0$ and $\xi = e_1$. Then, at x_0,

$$0 = G_i = u_{1i}\varphi\rho + u_1 u_i \varphi'\rho + u_1\varphi\rho_i\ ,$$

i.e.,

$$u_{1i} = -\frac{u_1}{\varphi\rho}(u_i\varphi'\rho + \varphi\rho_i) \tag{3.7}$$

and the matrix

$$\begin{aligned}\{G_{ij}\} = \big\{&u_{1ij}\varphi\rho + u_1u_{ij}\varphi'\rho + u_1u_iu_j\varphi''\rho + u_1\varphi\rho_{ij}\\ &+ (u_{1i}u_j + u_{1j}u_i)\varphi'\rho + \varphi(u_{1i}\rho_j + u_{1j}\rho_i) + u_1\varphi'(u_i\rho_j + u_j\rho_i)\big\}\end{aligned}$$

is nonpositive. Differentiating equation (3.1) gives

$$S^{ij}u_{ij1} = \nabla_1\psi\ .$$

Note that $S^{ij}u_{ij} = k\psi$. We obtain, by (3.7),

$$\begin{aligned}0 &\ge S^{ij}G_{ij}\\ &= \varphi\rho\nabla_1\psi + ku_1\psi\varphi'\rho + u_1\varphi''\rho S^{ij}u_iu_j + u_1\varphi S^{ij}\rho_{ij6}\\ &\quad + u_1\varphi' S^{ij}(u_i\rho_j + u_j\rho_i) + 2S^{ij}u_{1i}(u_j\varphi'\rho + \varphi\rho_j)\\ &= \varphi\rho\nabla_1\psi + ku_1\psi\varphi'\rho + u_1\rho\left(\varphi'' - \frac{2\varphi'^2}{\varphi}\right)S^{ij}u_iu_j + u_1\varphi S^{ij}\rho_{ij}\\ &\quad - u_1\varphi' S^{ij}(u_i\rho_j + u_j\rho_i) - \frac{2u_1\varphi}{\rho}S^{ij}\rho_i\rho_j\ .\end{aligned} \tag{3.8}$$

We have

$$\varphi'' - \frac{2\varphi'^2}{\varphi} \ge \frac{1}{16}M^{-5/2}.$$

Multiplying (3.8) by $M^{5/2}$ and noticing tht $\varphi' > 0$, we have

$$0 \ge \varphi\rho M^{5/2}\nabla_1\psi + \frac{1}{16}\rho S^{11}u_1^3 - \mathcal{S}\left(\frac{CM^2u_1}{r^2} + \frac{CMu_1^2}{r} + \frac{CM^2u_1}{\rho r^2}\right), \tag{3.9}$$

where $\mathcal{S} = \sum S^{ii}$ and C is independent of r and M.

Suppose now that $G(x_0)$ is so large that

$$\rho u_1 \geq 8\frac{M}{r} \quad \text{at } x_0\,.$$

It follows from (3.7) that

$$u_{11} \leq -\frac{\varphi'}{2\varphi}u_1^2 < 0 \quad \text{at } x_0\,.$$

We claim

$$S^{11} \geq \theta \mathscr{S} \quad \text{for some } \theta = \theta(n,k)\,. \tag{3.10}$$

In fact, we have

$$S_{k-1}(D^2u) = S^{11} + u_{11}S_{k-2;1}(\mu) - \sum_{i=2}^{n} u_{1i}^2 S_{k-3;1i}(\mu) \leq S^{11}\,.$$

Since $S_{k-1}(D^2u)$ is invariant under rotation of coordinates, (3.9) follows from (3.3). Multiplying (3.9) by ρ^2, we have

$$\rho u_1 \leq C_1 + C_2\frac{M}{r}\,, \tag{3.11}$$

provided

$$|\nabla_1 \psi| = o(u_1^3)\mathscr{S} \quad \text{as } u_1 \to \infty\,. \tag{3.12}$$

To estimate $\mathscr{S}$ we rotate the axes, which doesn't change the value of $\mathscr{S}$, so that under the new coordinates $y = (y_1, y_2, \ldots, y_n)$, D^2u is diagonal, and

$$u_{y_1y_1} \geq u_{y_2y_2} \geq \cdots \geq u_{y_ny_n}\,.$$

Then at the point where G reaches its maximum,

$$u_{y_ny_n} \leq u_{x_1x_1} \leq -\frac{\varphi'}{2\varphi}u_1^2\,.$$

From equation (3.1) we have

$$\psi = u_{y_ny_n}S_{k-1;n}(\lambda) + S_{k;n}(\lambda)\,, \quad \lambda = \lambda(D^2u)\,.$$

Since

$$S_{k-1;n}(\lambda) \geq C[S_{k;n}(\lambda)]^{(k-1)/k}\,, \quad 0 \leq u_{y_ny_n}S_{k-1;n}(\lambda) + C[S_{k-1;n}(\lambda)]^{k/(k-1)}\,.$$

So

$$S_{k-1;n}(\lambda) \geq C|u_{y_ny_n}|^{k-1} \geq C\left(\frac{\varphi'}{\varphi}\right)^{k-1} u_{x_1}^{2k-2} \quad \text{and} \quad \mathscr{S} \geq \frac{Cu_1^{2k-2}}{M^{k-1}} \quad \text{at } x_0\,,$$

where $C > 0$ is independent of M and r. Therefore (3.12) is satisfied if there exists a nonnegative function $h(t)$ with $h(t)/t \to 0$ as $t \to \infty$ such that

$$|\psi_x| + |\psi_z|\cdot|p| + |\psi_p|\cdot|p|^2 \leq h(|p|^{2k+1}) \quad \text{as } |p| \to \infty\,. \tag{3.13}$$

We have proven the following result:

THEOREM 3.2 *Let u be a k-admissible C^3-solution of* (3.1) *where* $\psi \geq 0$ *is Lipschitz-continuous and* $\Omega = B_r(0)$. *Under* (3.13) *we have*

$$|\nabla u(0)| \leq C\,,$$

where C depends only on n, k, r, h, and $\sup |u|$.

Here we have automatically extended the notion of k-admissibility to C^2-functions with $\lambda(D^2u) \in \overline{\Gamma}_k$, the closure of Γ_k. Observe that our argument does not require ψ to be strictly positive. Moreover, an examination of the above proof shows that one can take $C_1 = 0$ in (3.11) when $\psi \equiv 0$. As a result, we have the following Liouville property for entire solutions of the homogeneous Hessian equations:

THEOREM 3.3 *Let* $u \in C^3(\mathbb{R}^n)$ *be a k-admissible solution of*

$$S_k(D^2u) = 0 \quad \text{in } \mathbb{R}^n\,, \qquad \text{which satisfies } \lim_{|x|\to\infty} \frac{u(x)}{|x|} = 0\,.$$

Then u is a constant.

Gradient estimates on the boundary can be obtained by a construction of barriers [4, 9] when $u_{|\partial\Omega} \in C^{1,1}$, $\psi \in L^\infty(\Omega)$, and $\partial\Omega$ is strictly $(k-1)$–convex, as is always assumed. To get a global gradient estimate, we can use the auxiliary function G as above, where now ρ is replaced by the constant 1. When G attains its maximum at some point in Ω, at this point we have, from (3.8),

$$\varphi\nabla_1\psi + ku_1\psi\varphi' + u_1\left(\varphi'' - \frac{2\varphi'^2}{\varphi}\right)S^{ij}u_iu_j \leq 0\,.$$

Hence we have the following global gradient estimate:

THEOREM 3.4 *Consider* (3.1) *where* $\psi \geq 0$ *is Lipschitz-continuous and satisfies* (3.13). *Let u be a k-admissible solution of* (3.1) *with* $u|_{\partial\Omega} \in C^{1,1}$. *Then*

$$|\nabla u| \leq C\,,$$

where C depends on n, k, h, $\sup_\Omega |u|$, $\|u\|_{C^{1,1}(\partial\Omega)}$, *and* $\partial\Omega$.

For our problem (1.1), (3.13) is satisfied when $\psi(x, z)$ is Lipschitz-continuous at $z = 0$. When ψ is not Lipschitz-continuous at $z = 0$, as may happen in the sublinear case, we note that for any $u \in \Phi_0^k$ satisfying

$$\sup_\Omega |u| \leq C_1 \quad \text{and} \quad \|u\|_{L^1(\Omega)} \leq C_2\,,$$

by the subharmonicity of u, for any $\delta > 0$, there exists $\theta > 0$ depending only on δ, C_1, C_2, and Ω such that

$$u \leq -\theta \quad \text{in } \Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}\,. \tag{3.14}$$

Therefore, our interior estimate, Theorem 3.2, is still applicable.

4 Second-Order Estimates

In this section we establish an interior a priori estimate for the second-order derivatives of solutions to the problem

$$\begin{cases} S_k(D^2u) = \psi(x,u) & \text{in } \Omega \\ u = u_0(x) & \text{on } \partial\Omega\,, \end{cases} \tag{4.1}$$

where ψ satisfies

$$\psi(x,z) \geq \psi_0 > 0 \tag{4.2}$$

for some constant ψ_0 and Ω is a strictly $(k-1)$–convex domain. Let

$$F(D^2u) = \mu(S_k(D^2u))\,,$$

where μ is a positive, monotone increasing function such that $F(r)$ is concave in r. In this section we will take

$$\mu(t) = t^{1/k}\,.$$

It is proven in [4] that this choice of μ fulfills our requirement. We may rewrite the equation in (4.2) as

$$F(D^2u) = f(x,u)\,,$$

where $f(x,z) = \mu(\psi(x,z))$. Differentiating this equation with respect to x_γ gives

$$F_{ij}u_{ij\gamma} = f_\gamma\,, \quad F_{ij}u_{ij\gamma\gamma} + (F_{ij})_{rs}u_{ij\gamma}u_{rs\gamma} = f_{\gamma\gamma}\,,$$

where

$$F_{ij} = \frac{\partial F}{\partial r_{ij}}(D^2u) = \mu' S^{ij}(D^2u)\,.$$

When (D^2u) is diagonal at a given point, we have

$$(F_{ij})_{rs} = \begin{cases} \mu' S_{k-2;ir}(\lambda) + \mu'' S_{k-1;i}S_{k-1;r} & \text{if } i=j, r=s \\ -\mu' S_{k-2;ij}(\lambda) & \text{if } i \neq j, r=j, \text{ and } s=i \\ 0 & \text{otherwise} \end{cases}$$

at this point. Hence

$$\begin{aligned} \sum_i F_{ii}u_{ii\gamma\gamma} &= f_{\gamma\gamma} + \sum_{i,j=1}^n \mu' S_{k-2;ij}u_{ij\gamma}^2 \\ &\quad - \sum_{i,j=1}^n [\mu'' S_{k-1;i}S_{k-1;j} + \mu' S_{k-2;ij}]u_{ii\gamma}u_{jj\gamma} \\ &\geq f_{\gamma\gamma}\,. \end{aligned} \tag{4.3}$$

As contrast to linear elliptic equations, it is well-known that in general there is no interior $C^{1,1}$-regularity for solutions of (4.1) even when ψ is analytic [16, 27]. Nevertheless, under the additional assumption that u is strictly k-convex, i.e., if there exists an admissible function w such that $u < w$ in Ω and $w = u$ on $\partial\Omega$, we will derive an interior $C^{1,1}$-estimate for u. Our derivation is similar to those in [7, 16] where the Monge-Ampère equation is considered.

Consider the auxiliary function

$$G(x) = \rho^{\beta}(x)\varphi\left(\frac{1}{2}|Du|^2\right)u_{\xi\xi}\,,$$

where $\beta = 4$, $\varphi(t) = (1 - \frac{t}{M})^{-1/8}$, $M = 2\sup_{x\in\Omega}|Du|^2$, and

$$\rho = w - u\,.$$

Suppose that G attains its maximum at x_0 and in the direction $\xi = (1, 0, \dots, 0)$. Assume that D^2u is already diagonal at x_0 with $u_{11} \geq u_{22} \geq \cdots \geq u_{nn}$. Then at x_0,

$$0 = (\log G)_i = \beta\frac{\rho_i}{\rho} + \frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}}\,, \qquad j = 1, 2, \dots, n\,, \tag{4.4}$$

$$\begin{aligned} 0 &\geq \sum_i F_{ii}(\log G)_{ii} \\ &= \sum_i \beta F_{ii}\left[\frac{\rho_{ii}}{\rho} - \frac{\rho_i^2}{\rho^2}\right] + \sum_i F_{ii}\left[\frac{\varphi_{ii}}{\varphi} - \frac{\varphi_i^2}{\varphi^2}\right] + \sum_i F_{ii}\left[\frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2}\right]. \end{aligned} \tag{4.5}$$

We consider two cases separately.

Case 1. $u_{kk} \geq Ku_{11}$, where K is a small positive constant to be determined. By (4.4) we have

$$\frac{u_{11i}}{u_{11}} = -\left(\frac{\varphi_i}{\varphi} + \beta\frac{\rho_i}{\rho}\right). \tag{4.6}$$

Putting (4.6) into (4.5) yields

$$0 \geq \beta F_{ii}\left[\frac{\rho_{ii}}{\rho} - (1+2\beta)\frac{\rho_i^2}{\rho^2}\right] + F_{ii}\left[\frac{\varphi_{ii}}{\varphi} - 3\frac{\varphi_i^2}{\varphi^2}\right] + F_{ii}\frac{u_{11ii}}{u_{11}}\,. \tag{4.7}$$

Here and below, summations in i in these inequalities are understood. First, by equation (4.3),

$$F_{ii}u_{11ii} \geq f_{11} \geq -C(1 + u_{11})\,.$$

Next, we have

$$\begin{aligned} F_{ii}\left[\frac{\varphi_{ii}}{\varphi} - 3\frac{\varphi_i^2}{\varphi^2}\right] &= \left(\frac{\varphi''}{\varphi} - 3\frac{\varphi'^2}{\varphi^2}\right)F_{ii}u_i^2u_{ii}^2 + \frac{\varphi'}{\varphi}u_jF_{ii}u_{iij} + \frac{\varphi'}{\varphi}F_{ii}u_{ii}^2 \\ &\geq \frac{\varphi'}{\varphi}F_{ii}u_{ii}^2 + \frac{\varphi'}{\varphi}u_i f_i\,, \end{aligned} \tag{4.8}$$

which, after using (3.2),

$$F_{ii}u_{ii}^2 > F_{kk}u_{kk}^2 \geq \theta\mathcal{F}u_{11}^2\,,$$

where $\mathcal{F} = \sum_{i=1}^{n} F_{ii}$ and $\theta = \theta(n, k, K)$. Therefore, we have

$$F_{ii}\left[\frac{\varphi_{ii}}{\varphi} - 3\frac{\varphi_i^2}{\varphi^2}\right] \geq \theta \mathcal{F} u_{11}^2 - C\,.$$

Finally, by our special choice of ρ,

$$F_{ii}\rho_{ii} \geq -F_{ii}u_{ii} = -\mu' S^{ii} u_{ii} = -k\mu'\psi\,. \tag{4.9}$$

Putting these inequalities together, we obtain

$$0 \geq \theta \mathcal{F} u_{11}^2 - C\mathcal{F}\frac{\rho_i^2}{\rho^2} - \frac{k\beta\mu'\psi}{\rho} - C\,. \tag{4.10}$$

Note that when $u_{kk} \geq K u_{11}$, we have

$$\mathcal{F} \geq F_{nn} \geq \theta\mu' u_{11}u_{22}\cdots u_{k-1,k-1} \geq \theta_1 u_{11}^{k-1}\,.$$

Multiplying (4.10) by $\rho^{2\beta}\varphi^2$, we deduce $G(x_0) \leq C$.

Case 2. $u_{kk} \leq K u_{11}$ (and so $u_{jj} \leq K u_{11}$ for $j = k, k+1, \dots, n$) . In this case we have, by (4.4),

$$\frac{\rho_i}{\rho} = -\frac{1}{\beta}\left(\frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}}\right), \quad i = 2, 3, \dots, n\,. \tag{4.11}$$

Putting (4.6) for $i = 1$ and (4.11) for $i = 2, 3, \dots, n$ into (4.5), we obtain

$$\begin{aligned}
0 &\geq \left\{\sum_{i=1}^{n}\left[\beta F_{ii}\frac{\rho_{ii}}{\rho} + F_{ii}\left(\frac{\varphi_{ii}}{\varphi} - 3\frac{\varphi_i^2}{\varphi^2}\right)\right] - \beta(1+2\beta)F_{11}\frac{\rho_1^2}{\rho^2}\right\} \\
&\quad + \left\{\sum_{i=1}^{n} F_{ii}\frac{u_{11ii}}{u_{11}} - \left(1 + \frac{2}{\beta}\right)\sum_{i=2}^{n} F_{ii}\frac{u_{11i}^2}{u_{11}^2}\right\} \\
&=: I_1 + I_2\,.
\end{aligned} \tag{4.12}$$

By (4.8) and (4.9) we have

$$I_1 \geq \theta F_{ii}u_{ii}^2 - C\frac{F_{11}}{\rho^2} - \frac{k\beta\mu'\psi}{\rho} - C \geq \frac{1}{2}\theta F_{11}u_{11}^2 - \frac{k\beta\mu'\psi}{\rho} - C$$

provided $\rho^2 u_{11}^2$ is sufficiently large. By (3.4)

$$I_1 \geq \theta_1\mu'\psi u_{11} - \frac{k\beta\mu'\psi}{\rho} - C\,.$$

Next we claim

$$I_2 \geq \frac{f_{11}}{u_{11}}\,. \tag{4.13}$$

Granted the validity of (4.13), (4.12) reduces to

$$0 \geq \theta_1\mu'\psi u_{11} + \frac{f_{11}}{u_{11}} - \frac{k\beta\mu'\psi}{\rho} - C\,.$$

Multiplying the above inequality by $\rho^\beta\varphi$ we obtain $G(x_0) \leq C$.

To prove (4.13) we first note that by the concavity of F,

$$-\sum_{i,j=1}^{n}[\mu'' S_{k-1,i}S_{k-1,j}+\mu' S_{k-2,ij}]u_{ii1}u_{jj1}=-\sum\frac{\partial^2}{\partial\lambda_i\partial\lambda_j}\mu(S_k(\lambda))u_{ii1}u_{jj1}\geq 0\,.$$

Hence by (4.3),

$$\begin{aligned}u_{11}I_2 &\geq f_{11}+\sum_{i,j=1}^{n}\mu' S_{k-2;ij}u_{ij1}^2-\left(1+\frac{2}{\beta}\right)\sum_{i=2}^{n}F_{ii}\frac{u_{11i}^2}{u_{11}}\\ &\geq f_{11}+\sum_{i=2}^{n}\mu'\left(2S_{k-2;1i}-\left(1+\frac{2}{\beta}\right)\frac{S_{k-1;i}}{u_{11}}\right)u_{11i}^2\,.\end{aligned}$$

Since $\beta=4$, we only need to show

$$S_{k-2;1i}-\frac{3}{4}\frac{S_{k-1;i}}{u_{11}}\geq 0\,.$$

But this follows from (3.5) when K is sufficiently small. Hence (4.13) holds. We have thus proven the following result:

THEOREM 4.1 *Consider* (4.1) *where* $\psi\in C^{1,1}(\overline{\Omega}\times\mathbb{R})$ *satisfies* (4.2). *Let* $u\in C^{3,1}(\Omega)\cap C^{0,1}(\overline{\Omega})$ *be a k-admissible solution of* (4.1). *Suppose that there is an admissible function* w *such that* $w>u$ *in* Ω *and* $w=u_0$ *on* $\partial\Omega$. *Then*

$$(w-u)^4(x)|D^2u(x)|\leq C\,, \tag{4.14}$$

where C *depends only on* n, k, $\sup_\Omega|Du(x)|$, *and* ψ.

Now Theorem 1.5 follows from this theorem since we can take $w\equiv 0$ when $u_0\equiv 0$.

We point out two facts. First, the power 4 in (4.14) can be improved to any constant larger than 1. Indeed, for any $\beta>1$, let $0<\delta<\beta-1$. Following the above derivation, we have, instead of (4.12),

$$\begin{aligned}0\geq&\left\{\sum_{i=1}^{n}\left[\beta F_{ii}\frac{\rho_{ii}}{\rho}+F_{ii}\left(\frac{\varphi_{ii}}{\varphi}-C_\delta\frac{\varphi_i^2}{\varphi^2}\right)\right]-\beta(1+2\beta)F_{11}\frac{\rho_1^2}{\rho^2}\right\}\\ &+\left\{\sum_{i=1}^{n}F_{ii}\frac{u_{11ii}}{u_{11}}-\left(1+\frac{1+\delta}{\beta}\right)\sum_{i=2}^{n}F_{ii}\frac{u_{11i}^2}{u_{11}^2}\right\}.\end{aligned}$$

So the above argument is still valid, so long as we choose $\varphi=\varphi(t)$ such that $\varphi''\varphi-C_\delta\varphi'^2>0$ for $t<\|u\|_{L^\infty}$. For the Monge-Ampère equation ($k=n$), due to its special structure the power in (4.14) can be made to 1. Second, Theorem 4.1, whose validity relies on (4.2), cannot be applied to prove Theorems 1.2 and 1.3 directly. Instead, we apply it to the subdomain Ω_δ in (3.14). An examination of the proof of Theorem 4.1 shows that the following result holds:

THEOREM 4.2 *Consider* (1.1) *where* $\psi \in C^{1,1}(\overline{\Omega \times \mathbb{R}^-})$. *Suppose there exists a nonincreasing function* h *satisfying* $h(0) = 0$ *and* $h(z) > 0$ *for* $z < 0$ *such that* $\psi(x, z) > h(z)$ *for all* $z < 0$. *Then for any* $\delta > 0$, *there is a constant* $C > 0$ *depending on* n, k, δ, h, $M = \sup_\Omega |u|$, $\|\psi\|_{C^{1,1}(\overline{\Omega}\times(-M,-\delta/2))}$, *and* $\sup\{|Du(x)| : u(x) < -\delta/2\}$ *such that*

$$\sup\{|D^2u(x)| : u(x) \le -\delta\} \le C\,. \tag{4.15}$$

Estimate (4.15) shows that equation (1.1) is uniformly elliptic in any compact subdomain of Ω. If one further assumes that $\partial\Omega$ is $C^{3,1}$ and strictly $(k-1)$–convex, then the following global second-order derivative estimate holds [29]:

$$\sup_{x\in\overline{\Omega}} |D^2u(x)| \le C\,,$$

where C depends on n, k, ψ, $\|u\|_{C^1(\overline{\Omega})}$, and the boundary $\partial\Omega$.

When ψ satisfies (4.2), first- and second-order derivative estimates have been obtained in [4, 9]. Hence (1.1) is uniformly elliptic and further regularity follows from Krylov's regularity theory [12]. We state the result as follows:

THEOREM 4.3 *Consider* (1.1) *where* $\psi \in C^{1,1}(\overline{\Omega \times \mathbb{R}^-})$ *and* Ω *is of class* $C^{3,1}$ *and strictly* $(k-1)$*–convex. Suppose that* (4.2) *holds. Then its solution satisfies*

$$\|u\|_{C^{3,\alpha}(\overline{\Omega})} \le C$$

for some $\alpha \in (0, 1)$, *where* C *depends only on* n, k, Ω, $\sup_\Omega |u|$, *and* ψ *up to second order.*

When ψ is further assumed to be monotone increasing in z, one may use the method of continuity to show that (1.1) admits a unique solution.

5 Parabolic Hessian Equations

Let Ω be a strictly $(k-1)$–convex domain in $\mathbb{R}^n$ with a $C^{3,1}$-boundary. Let $Q = \Omega \times (0,\infty)$ and $Q_T = \Omega \times (0, T]$. We denote by $\partial^* Q_T$ the parabolic boundary of Q_T. In this section we consider the parabolic equation

$$\begin{cases} F(D^2u) - u_t = f(x,t,u) & \text{in } Q_T \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u = u_0 & \text{on } \{t = 0\} \end{cases} \tag{5.1}$$

where $u_0 \in \Phi_0^k(\Omega)$, $f \in C^2(\overline{Q}_T \times \mathbb{R})$, and $F(D^2u) = \mu(S_k(D^2u))$. The function μ is chosen to satisfy $\mu' > 0$, $\mu'' < 0$,

$$\lim_{t\to\infty} \mu(t) = +\infty\,, \tag{5.2}$$

$$\lim_{t\to 0} \mu(t) = -\infty\,, \tag{5.3}$$

and such that F is concave with respect to its arguments. Moreover, we require

$$\mu(t) = \begin{cases} t^{1/p} & t \geq 1 \\ \log t & t \text{ small} \end{cases} \quad \text{for some } p > k\,. \tag{5.4}$$

For the parabolic equation (5.1), a function u in $C^{2,1}_{x,t}(Q_T)$ is said to be k-admissible if $u(\cdot, t)$ is k-admissible for each $t \in [0, T]$.

In this section we will just prove some results that are needed for the proof of Theorems 1.2 and 1.3. We refer the reader to [11, 22, 29] for discussions on various parabolic Hessian equations. First we establish a gradient estimate for solutions of (5.1).

THEOREM 5.1 *Consider* (5.1) *where* $F(r) = \mu(S_k(r))$, μ *is specified as above,* $u_0 \in \Phi^k_0(\Omega)$, *and* $f \in C^{0,1}(\overline{Q}_T \times \mathbb{R})$. *Suppose further that the compatibility condition* $F(D^2u_0) = f(x, t, u_0)$ *on* $\partial\Omega \cap \{t = 0\}$ *holds, and*

$$|f(x, t, z)| \leq C_0(1 + |z|) \quad \forall (x, t, z) \in Q_T \times \mathbb{R}\,.$$

Then for any k-admissible solutions u in $C^{4,2}(Q)$, *we have, for* $0 < t < T$,

$$u(x, t) \geq -e^{C_1 t} \sup_\Omega |u_0(x)|\,, \tag{5.5}$$

$$|\nabla_x u(x, t)| \leq C_2\left(1 + \frac{1}{r} M_t^{(p+k)/2k}\right), \tag{5.6}$$

$$|u_t(x, t)| \leq C_3(1 + M_t)\,, \tag{5.7}$$

where $M_t = \sup_{Q_t} |u|$, $r = \min\{1, \operatorname{dist}(x, \partial\Omega)\}$. *Here* C_1 *depends only on n, k, p, and* C_0; *and* C_2 *and* C_3 *depend additionally on* u_0 *and the gradient of* f.

PROOF: (5.5) is obvious as the right-hand side is a lower barrier. To prove (5.6) and (5.7) we assume for simplicity that $M_t \geq 1$. First we prove (5.7). Let

$$G = \frac{u_t}{M - u}\,,$$

where $M = 2M_t$. If G attains its minimum at the parabolic boundary $\partial^* Q_t$, we have $u_t \geq -C$ for some $C > 0$ depending on the initial value u_0. Hence we may assume G attains its minimum at some point in Q_t. At this point we have

$$u_{tt} + (M - u)^{-1}u_t^2 \leq 0\,, \quad u_{jt} + (M - u)^{-1}u_t u_j = 0\,, \quad j = 1, 2, \ldots, n\,, \tag{5.8}$$

and the matrix

$$\begin{aligned} &\left\{u_{ijt} + (M - u)^{-1}(u_{it}u_j + u_{jt}u_i + u_t u_{ij}) + 2(M - u)^{-2}u_i u_j u_t\right\} \\ &\qquad = \left\{u_{ijt} + (M - u)^{-1}u_t u_{ij}\right\} \geq 0\,. \end{aligned} \tag{5.9}$$

Differentiating equation (5.1) gives

$$F_{ij}u_{ijt} - u_{tt} = f_t + f_u u_t\,, \quad F_{ij}u_{rij} - u_{rt} = f_r + f_u u_r\,. \tag{5.10}$$

We may assume $u_t \le 0$ at this point. From (5.8), (5.9), and (5.10), we have

$$(M-u)^{-1}u_t^2 \le -F_{ij}u_{ijt} + f_t + f_u u_t \le (M-u)^{-1}u_t F_{ij}u_{ij} + f_t + f_u u_t \\ \le f_t + f_u u_t \,.$$

Hence $u_t \ge -CM$.

Similarly, let

$$G = \frac{u_t}{M+u}\,.$$

If G attains its maximum on $\partial^* Q_t$, we have $u_t \le C$. So, assume it attains its minimum at some point in Q_t. At this point we have

$$u_{tt} - (M+u)^{-1}u_t^2 \ge 0\,, \quad u_{jt} - (M+u)^{-1}u_t u_j = 0\,, \quad \text{for } j = 1, 2, \dots, n\,,$$

and

$$\left\{u_{ijt} - (M+u)^{-1}u_t u_{ij}\right\} \le 0\,.$$

Hence we have

$$\begin{aligned}(M+u)^{-1}u_t^2 &\le F_{ij}u_{ijt} - f_t - f_u u_t \\ &\le (M+u)^{-1}u_t F_{ij}u_{ij} - f_t - f_u u_t \\ &= (M+u)^{-1}k u_t \mu' S_k(D^2u) - f_t - f_u u_t\,.\end{aligned} \tag{5.11}$$

In case $S_k(D^2u) \le 1$, by (5.1) we have

$$u_t = F(D^2u) - f \le 1 - f\,.$$

In case $S_k(D^2u) \ge 1$,

$$\mu' S_k(D^2u) = \frac{1}{p}\mu(S_k(D^2u)) = \frac{1}{p}(u_t + f)\,.$$

It follows from (5.11) that

$$\frac{p-k}{p}\frac{u_t^2}{M+u} \le \frac{kfu_t}{p(M+u)} - f_t - f_u u_t\,.$$

Hence in both cases $u_t \le CM$ holds.

Next we prove (5.6). For simplicity let's take $t = T$. Our proof is somewhat parallel to the proof in the elliptic case in Section 3. Let us assume $B_r(0)$ is a ball inside Ω and consider

$$G(x,t,\xi) = \rho(x)\varphi(u)u_\xi\,,$$

where $\rho(x) = (1-|x|^2/r^2)$, $\varphi(u) = (M-u)^{-\alpha}$, and $\alpha \in (0,1)$ is a small constant to be chosen later. Suppose that

$$\sup\left\{G(x,t,\xi) : (x,t) \in B_r(0) \times [0,T], |\xi| = 1\right\}$$

is attained at (x_0, t_0) and $\xi_0 = (1, 0, \dots, 0)$. Then at (x_0, t_0) we have

$$0 = (\log G)_i = \frac{\rho_i}{\rho} + \frac{\varphi_i}{\varphi} + \frac{u_{1i}}{u_1} \tag{5.12}$$

and

$$\begin{aligned}0 &\geq F_{ij}(\log G)_{ij} - (\log G)_t \\ &= F_{ij}\left(\frac{\rho_{ij}}{\rho} - \frac{\rho_i\rho_j}{\rho^2}\right) + F_{ij}\left(\frac{\varphi_{ij}}{\varphi} - \frac{\varphi_i\varphi_j}{\varphi^2}\right) - \frac{\varphi_t}{\pi} \\ &\quad + F_{ij}\left(\frac{u_{1ij}}{u_1} - \frac{u_{1i}u_{1j}}{u_1^2}\right) - \frac{u_{1t}}{u_1}\,.\end{aligned}$$

By (5.12) we obtain

$$\begin{aligned}0 &\geq F_{ij}\left(\frac{\rho_{ij}}{\rho} - 3\frac{\rho_i\rho_j}{\rho^2}\right) + F_{ij}\left(\frac{\varphi_{ij}}{\varphi} - 3\frac{\varphi_i\varphi_j}{\varphi^2}\right) - \frac{\varphi_t}{phi} + \frac{1}{u_1}(F_{ij}u_{1ij} - u_{1t}) \\ &\geq -\frac{C}{\rho^2}\mathcal{F} + \left(\frac{\varphi''}{\varphi} - C\frac{\varphi'^2}{\varphi^2}\right)F_{11}u_1^2 + \frac{\varphi'}{\varphi}(F_{ij}u_{ij} - u_t) + \frac{f_1}{u_1}\,.\end{aligned}$$

Choosing $\alpha \in (0, 1)$ so small that

$$\frac{\varphi''}{\varphi} - C\frac{\varphi'^2}{\varphi} \geq \frac{\theta}{M^2} > 0\,,$$

and noticing that $F_{ij}u_{ij} \geq 0$ and $\varphi' \geq 0$, we have

$$0 \geq \frac{\theta}{M^2}F_{11}u_1^2 - \frac{C}{\rho^2}\mathcal{F} - \frac{\varphi'}{\varphi}u_t + \frac{f_1}{u_1}\,.$$

Therefore, we have either

$$F_{11}u_1^2 \leq \frac{CM^2}{\rho^2}\mathcal{F} \tag{5.13}$$

or

$$F_{11}u_1^2 \leq CM^2\left(\frac{\varphi'}{\varphi}u_t - \frac{f_1}{u_1}\right) \leq CM^2\,. \tag{5.14}$$

By (5.12) we have

$$u_{11} = -u_1\left(\frac{\rho_1}{\rho} + \frac{\varphi'}{\varphi}u_1\right).$$

We may assume

$$u_{11} \leq -\frac{C}{M}u_1^2\,,$$

for otherwise $u_1\rho \leq 2|\rho_1\varphi/\varphi'|$ would hold and (5.6) would follow. By a rotation of coordinates we may suppose $u_{ij} = 0$ at (x_0, t_0) for $i \neq j$ and $i, j \geq 2$, and $u_{22} \leq u_{33} \leq \cdots \leq u_{nn}$. Let's write $\lambda_i = u_{ii}$. Then $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \overline{\Gamma}_k$ since for $j \leq k$,

$$S_j(\lambda) = S_j(D^2u) + S_{j-2;1i}(\lambda)u_{1i}^2 \geq S_j(D^2u) \geq 0\,. \tag{5.15}$$

It follows that

$$S_{k-1}(D^2u) = S_k^{11}(D^2u) + u_{11}S_{k-1}^{11}(D^2u) - u_{1i}^2S_{k-3;1i}(\lambda) \leq S_k^{11}(D^2u)$$

and

$$S^{11}(D^2u) \geq S_{k-1}(D^2u) = \frac{1}{n-k+1} \sum S_k^{ii}(D^2u)\,.$$

So $F_{11} \geq C_{n,k}\mathcal{F}$ holds. Putting this into (5.13), we see that (5.6) is valid.

Next, observe that by (5.15)

$$0 \leq S_k(\lambda) = \lambda_1 S_{k-1;1}(\lambda) + S_{k;1}(\lambda) \leq \lambda_1 S_{k-1;1}(\lambda) + C S_{k-1;1}^{k/(k-1)}(\lambda)\,.$$

As a result,

$$S_{k-1;1}(\lambda) \geq C|\lambda_1|^{k-1}\,;$$

that is,

$$S^{11}(D^2u) \geq \frac{C u_1^{2k-2}}{M^{k-1}}\,. \tag{5.16}$$

From (5.7) and equation (5.1), we have

$$S_k(D^2u) \leq CM^p \ \text{ at } (x_0,t_0) \quad \text{and so} \quad \mu'(S_k(D^2u)) \geq \frac{C}{M^{p-1}} \ \text{ at } (x_0,t_0)$$

by our choice of μ. Consequently, by (5.16),

$$F_{11} \geq \frac{C u_1^{2k-2}}{M^{p+k-2}}\,.$$

Putting this inequality into (5.14) yields the desired estimate

$$u_1 \leq CM^{(p+k)/2k} \quad \text{at } (x_0,t_0)\,.$$

Since G attains its maximum at (x_0,t_0),

$$|u_\xi(x,t)| \leq \frac{\rho\varphi(x_0,t_0)}{\rho\varphi(x,t)} u_1(x_0,t_0) \leq \frac{C}{r} M^{(p+k)/2k}.$$

So (5.6) holds. The proof of Theorem 5.1 is complete. □

Theorem 5.1 gives an interior estimate for the spatial derivatives of the solutions. A global gradient estimate can be obtained by modifying our proof. In view of (5.7), we can write (5.1) as an elliptic equation with bounded right-hand side. By constructing barriers, one can show that

$$|\nabla_x u(x,t)| \leq C\big(1 + M_t^{p/k}\big)\,, \quad x \in \partial\Omega\,.$$

Now, the global gradient estimate at interior points follows by the above argument, where we now take $\rho = 1$ in the definition of G.

We state a global existence and regularity result for (5.1) for later use.

THEOREM 5.2 *In addition to the hypotheses in Theorem* 5.1, *let's assume further that* $u_0 \in C^{3,1}(\overline{\Omega})$ *and* $f \in C^{1,1}(\overline{Q}_T \times \mathbb{R})$. *Then for any* $T > 0$, *there is a unique* k*-admissible solution* $u \in C_{x,t}^{3+\alpha,1+\alpha/2}(\overline{Q}_T)$, $\alpha \in (0,1)$, *of* (5.1) *that satisfies the a priori estimate*

$$\|u\|_{C_{x,t}^{3+\alpha,1+\alpha/2}(Q_T)} \leq C\,,$$

where C depends on n, k, α, T, $\|u\|_{L^\infty(Q_T)}$, u_0, and f. The constant C can be chosen independently of T if f and its derivatives up to second order are uniformly bounded.

Theorem 5.2, whose proof is omitted, can be proven in the same way as theorem A.1 in [25, 29], where the special case $\mu(t) = \log t$ is considered. Similar results can also be found in [11, 22]. Note that we do not require the right-hand side to be nonnegative, due to the condition $\mu(t) \to -\infty$ as $t \to 0$.

6 The Sublinear Case

In this section we prove Theorem 6.3, which contains Theorem 1.3. As we stated in the introduction, for any strictly $(k-1)$–convex domain Ω, there exists a unique $\lambda_1 > 0$ such that the problem

$$\begin{cases} S_k(D^2u) = \lambda_1 |u|^k & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a nonzero admissible solution φ, called the eigenfunction of the k-Hessian operator, which is unique up to the multiplication of a positive number. In fact, the (first) eigenvalue λ_1 has the following variational characterization:

$$\lambda_1 = \inf \left\{ E_k(u) : \|u\|_{L^{k+1}(\Omega)} = 1, u \in \Phi_0^k \right\}.$$

Recall that

$$J(u) = \frac{-1}{k+1} \int_\Omega u S_k(D^2u)dx - \int_\Omega \Psi(x,u)dx\,,$$

where $\Psi(x,z) = \int_z^0 \psi(x,s)ds$. Let $u(x,t)$ be a solution of

$$\begin{cases} F(D^2u) - u_t = f(x,u) & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \partial\Omega \times [0,\infty) \\ u = u_0 & \text{on } \{t = 0\}\,, \end{cases} \tag{6.1}$$

where $F(r) = \mu(S_k(r))$, $f(x,z) = \mu(\psi(x,z))$, and μ has been specified in Section 5. We have

$$\begin{aligned} \frac{d}{dt} J(u(\cdot,t)) &= -\int_\Omega u_t \big(S_k(D^2u) - \psi(x,u)\big)dx \\ &= -\int_\Omega \big(S_k(D^2u) - \psi(x,u)\big)\big(\mu(S_k(D^2u)) - \mu(\psi(x,u))\big)dx \\ &\le 0 \end{aligned} \tag{6.2}$$

Hence (6.1) is a negative gradient flow of the functional J.

In this section we consider the sublinear case (1.7), that is,

$$\lim_{z\to-\infty}\frac{\psi(x,z)}{|z|^k}<\lambda_1 \quad \text{uniformly for } x\in\overline{\Omega}\,.$$

Under this assumption, we have

$$\Psi(x,z)\le C_1+\frac{(1-\theta)\lambda_1}{k+1}|z|^{k+1} \quad \text{for some } \theta>0\,. \tag{6.3}$$

By the variational characterization of the eigenvalue, we have

$$J(u)\ge\frac{-\theta}{k+1}\int\limits_{\Omega} uS_k(D^2u)-C_2\,.$$

Hence J is bounded from below and $J(u)\to\infty$ as $\|u\|_{\Phi_0^k}\to\infty$ in the sublinear case.

LEMMA 6.1 *Consider* (1.1) *where* (1.7) *holds. There exists $C>0$ depending only on n, k, C_1, θ* (*in* (6.3))*, and Ω such that for any admissible solution of* (1.1)*,*

$$\sup_{x\in\Omega}|u(x)|\le C\,.$$

PROOF: By (1.7) we have

$$\psi(x,z)\le(K-(\lambda_1-\theta)z)^k\,,\quad z\le 0\,,$$

for some constant $K=K(\theta,C_1)>0$. If the lemma is not true, there is a sequence $\{\psi_m\}$ satisfying the inequality above such that the equation (1.1) for $\psi=\psi_m$ has a solution $u_m\in\Phi_0^k$ with

$$M_m=\sup_{\Omega}|u_m|\to\infty \quad \text{as } m\to\infty\,.$$

Let $v_m=u_m/M_m$. We have $v_m\to v$ by (1.10), and by the weak convergence of the Hessian measures [23], v is a subsolution of

$$S_k(D^2u)=|(\lambda_1-\theta)u|^k\,. \tag{6.4}$$

By constructing appropriate barriers, one infers from the comparison principle [21, 23] that v is Lipschitz-continuous on $\partial\Omega$. Let $a>1$ be sufficiently large such that $w=a\varphi<v$ in Ω, where φ is the eigenfunction of the Hessian operator. Then v and w are, respectively, a supersolution and a subsolution of (6.4). It follows that there is a solution $\varphi_1\in\Phi_0^k(\Omega)$ of (6.4) satisfying $v\ge\varphi_1\ge w$. However, this is in conflict with the uniqueness of the first eigenvalue [29]. So the lemma must hold. □

First we prove an existence result assuming ψ is strictly positive.

THEOREM 6.2 *Let Ω be of class $C^{3,1}$, $\psi\in C^{1,1}(\overline{\Omega\times\mathbb{R}^-})$, $\psi\ge\psi_0>0$, and* (1.7) *holds. Then* (1.1) *has a k-solution $u\in C^{3,\alpha}(\overline{\Omega})$ that is a minimizer of the functional J in Φ_0^k.*

PROOF: We choose a sequence of positive functions $\{\psi_m(x,z)\}$ in $C^{1,1}(\overline{\Omega}\times\mathbb{R})$ such that $\psi_m \le \psi$, $\psi_m(x,z) = \psi(x,z)$ when $|z| \le m$, and $\psi_m(x,z)$ is independent of z when $|z| > 2m$. Let

$$J_m(u) = \frac{-1}{k+1}\int_\Omega uS_k(D^2u)dx - \int_\Omega \Psi_m(x,u)dx$$

where $\Psi_m(x,z) = \int_z^0 \psi_m(x,s)ds$. Let $R > 0$ be so large that

$$\inf\left\{J_m(u) : u \in \Phi_0^k,\ \|u\|_{\Phi_0^k} \ge R\right\} > 1 + \inf\left\{J_m(u) : u \in \Phi_0^k\right\}. \tag{6.5}$$

For $K > 0$ small, we choose $u_0 \in \Phi_0^k$ such that $J_m(u_0) < K + \inf_{\Phi_0^k} J_m(u)$. By Theorem 5.2 there is a solution $u = u_{m,K}$ of the problem

$$\begin{cases} F(D^2u) - u_t = f_m(x,u) & \text{in } \Omega\times(0,\infty) \\ u = 0 & \text{on } \partial\Omega\times[0,\infty) \\ u = u_0 & \text{on } \{t=0\} \end{cases}$$

where $f_m(x,u) = \mu(\psi_m(x,u))$.

We point out that all hypotheses in Theorem 5.2 can be verified easily except the compatibility condition. This condition can be satisfied by a slight modification of the initial function [29]. Indeed, for any initial function, $u_0 \in \Phi_0^k(\Omega)\cap C^{3,1}(\overline{\Omega})$ such that $S_k(D^2u_0) > 0$ on $\overline{\Omega}$. Let $g \in C^{1,1}(\overline{\Omega})$ such that $g = S_k(D^2u_0)$ in Ω_δ (for some $\delta > 0$ sufficiently small) and $g = f_m(x,0) = f(x,0)$ on $\partial\Omega$, and let $\widetilde{u}_0 \in \Phi_0^k(\Omega)$ be the solution of $S_k(D^2u) = g$. Then $\widetilde{u}_0$ satisfies the compatibility condition on $\partial\Omega\times\{t=0\}$, and we also have $J_m(\widetilde{u}_0) < K + \inf_{\Phi_0^k} J_m(u)$. Hence, we may replace u_0 by $\widetilde{u}_0$, if necessary, and then apply Theorem 5.2.

By (6.2) and (6.5), $\|u(\cdot,t)\|_{\Phi_0^k} < R$ for all $t > 0$. Since for each fixed m, ψ_m is uniformly bounded, by Theorem 5.2 we have

$$\|u\|_{C^{2,1}_{x,t}(\Omega\times\mathbb{R})} \le C\,.$$

It implies that

$$\frac{d}{dt}J_m(u(\cdot,t)) \to 0 \quad \text{as } t\to\infty\,.$$

Hence we can select a sequence $\{t_j\} \to \infty$ such that $\{u(x,t_j)\}$ converges to a solution $w = w_{m,K} \in \Phi_0^k(\Omega)$ of $S_k(D^2u) = \psi_m$.

By Lemma 6.1, $\{w_{m,K}\}$ is uniformly bounded for all sufficiently small $K > 0$. Hence for m sufficiently large, $w_{m,K}$ is indeed a solution of (1.1), since $\psi_m = \psi$ when $|z| < m$. Theorem 6.2 follows by sending $K \to 0$ by using the a priori estimate in Theorem 4.3. $\square$

The strict positivity condition in Theorem 6.2 can be relaxed to (1.8), namely,

$$\lim_{z\to 0^-}\frac{\psi(x,z)}{|z|^k} > \lambda_1$$

uniformly for $x \in \overline{\Omega}$. It implies

$$\inf_{u \in \Phi_0^k} J(u) < 0$$

since $J(\delta\varphi) < 0$ when $\delta > 0$ is small and φ is the eigenfunction of the Hessian operator. We have the following precise statement of Theorem 1.3:

THEOREM 6.3 *Consider* (1.1) *where* $\psi \in C(\overline{\Omega \times \mathbb{R}^-}) \cap C^{1,1}(\overline{\Omega} \times \mathbb{R}^-)$ *and* Ω *is of class* $C^{3,1}$. *Suppose* $\psi(x, z) > 0$ *when* $z < 0$ *and it satisfies* (1.7) *and* (1.8) *uniformly in* $\overline{\Omega}$. *Then* (1.1) *has a k-admissible solution* $u \in C^{3,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ *for some* $\alpha \in (0, 1)$ *that is negative in* Ω. *Moreover, it is a minimizer of the functional* J *over* Φ_0^k.

PROOF: Theorem 6.3 follows from Theorem 6.2 by approximation. Indeed, let $\{\psi_n\}$ be a sequence of positive functions in $C^{1,1}(\overline{\Omega \times \mathbb{R}^-})$ that converges to ψ in $C(\overline{\Omega \times \mathbb{R}^-})$. By our assumption on ψ we may assume each $\psi_n > h$ for some h satisfying $h(0) = 0$ and $h(z) > 0$ for negative z. By Theorem 6.2, for each n there is a minimizer u_n in Φ_0^k of the corresponding functional J_n. Without loss of generality we may assume

$$J_n(u_n) < -c_0 < 0$$

for some c_0 independent of n. By Lemma 6.1, $\{u_n\}$ is uniformly bounded. By (3.14), Theorem 3.2, and Theorem 4.2, $\{u_n\}$ is also uniformly bounded in $C^2_{\mathrm{loc}}(\Omega)$. Hence $\{u_n\}$ subconverges to a solution u of (1.1), and u is a minimizer of the functional J in Φ_0^k. Note that by (6.5), $\{\|u_n\|_{\Phi_0^k(\Omega)}\}$ is uniformly bounded, and the Poincaré-type inequality (1.11) ensures that in fact u is continuous in $\overline{\Omega}$ and vanishes on $\partial\Omega$. Alternatively, one can construct a Lipschitz-continuous subsolution $\underline{u} \in \Phi_0^k(\Omega)$ of (1.1) such that for all large n, there holds $u_n \geq \underline{u}$, which forces $u = 0$ on $\partial\Omega$. $\square$

We remark that the strict positivity condition $\psi(x, z) > 0$ in Theorem 6.3 can be replaced by $\psi(x, z) \geq 0$. Indeed, by Lemma 6.1, the sequence of solutions, $\{u_n\}$, is uniformly bounded. By (1.10) we may assume, by passing to a subsequence, $u_n \to u$. Then u is a weak solution of (1.1) [21, 23].

7 The Superlinear Case

In this section we finally prove Theorem 1.2. We will use the idea in the proof of the mountain pass lemma, making use of the gradient flow generated by the parabolic Hessian equation (6.1) to prove Theorem 1.2. Because the right-hand side ψ of (1.1) equals zero on the boundary, we need to approximate ψ by positive functions so that Theorem 5.2 is applicable.

The proof of the theorem is quite long. We divide it into six steps.

Step 1

In the first step we show that c_δ, obtained by the min-max principle (7.5), is positive for sufficiently small $\delta > 0$.

First, let's assume that ψ satisfies a growth condition stronger than (1.5),

$$\lim_{z\to-\infty} \frac{\psi(x,z)}{|z|^p} = 0 \quad \text{uniformly for } x \in \overline{\Omega}, \tag{7.1}$$

where $p < k^* - 1$ and is sufficiently close to $k^* - 1$ when $n > 2k$ and $p > k$ is a large positive number when $n \le 2k$.

For small $\delta > 0$, let $\eta_\delta \in C^2(\Omega)$ be a nonnegative function satisfying $0 \le \eta_\delta(x) \le 1$, $\eta_\delta(x) = 0$ when $\operatorname{dist}(x, \partial\Omega) \le \delta$, and $\eta_\delta(x) = 1$ when $\operatorname{dist}(x, \partial\Omega) \ge 2\delta$. We define $\eta_0(x) \equiv 1$ if $\delta = 0$. Let

$$\psi_\delta(x,t) = \eta_\delta(x)\psi(x,t) + \delta^2 . \tag{7.2}$$

Let's first consider the following modification of (1.1):

$$\begin{cases} S_k(D^2u) = \psi_\delta(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases} \tag{7.3}$$

Let

$$J_\delta(u) = \frac{-1}{k+1}\int_\Omega uS_k(D^2u)dx - \int_\Omega \Psi_\delta(x,u)dx$$

where $\Psi_\delta(x,u) = \int_u^0 \psi_\delta(x,t)dt$. Let $u_1 \equiv 0$. By (1.4) there exists $u_2 \in \Phi_0^k(\Omega)$ such that $J_\delta(u_2) < -1$ for all small $\delta > 0$. Let $\widetilde{u}_1$ and $\widetilde{u}_2$ be smooth admissible functions sufficiently close to u_1 and u_2, respectively, such that $S_k(D^2\widetilde{u}_i) > 0$ on $\overline{\Omega}$ $(i = 1, 2)$, $J_\delta(\widetilde{u}_1)$ is sufficiently small, and $J_\delta(\widetilde{u}_2) < -1$. Denote by Γ the set of "admissible paths" connecting u_1 and u_2, namely,

$$\begin{aligned}\Gamma = \Big\{\gamma \in C([0,1], \Phi_0^k \cap C^{3,1}(\overline{\Omega})) :\\ \gamma(0) = \widetilde{u}_1, \gamma(1) = \widetilde{u}_2, S_k(D^2\gamma(s)) > 0 \text{ on } \overline{\Omega}\Big\} .\end{aligned} \tag{7.4}$$

Let

$$c_\delta = \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} J_\delta(\gamma(s)) . \tag{7.5}$$

It is easy to see that

$$c_0 \ge \overline{\lim}_{\delta\to 0} c_\delta .$$

Since $\psi_\delta(x,t) \le \psi(x,t) + \delta^2$, we have

$$c_0 = \lim_{\delta\to 0} c_\delta .$$

We claim $c_0 > 0$. Indeed, by (1.3) there exists a sufficiently small $\theta > 0$ such that

$$\Psi(x,u) \le \frac{\lambda_1(1-\theta)}{k+1}|u|^{k+1} + C|u|^{p+1} .$$

Hence

$$J(u) \geq \frac{1}{k+1}\|u\|_{\Phi_0^k}^{k+1} - \int_\Omega \Psi(x,u)dx$$
$$\geq \frac{1}{k+1}\|u\|_{\Phi_0^k}^{k+1} - \int_\Omega \left[\frac{\lambda_1(1-\theta)}{k+1}|u|^{k+1} + C|u|^{p+1}\right]dx\,.$$

By (6.2) and the Hessian Sobolev inequality,

$$J(u) \geq \frac{\theta}{k+1}\|u\|_{\Phi_0^k}^{k+1} - C\|u\|_{\Phi_0^k}^{p+1}\,.$$

Hence for some small $\sigma > 0$, we have

$$J(u) \geq \frac{\theta}{2(k+1)}\sigma^{k+1} > 0 \quad \forall u \in \Phi_0^k\,, \quad \|u\|_{\Phi_0^k} = \sigma\,.$$

In particular, let $\sigma = \theta$ (if $\theta > 0$ is small); we obtain

$$c_0 \geq \frac{\theta^{k+2}}{2(k+1)}\,.$$

We also have $c_\delta \geq \frac{1}{2}c_0 > 0$ for all small $\delta > 0$.

Remark. We can choose $\widetilde{u}_1$ such that $J_\delta(\widetilde{u}_1) < \theta^{k+2}/8(k+1)$ for all small $\delta > 0$.

Step 2

In this step we show that c_δ is a critical value of J_δ; i.e., there is a solution u_δ of (7.3) with $J_\delta(u_\delta) = c_\delta$ under the boundedness assumption (7.8) below.

Let $\gamma \in \Gamma$ satisfy

$$\sup_{s\in[0,1]} J_\delta(\gamma(s)) \leq c_0 + K$$

where $0 < K < c_0/4$. Let's consider the problem

$$\text{(7.6)} \qquad \begin{cases} F(D^2u) - u_t = \mu[\psi_\delta(x,u)] & \text{in } Q = \Omega\times[0,\infty)\,, \\ u(x,0) = \gamma(s),\ u(\cdot,t) \in \Phi_0^k\,, \end{cases}$$

where $F(D^2u) = \mu[S_k(D^2u)]$ and μ is specified in Section 5, where now the exponent p in (5.4) is the one in (7.1). We further assume that μ satisfies

$$\text{(7.7)} \qquad (t-s)(\mu(t)-\mu(s)) \geq (t-s)(t^{1/p} - s^{1/p}) \quad \text{for } t,s > 0\,.$$

Observe that by (7.1)

$$\mu[\psi_\delta(\cdot,z)] \leq C(1+z) \quad \text{for large } z.$$

By Theorem 5.2, there exists a global solution $u^s(x,t)$, $s \in [0,1]$, for (7.6). We may assume directly that functions on the path γ satisfy the compatibility condition in Theorem 5.2, for otherwise we could make a modification of the functions, as we did in the proof of Theorem 6.2.

For any given $\tau > 0$, $\gamma^\tau(s) := u^s(\cdot, \tau)$ is a path in $\Phi_0^k(\Omega)$ but not in Γ. Note that

$$\gamma_1 = \left\{u^s(\cdot, t) : s = 0, 0 \le t \le \tau\right\}, \quad \gamma_2 = \left\{u^s(\cdot, t) : s = 1, 0 \le t \le \tau\right\},$$

are two paths in $\Phi_0^k(\Omega)$. Similar to (6.2) we have

$$\frac{d}{dt} J_\delta(u^s(\cdot, t)) \le 0\,.$$

That is, $u^s(\cdot, t)$ is a negative gradient flow for the functional J_δ. Connect γ_1, γ^τ, and γ_2 together to form a path in Γ and denote it by $\widetilde{\gamma}^\tau$.

Let $I_t = \{s \in [0, 1] : J_\delta(u^s(\cdot, t)) \ge c_\delta - K\}$. Clearly I_t is a closed subset of $[0, 1]$, and $I_t \subset I_{t'}$ for any $t \ge t'$. Let $I_\infty = \bigcap_{t\ge 0} I_t$. I_∞ cannot be empty, otherwise there would exist some $\tau > 0$ such that $I_\tau = \emptyset$, i.e., $J_\delta(u^s(\cdot, \tau)) \le c_\delta - K$ for all $s \in [0, 1]$. It would follow that $\sup\{J_\delta(u) : u \in \widetilde{\gamma}^\tau\} \le c_\delta - K$, a contradiction to the definition of c_δ.

For any fixed $s_0 \in I_\infty$, let us assume for a moment that

$$|u^{s_0}(x, t)| \le M_0 \quad \forall t \ge 0\,. \tag{7.8}$$

Then by Theorem 5.2, we conclude that

$$\frac{d}{dt} J_\delta(u^{s_0}(\cdot, t)) \to 0 \quad \text{as } t \to \infty\,,$$

since $J_\delta(u^{s_0}(\cdot, t)) \ge c_\delta - K$. Moreover, we can select a sequence $\{t_j\}, t_j \to \infty$, such that $\{u^{s_0}(\cdot, t_j)\}$ converges to a solution u_δ of (7.3) satisfying

$$c_\delta - K \le J_\delta(u_\delta) \le c_\delta + K\,. \tag{7.9}$$

In particular, $u_\delta \not\equiv 0$. By the maximum principle, u_δ is a negative solution of (7.3).

Step 3

Let

$$K^0 = \left\{t : \frac{d}{dt} J_\delta(u^{s_0}(\cdot, t)) < -K\right\}.$$

Since

$$c_\delta - K \le J_\delta(u^{s_0}(\cdot, t)) \le c_\delta + K \quad \text{for all } t \ge 0\,,$$

we have $\mathrm{mes}(K^0) < 2$. In this step we prove that for any $t \notin K^0$, the following bounds hold:

$$\int_\Omega (-u^{s_0}) S_k(D^2 u^{s_0}) dx \le C\,, \tag{7.10}$$

$$\int_\Omega |u^{s_0} \psi_\delta(x, u^{s_0})| dx \le C\,. \tag{7.11}$$

In the following we drop the superscript s_0 in u^{s_0} for simplicity.

For any $t \notin K^0$, by (6.2) we have

$$\int_\Omega \big(S_k(D^2u) - \psi_\delta(x,u)\big)\big(\mu(S_k(D^2u)) - \mu(\psi_\delta(x,u))\big)dx$$
$$= -\frac{d}{dt} J_\delta(u(\cdot,t)) \le K\,.$$

It follows, by (7.7), that

$$\int_\Omega \big(S_k(D^2u) - \psi_\delta(x,u)\big)\big(S_k^{1/p}(D^2u) - \psi_\delta^{1/p}(x,u)\big)dx \le K\,.$$

Let $\alpha = S_k^{1/p}(D^2u)$ and $\beta = \psi_\delta^{1/p}(x,u)$. Then

$$\int_\Omega |\alpha-\beta|^{p+1}\,dx \le C\int_\Omega (\alpha^p - \beta^p)(\alpha-\beta)dx \le CK\,.$$

We have

$$\begin{aligned}
&\left|\int_\Omega u(\alpha^p - \beta^p)dx\right| \\
&\quad \le C\int_\Omega |u|\cdot|\alpha-\beta|\cdot|\alpha^{p-1}+\beta^{p-1}|dx \\
&\quad \le C\left[\int_\Omega |\alpha-\beta|^{p+1}\,dx\right]^{\frac{1}{p+1}}\left[\int_\Omega |u|^{p+1}\,dx\right]^{\frac{1}{p(p+1)}}\left[\int_\Omega |u|(\alpha^p+\beta^p)dx\right]^{\frac{p-1}{p}} \\
&\quad \le CK^{1/(p+1)}\|u\|_{L^{p+1}}^{1/p}\left[\left(\int_\Omega |u|\alpha^p\,dx\right)^{\frac{p-1}{p}} + \left(\int_\Omega |u|\beta^p\,dx\right)^{\frac{p-1}{p}}\right].
\end{aligned}$$

By (7.2) and (1.6),

$$\begin{aligned}
\Psi_\delta(x,u) &= -\delta^2 u + \Psi(x,u)\eta_\delta(x) \\
&\le \delta^2|u| + \frac{1-\theta}{k+1}|u|\psi(x,u)\eta_\delta(x) + C \\
&= \delta^2\left(1+\frac{1-\theta}{k+1}\right)|u| + \frac{1-\theta}{k+1}|u|\psi_\delta(x,u) + C\,.
\end{aligned}$$

Hence

$$
\begin{aligned}
J_\delta(u(\cdot,t)) &= \int_\Omega \left[\frac{-u}{k+1} S_k(D^2u) - \Psi_\delta(x,u)\right] dx \\
&\geq \int_\Omega \left[\frac{-u}{k+1} S_k(D^2u) - \frac{1-\theta}{k+1}|u|\psi_\delta(x,u)\right] dx \\
&\quad - C\left(1 + \delta^2 \int_\Omega |u| dx\right) \\
&\geq -\frac{1}{k+1}\left|\int_\Omega u(\alpha^p - \beta^p) dx\right| + \frac{\theta}{k+1}\int_\Omega |u|\psi_\delta(x,u) \\
&\quad - C\left(1 + \delta^2 \int_\Omega |u| dx\right).
\end{aligned}
\tag{7.12}
$$

Using the Hessian Sobolev inequality, we have

$$
\begin{aligned}
&\int_\Omega |u|\psi_\delta(x,u) dx \\
&\quad \leq C\left|\int_\Omega u(\alpha^p - \beta^p) dx\right| + C\left(1 + \delta^2 \int_\Omega |u| dx\right) \\
&\quad \leq CK^{1/(p+1)} \|u\|_{L^{p+1}}^{1/p} \left[\left(\int_\Omega |u|\alpha^p\, dx\right)^{\frac{p-1}{p}} + \left(\int_\Omega |u|\beta^p\, dx\right)^{\frac{p-1}{p}}\right] \\
&\qquad + C\left(1 + \delta 2 \int_\Omega |u| dx\right) \\
&\quad \leq CK^{1/(p+1)} \left[\int_\Omega |u|\alpha^p\, dx + \left(\int_\Omega |u|\alpha^p\, dx\right)^{\frac{1}{p}} \left(\int_\Omega |u|\beta^p\, dx\right)^{\frac{p-1}{p}}\right] \\
&\qquad + C\left(1 + \delta^2 \int_\Omega |u| dx\right).
\end{aligned}
$$

Further, by the Hölder inequality, we have

$$
\int_\Omega |u|\psi_\delta(x,u) dx \leq CK^{1/(p+1)} \int_\Omega |u|\alpha^p dx + C\left(1 + \delta^2 \int_\Omega |u| dx\right).
$$

Another application of the Sobolev inequality gives

$$\int_\Omega |u|\psi_\delta(x,u)dx \le C\left[1+\left(\delta^2+K^{1/(p+1)}\right)\int_\Omega |u|\alpha^p\, dx\right]. \tag{7.13}$$

Combining (7.12) and (7.13) and choosing $K, \delta > 0$ small, we obtain (7.10) and (7.11) for any $t \notin K^0$.

Step 4

PROOF OF (7.8): Let $M_t = \sup_{x\in\Omega_\delta} |u(x,t)|$ and $\widetilde{M}_t = \sup_{x\in\Omega} |u(x,t)|$. If M_t is not uniformly bounded, we can find a sequence $\{t_j\}$, $t_j \to \infty$, such that $M_{t_j} \to \infty$ and

$$M_{t_j} \ge M_t \quad \text{for } t < t_j\,. \tag{7.14}$$

Since $\psi_\delta = \delta^2$ in $\Omega - \Omega_\delta$ by (7.14) and the maximum principle, we have $\widetilde{M}_t \le M_{t_j} + C\delta^2$ for $t \in (0, t_j)$. By (5.5) we have

$$\widetilde{M}_t \ge \widetilde{M}_{t_j} e^{C_1(t-t_j)}\,, \quad t \le t_j\,.$$

Hence $M_t \ge CM_{t_j}$ for $t \in (t_j - 2, t_j)$.

Let $\tau \in (t_j - 2, t_j)$ but $\notin K^0$ and $y \in \Omega_\delta$ such that $u(y,\tau) = -M_\tau$. By the interior gradient estimate (5.6), we have

$$u(x,\tau) \le -\frac{1}{2}M_\tau \quad \text{if } x \in B_K(y)\,,$$

where $K = \theta M_\tau^\beta$, $\theta > 0$ depends only on $r = \operatorname{dist}(y, \partial\Omega)$ ($r \ge \delta$) and the constant C_2 in (5.6), and

$$\beta = 1 - \frac{p+k}{2k} = \frac{k-p}{2k}\,.$$

When M_τ is large, $K < \delta$ and $B_K(y) \subset \Omega$.

By (7.10) and the Sobolev inequality, we have

$$\|u(\cdot,\tau)\|_{L^q(B_K(y))} \le \|u(\cdot,\tau)\|_{L^q(\Omega)} \le C\,,$$

where we can take $q = k^*$ if $k < n/2$ and q arbitrarily large if $k \ge n/2$. On the other hand, from what we have just shown,

$$\|u(\cdot,\tau)\|^q_{L^q(B_K(y))} \ge CK^n M_\tau^q \ge CM_\tau^{q+n\beta}\,.$$

When $k \ge n/2$, we choose q large so that $q + n\beta > 0$ and reach a contradiction. When $k < n/2$, then $q = k^*$ and $p < k^* - 1$; again we have $q + n\beta > 0$ and the same contradiction. So (7.8) must hold. □

Step 5

We have obtained a solution u_δ of (7.3) that satisfies (7.9). In this step we prove

$$M_\delta = \sup\{|u_\delta(x)|, x \in \Omega\} \tag{7.15}$$

is uniformly bounded and $\{u_\delta\}$ converges to a solution of (1.1).

Similar to (7.10) and (7.11) (notice that u is independent of t in their proof), we have

$$\left| \int_\Omega u_\delta S_k(D^2 u_\delta) dx \right| \le C\,, \tag{7.16}$$

$$\left| \int_\Omega u_\delta \psi_\delta(x, u_\delta) dx \right| \le C \tag{7.17}$$

for some C independent of δ.

When $k > n/2$, we can combine (7.16) and the Sobolev inequality to obtain the boundedness of M_δ. When $k = n/2$, by the same reasoning we have $\|\psi_\delta\|_{L^p(\Omega)} \le C$ for some $p > 1$. Hence M_δ is uniformly bounded. See the discussion in the last paragraph of Section 2.

When $k < n/2$, we need a rescaling argument. Suppose on the contrary that $M_\delta \to \infty$ (taking a subsequence if necessary). Let the supremum M_δ be attained at x_δ, and let

$$v_\delta(y) = M_\delta^{-1} u(R_\delta^{-1} y + x_\delta)\,, \quad R_\delta^{-1} y + x_\delta \in \Omega\,,$$

where $R_\delta = M_\delta^{(p-k)/2k}$. Then $v_\delta(0) = -1$, $-1 \le v_\delta(y) \le 0$ in D_δ, and v_δ satisfies

$$S_k(D_y^2 v) = \widetilde{\psi}_\delta(y) =: M_\delta^{-p} \psi_\delta(R_\delta^{-1} y + x_\delta) \quad \text{in } D_\delta\,,$$

where $D_\delta = \{y : R_\delta^{-1} y + x_\delta \in \Omega\}$. By (7.1) we have

$$\widetilde{\psi}_\delta(y) \to 0 \quad \text{uniformly for } y \in D_\delta \text{ as } \delta \to 0\,. \tag{7.18}$$

A direct computation shows that

$$\int_{D_\delta} |v_\delta(y, t)|^{p+1}\, dy = M_\delta^{-c_1} \int_\Omega |u(x, t)|^{p+1}\, dx$$

where

$$c_1 = p + 1 - \frac{n(p-k)}{2k} > 0\,.$$

Hence

$$\int_{D_\delta} |v_\delta(y, t)|^{p+1}\, dy \le C\,. \tag{7.19}$$

Let

$$E_\delta = \left\{ y \in D_\delta : v_\delta(y) \le -\tfrac{1}{2} \right\}.$$

By (7.19),

$$\mathrm{mes}(E_\delta) \le C\,.$$

Applying Theorem 2.1 to $v_\delta(y) + \frac{1}{2}$ on the domain E_δ, we conclude by (7.18) that

$$v_\delta(y) \ge -\tfrac{3}{4} \quad \text{for large } M_\delta\,.$$

But, on the other hand, $v_\delta(0) = -1$ by definition. We reach a contradiction. Hence $\{M_\delta\}$ is uniformly bounded.

By (3.14) and Theorem 4.2, we can now select a subsequence of $\{u_\delta\}$ that converges to a solution $u = u_{(K)}$ of (1.1) such that

$$J(u_{(K)}) = \lim_{\delta\to 0} J_\delta(u_\delta)\,.$$

By (7.9) we have $c_0 - K \le J(u_{(K)}) \le c_0 + K$. Sending $K \to 0$ and again employing Theorem 4.2, we conclude that (choosing a subsequence if necessary) $u_{(K)} \to u \in C^{3,\alpha}(\Omega)$, and u is a solution of (1.1) satisfying

$$J(u) = c_0\,.$$

By Theorem 3.4, we have $u \in C^{0,1}(\overline{\Omega})$.

Step 6

Finally, we remove assumption (7.1). We may select a sequence $\{\psi_j(x,z)\}$ satisfying (1.3) through (1.6) such that $\psi_j(x,z) = \psi(x,z)$ when $|z| < j$ and each $\psi_j(x,z)$ satisfies (7.1). By the above argument there exists a solution $u_j \in \Phi_0^k$ of $S_k(D^2u) = \psi_j$ such that $J_j(u_j) = c_j$, where

$$J_j(u) = -\frac{1}{k+1}\int_\Omega u S_k(D^2u) - \int_\Omega \Psi_j(x,u)\,,$$

$\Psi_j(x,z) = \int_z^0 \psi(x,s)ds$, and c_j is a critical value of J_j, defined by a corresponding min-max scheme such as c_δ in (7.6). As in to step 1 it is easy to show that $c' \le c_j \le c''$ for c' and c'' independent of j. We claim $\|u_j\|_{L^\infty(\Omega)}$ is uniformly bounded. Indeed, by

$$J(u_j) = \int_\Omega \left[\frac{-u_j}{k+1} S_k(D^2u_j) - \Psi_j(x,u_j)\right]$$

and

$$0 = \int_\Omega [-u_j S_k(D^2u_j) + u_j\psi_j(x,u_j)]\,,$$

we have,

$$\left|\int_\Omega \left[\frac{1}{k+1}u_j\psi_j(x,u_j) + \Psi_j(x,u_j)\right]\right| \le C\,.$$

By (1.6),

$$\left| \int_\Omega u_j \psi_j(x, u_j) \right| \le C \tag{7.20}$$

where C is independent of j. Consequently, we also have

$$\int_\Omega (-u_j) S_k(D^2 u_j) \le C\,. \tag{7.21}$$

With (7.20) and (7.21) we can repeat the argument of step 5 to conclude the uniform boundedness of $\{\|u_j\|_{L^\infty(\Omega)}\}$. Similarly, by Theorem 4.2 there is a subsequence of $\{u_j\}$ that converges to a solution of (1.1) such that $J(u) = c_0$, i.e., $u \not\equiv 0$. The regularity of u follows from Theorems 3.4 and 4.2. This completes the proof of Theorem 1.2.

As a final remark we point out that condition (1.6) is often referred to as the subcritical growth condition, which is used for the uniform estimates in steps 4 and 5; see (7.8) and (7.15). Such estimates are not true if the subcritical growth condition is violated. For the semilinear elliptic equation (1.2), one can work in the Sobolev space $W^{1,2}(\Omega)$. Then condition (1.6) (with $k = 1$) ensures that the functional J satisfies the Palais-Smale condition [5], and there is no need for such uniform estimates.

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