INTERIOR CURVATURE BOUNDS FOR A CLASS OF CURVATURE EQUATIONS

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Abstract

We derive interior curvature bounds for admissible solutions of a class of curvature equations subject to affine Dirichlet data, generalizing a well-known estimate of Pogorelov for equations of Monge-Ampère type. For equations for which convexity of the solution is the natural ellipticity assumption, the curvature bound is proved for solutions with $C^{1,1}$ Dirichlet data. We also use the curvature bounds to improve and extend various existence results for the Dirichlet and Plateau problems.

1. Introduction

In this paper we derive interior curvature bounds for admissible solutions of a class of curvature equations subject to affine Dirichlet data, generalizing the well-known interior second derivative bound of Pogorelov [P] for equations of Monge-Ampère type (see also Ivochkina [I1]). This estimate has recently been extended to k-Hessian equations by Chou and Wang [CW]. In addition, in the case that convexity of the solution is the natural ellipticity assumption, we prove an interior curvature bound for convex solutions subject to $C^{1,1}$ Dirichlet data. This is a generalization of the interior second derivative bound of Trudinger and Urbas [TU2] for solutions of Monge-Ampère equations (see also Caffarelli [C] and Urbas [U1]).

The interior curvature bound permits us to extend in a straightforward way some of the existence theorems of Caffarelli, Nirenberg, and Spruck [CNS1] and Ivochkina, Trudinger, and Lin [I2], [I3], [ILT], [LT] to less regular boundary data than are required for the existence of globally smooth solutions. Moreover, the existence results of Trudinger [T1] for Lipschitz continuous viscosity solutions can be improved to yield locally smooth solutions, in the case of constant Dirichlet data, and more generally, affine Dirichlet data. In addition, we use the curvature bound to extend the recent work of Trudinger and Wang [TW] and Guan and Spruck [GS2] on the Plateau

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problem for hypersurfaces of constant Gauss curvature to hypersurfaces of constant curvature ratio H_n/H_l where for any integer k between 1 and n, H_k denotes the kth mean curvature of the hypersurface. This is explained more fully below.

We assume that $f \in C^2(\Gamma) \cap C^0(\overline{\Gamma})$ is a symmetric function defined on an open, convex, symmetric region $\Gamma \subset \mathbf{R}^n$, $\Gamma \neq \mathbf{R}^n$, with $0 \in \partial\Gamma$ and such that $\Gamma + \Gamma_+ \subset \Gamma$, where Γ_+ is the positive cone in $\mathbf{R}^n$. We assume that f satisfies

$$f > 0 \quad \text{in } \Gamma, \qquad f = 0 \quad \text{on } \partial\Gamma; \tag{1.1}$$

$$f \text{ is concave in } \Gamma; \tag{1.2}$$

$$\sum_{i=1}^n f_i \geq \sigma_0 \quad \text{on } \Gamma_{\mu_1,\mu_2}, \tag{1.3}$$

and

$$\sum_{i=1}^n f_i\lambda_i \geq \sigma_1 \quad \text{on } \Gamma_{\mu_1,\mu_2}, \tag{1.4}$$

where $\Gamma_{\mu_1,\mu_2} = \{\lambda \in \Gamma : \mu_1 \leq f(\lambda) \leq \mu_2\}$ for any $\mu_2 \geq \mu_1 > 0$, and σ_0, σ_1 are positive constants depending on μ_1 and μ_2.

Remarks 1.1
(i) (1.1) and (1.2) imply the degenerate ellipticity condition

$$f_i = \frac{\partial f}{\partial \lambda_i} \geq 0 \quad \text{in } \Gamma \text{ for} \quad i = 1, \ldots, n. \tag{1.5}$$

(ii) In the presence of (1.1) and (1.2), conditions (1.3) and (1.4) can be derived from various other assumptions. For example, if

$$f(t, \ldots, t) \to \infty \quad \text{as } t \to \infty, \tag{1.6}$$

then from the concavity of f and the fact that $\sum f_i\lambda_i \geq 0$ we have

$$f(t, \ldots, t) \leq f(\lambda) + \sum f_i(\lambda)(t - \lambda_i) \leq f(\lambda) + t\sum f_i(\lambda),$$

from which (1.3) follows on any Γ_{0,μ_2} by fixing t large enough. In fact, we see that (1.3) and (1.6) are equivalent. Similarly, if for any $\mu_1 \geq \mu_2 > 0$ there is a constant $\theta = \theta(\mu_1, \mu_2) > 0$ such that

$$f(2\lambda) \geq \theta + f(\lambda) \quad \text{for all } \lambda \in \Gamma_{\mu_1,\mu_2}, \tag{1.7}$$

then

$$f(2\lambda) \leq f(\lambda) + \sum f_i(\lambda)\lambda_i,$$

which implies (1.4) with $\sigma_1 = \theta$. Clearly, (1.7) is satisfied with $\theta = (2^\alpha - 1)\mu_1$ if f is homogeneous of some degree $\alpha \in (0, 1]$.

(iii) The main examples of functions f satisfying (1.1) to (1.4) are those corresponding to the kth mean curvature operators H_k , for which we take

$$f(\lambda) = S_k(\lambda)^{1/k} = \Big(\sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} \Big)^{1/k}, \tag{1.8}$$

and those corresponding to the curvature quotients H_k/H_l, $1 \le \cdots \le l < k \le \cdots \le n$, for which we take

$$f(\lambda) = \Big(\frac{S_k(\lambda)}{S_l(\lambda)} \Big)^{1/(k-l)}. \tag{1.9}$$

In both cases $\Gamma = \Gamma_k$ is defined to be the connected component of the set $\{\lambda : f(\lambda) > 0\}$ containing Γ_+. For these examples the concavity condition (1.3) is verified in [CNS1] and [T1].

(iv) These conditions, augmented by one further condition (see (1.14)) which we do not need for proving the curvature bound, are essentially the conditions formulated in [T1], for which the existence of viscosity solutions of the Dirichlet problem for the corresponding curvature equations has been established under appropriate further hypotheses on the data (in [T1] Γ is assumed to be a cone, but this is not necessary).

The curvature operator F corresponding to f is defined by

$$F[u] = f(\lambda_1, \ldots, \lambda_n) \tag{1.10}$$

where $\lambda_1, \ldots, \lambda_n$ are the principal curvatures of the graph of the function $u \in C^2(\Omega)$ defined on a domain $\Omega \subset \mathbf{R}^n$. A function $u \in C^2(\Omega)$ is said to be *admissible* (or Γ-*admissible*) if $\lambda = (\lambda_1, \ldots, \lambda_n)$ belongs to Γ at each point of Ω.

THEOREM 1.1
Let Ω be a bounded domain in $\mathbf{R}^n$, and let $u \in C^4(\Omega) \cap C^{0,1}(\overline{\Omega})$ be an admissible solution of

$$F[u] = g(x, u) \quad \text{in } \Omega, \qquad u = \phi \quad \text{on } \partial\Omega, \tag{1.11}$$

where $g \in C^{1,1}(\overline{\Omega} \times \mathbf{R})$ is a positive function and ϕ is affine. Then there exist positive constants β, depending only on $\sup_\Omega |Du|$, *and C, depending only on n, $\|u\|_{C^1(\overline{\Omega})}$, g and its first and second derivatives, $\mu_1 = \inf_\Omega g(x, u)$, $\mu_2 = \sup_\Omega g(x, u)$, and the structure constants σ_0, σ_1 in (1.3) and (1.4), such that the second fundamental form $\mathbf{A}$ of graph u satisfies*

$$|\mathbf{A}| \le \frac{C}{(\phi - u)^\beta}. \tag{1.12}$$

Remark 1.2
To see that this is indeed an interior curvature estimate we need to verify that $\phi - u \geq c(\Omega') > 0$ for any $\Omega' \subset\subset \Omega$. To do this we fix any point $x_0 \in \Omega$, and let $X_0 = (x_0, \phi(x_0))$. Let v be the function whose graph is a hemisphere of radius R lying above graph ϕ, such that $v(x_0) = \phi(x_0)$ and $Dv(x_0) = D\phi(x_0)$. Then for large enough R and small enough $\epsilon > 0$ we have $F[v - \epsilon] < F[u]$ in $\Omega_\epsilon = \{x \in \Omega : v(x) - \epsilon < \phi(x)\} \subset\subset \Omega$, and $v - \epsilon = \phi \geq u$ on $\partial\Omega_\epsilon$. By the comparison principle we then have $u \leq v - \epsilon$ in Ω_ϵ. Consequently $(\phi - u)(x_0) \geq \epsilon$.

We see that by making some minor modifications and extensions to the proof of Theorem 1.1 we are able to relax the assumptions on the boundary data in certain special cases. We defer the statements of these results to the later sections.

The main application of the curvature bound of Theorem 1.1 is to extend and improve various existence results for the Dirichlet problem for curvature equations, in particular, for the equations of prescribed kth mean curvature H_k and prescribed curvature quotients H_k/H_l with $k > l$. To obtain the existence of classical solutions we need to strengthen the degenerate ellipticity condition (1.5) to the strict ellipticity condition

$$f_i > 0 \quad \text{in } \Gamma \text{ for } i = 1, \ldots, n. \tag{1.13}$$

In addition, for proving gradient estimates we need to assume that

$$f_i \geq \sigma_2 \sum_{j=1}^{n} f_j \quad \text{if } \lambda_i \leq 0,\ \lambda \in \Gamma_{\mu_1,\mu_2}, \tag{1.14}$$

for any $\mu_2 \geq \mu_1 > 0$, where σ_2 is positive constant depending on μ_1 and μ_2.

A typical result is the following theorem, which improves a result of Trudinger [T1] on the existence of viscosity solutions, in the case of zero Dirichlet data. Various extensions and modifications of this result are mentioned in Sections 3 and 4.

THEOREM 1.2
Let f satisfy (1.1) to (1.4), together with (1.13) and (1.14). Let Ω be a bounded domain in $\mathbf{R}^n$, let $g \in C^{1,1}(\overline{\Omega} \times \mathbf{R})$ be a positive function satisfying $g_z \geq 0$, and suppose there is an admissible function $\underline{u} \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega})$ satisfying

$$F[\underline{u}] \geq g(x, \underline{u}) \quad \text{in } \Omega, \qquad \underline{u} = 0 \quad \text{on } \partial\Omega. \tag{1.15}$$

Then the problem

$$F[u] = g(x, u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.16}$$

has a unique admissible solution $u \in C^{3,\alpha}(\Omega) \cap C^{0,1}(\overline{\Omega})$ for all $\alpha \in (0, 1)$.

As a further application of the a priori curvature estimate we also consider a Plateau type problem for locally convex Weingarten hypersurfaces. Let Σ be a finite collection of disjoint, smooth, closed, co-dimension 2 submanifolds of $\mathbf{R}^{n+1}$. Let f be as above with $\Gamma = \Gamma_+$. We consider the following.

Suppose Σ bounds a locally uniformly convex hypersurface $\mathscr{M}_0$ with $f(\lambda^0) \geq k$, where $\lambda^0 = (\lambda_1^0, \dots, \lambda_n^0)$ are the principal curvatures of $\mathscr{M}_0$ and k is a positive constant. Does Σ bound a locally convex hypersurface $\mathscr{M}$ with $f(\lambda) = k$, where $\lambda = (\lambda_1, \dots, \lambda_n)$ are the principal curvatures of $\mathscr{M}$?

In the Gauss curvature case $f(\lambda) = \prod \lambda_i$, this problem was studied in [HRS] for Euclidean graphs over annular domains and in [GS1] for radial graphs over subdomains of the sphere. In [S] it was conjectured to have an affirmative answer in the general case. For Weingarten hypersurfaces with curvature function

$$f(\lambda) = \left(\frac{S_n(\lambda)}{S_l(\lambda)}\right)^{1/(n-l)}, \qquad l = 1, \dots, n-1, \tag{1.17}$$

this problem was studied in [IT] in the more general setting that the hypersurface can be represented as a section of a locally Euclidean line bundle; this includes Euclidean and radial graphs as special cases. In all these papers, however, $\mathscr{M}_0$ is a graph of some kind, so the problem can be reduced to a Dirichlet problem, for which the existence of a solution is proved under certain conditions guaranteeing a priori solution and gradient estimates.

The graph condition on $\mathscr{M}_0$ was removed in [TW] and [GS2] for the Gauss curvature case, thereby confirming the conjecture made in [S]. In Section 5 we extend this result to Weingarten hypersurfaces with curvature function f given by (1.17). In fact, the existence of locally smooth solutions is valid for the class of functions f satisfying (1.1) to (1.4) with $\Gamma = \Gamma_+$, together with the strict ellipticity condition (1.13), but for the existence of globally smooth solutions we need to impose further assumptions in order to derive curvature estimates at the boundary. To avoid these technicalities we prove boundary regularity only for the case (1.17).

THEOREM 1.3
Let f be given by $f(\lambda) = S_n(\lambda)/S_l(\lambda)$ for some $l = 0, \dots, n-1$. Let Σ be as above, and suppose that Σ bounds a locally uniformly convex hypersurface $\mathscr{M}_0$ with $f(\lambda^0) \geq k$ at each point of $\mathscr{M}_0$, where k is a positive constant. Then Σ bounds a smooth, locally convex hypersurface $\mathscr{M}$ with $f(\lambda) = k$ at each point of $\mathscr{M}$.

2. Proof of the curvature bound

We compute using a local orthonormal frame field $\hat{\mathbf{e}}_1, \dots, \hat{\mathbf{e}}_n$ defined on $\mathscr{M} = \operatorname{graph} u$ in a neighbourhood of the point at which we are computing. The standard basis of $\mathbf{R}^{n+1}$ is denoted by $\mathbf{e}_1, \dots, \mathbf{e}_{n+1}$. Covariant differentiation on $\mathscr{M}$ in the direction $\hat{\mathbf{e}}_i$ are denoted by ∇_i. The components of the second fundamental form $\mathbf{A}$ of $\mathscr{M}$ in the basis $\hat{\mathbf{e}}_1, \dots, \hat{\mathbf{e}}_n$ are denoted by $[h_{ij}]$. Thus

$$h_{ij} = \langle D_{\hat{\mathbf{e}}_i} \hat{\mathbf{e}}_j, \nu \rangle, \tag{2.1}$$

where D and $\langle \cdot, \cdot \rangle$ denote the usual connection and inner product on $\mathbf{R}^{n+1}$, and ν denotes the upward unit normal vector field

$$\nu = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}. \tag{2.2}$$

The differential equation in (1.11) can then be expressed as

$$F[\mathbf{A}] = g(X). \tag{2.3}$$

As usual we denote first and second partial derivatives of F with respect to h_{ij} by F_{ij} and $F_{ij,kl}$.

The following facts are well known (see [U3]). We assume summation from 1 to n over repeated Latin indices unless otherwise indicated.

LEMMA 2.1
The second fundamental form h_{ab} satisfies

$$\begin{aligned} F_{ij} \nabla_i \nabla_j h_{ab} = -F_{ij,kl} \nabla_a h_{ij} \nabla_b h_{kl} + F_{ij} h_{ij} h_{ak} h_{bk} \\ - F_{ij} h_{ik} h_{jk} h_{ab} + \nabla_a \nabla_b g. \end{aligned}$$

LEMMA 2.2
For any $\alpha = 1, \dots, n+1$ we have

$$F_{ij} \nabla_i \nabla_j \nu_\alpha + F_{ij} h_{ik} h_{jk} \nu_\alpha = -\langle \nabla g, \mathbf{e}_\alpha \rangle.$$

The following lemma is stated without proof in [An]; a proof is given in [G].

LEMMA 2.3
For any symmetric matrix $\eta = [\eta_{ij}]$ we have

$$F_{ij,kl} \eta_{ij} \eta_{kl} = \sum_{i,j} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \eta_{ii} \eta_{jj} + \sum_{i \neq j} \frac{f_i - f_j}{\lambda_i - \lambda_j} \eta_{ij}^2.$$

The second term on the right hand side is nonpositive if f is concave, and is interpreted as a limit if $\lambda_i = \lambda_j$.

Proof of Theorem 1.1
First, we may assume that $u \in C^4(\overline{\Omega})$ by replacing u by $u+\epsilon$ and Ω by $\{x \in \Omega : u(x)+\epsilon < \phi(x)\}$ for small enough $\epsilon > 0$.

We now let $\eta = \phi - u$. As observed in Remark 1.2, $\eta > 0$ in Ω. For a function Φ and a constant $\beta \geq 1$ to be chosen, we consider the function

$$\widetilde{W}(X,\xi) = \eta^\beta\big(\exp\Phi(\nu_{n+1})\big)h_{\xi\xi}$$

for all $X \in \mathscr{M}$ and all unit $\xi \in T_X\mathscr{M}$. Then $\widetilde{W}$ attains its maximum at an interior point $X_0 \in \mathscr{M}$, in a direction $\xi_0 \in T_{X_0}\mathscr{M}$ which we may take to be $\hat{\mathbf{e}}_1$. We may assume that $[h_{ij}]$ is diagonal at X_0 with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. The concavity of f then implies that $f_1 \leq \cdots \leq f_n$, and therefore also

$$f_n \geq \frac{1}{n}\mathscr{T} := \frac{1}{n}\sum f_i.$$

We may assume without loss of generality that the frame $\hat{\mathbf{e}}_1,\ldots,\hat{\mathbf{e}}_n$ has been chosen so that $\nabla_i\hat{\mathbf{e}}_j = 0$ at X_0 for all $i, j = 1,\ldots,n$. The existence of such a frame follows easily from the existence of Riemannian normal coordinates (see [Y], Section 1.5). Let $\zeta = \hat{\mathbf{e}}_1$. Then the function

$$W(X) = \eta^\beta\big(\exp\Phi(\nu_{n+1})\big)h_{ab}\zeta_a\zeta_b,$$

which is defined near X_0, has an interior maximum at X_0. We need to compute the equation satisfied by $Z := h_{ab}\zeta_a\zeta_b$. Using the special choice of frame and the fact that h_{ab} is diagonal at X_0 in this frame, we find that

$$\nabla_i Z = \nabla_i h_{11} \quad \text{and} \quad \nabla_i\nabla_j Z = \nabla_i\nabla_j h_{11} \quad \text{at} \quad X_0.$$

Thus at X_0, the scalar function Z satisfies the same equation as the component h_{11} of the tensor h_{ab}. Therefore

$$\frac{\nabla_i W}{W} = \beta\frac{\nabla_i\eta}{\eta} + \Phi'\nabla_i\nu_{n+1} + \frac{\nabla_i h_{11}}{h_{11}} = 0 \quad \text{at } X_0, \tag{2.4}$$

and

$$\begin{aligned}\frac{\nabla_i\nabla_j W}{W} - \frac{\nabla_i W\nabla_j W}{W^2} &= \beta\Big\{\frac{\nabla_i\nabla_j\eta}{\eta} - \frac{\nabla_i\eta\nabla_j\eta}{\eta^2}\Big\}\\ &\quad + \Phi''\nabla_i\nu_{n+1}\nabla_j\nu_{n+1} + \Phi'\nabla_i\nabla_j\nu_{n+1}\\ &\quad + \frac{\nabla_i\nabla_j h_{11}}{h_{11}} - \frac{\nabla_i h_{11}\nabla_j h_{11}}{h_{11}^2}\end{aligned} \tag{2.5}$$

is nonpositive in the sense of matrices at X_0. Using Lemmas 2.1 and 2.2, we therefore have, at X_0,

$$
\begin{aligned}
0 \geq \beta F_{ij}\Big\{\frac{\nabla_i\nabla_j\eta}{\eta} - \frac{\nabla_i\eta\nabla_j\eta}{\eta^2}\Big\} &+ \Phi'' F_{ij}\nabla_i\nu_{n+1}\nabla_j\nu_{n+1} \\
&- (\Phi'\nu_{n+1}+1)F_{ij}h_{ik}h_{jk} + F_{ij}h_{ij}h_{11} + \frac{\nabla_1\nabla_1 g}{h_{11}} \\
&- \Phi'\langle\nabla g, \mathbf{e}_{n+1}\rangle - \frac{1}{h_{11}}F_{ij,kl}\nabla_1 h_{ij}\nabla_1 h_{kl} - F_{ij}\frac{\nabla_i h_{11}\nabla_j h_{11}}{h_{11}^2}.
\end{aligned}
\tag{2.6}
$$

Using Gauss's formula

$$\nabla_i\nabla_j X_\alpha = h_{ij}\nu_\alpha, \tag{2.7}$$

we compute

$$
\begin{aligned}
\nabla_1\nabla_1 g(X) &= \sum_{\alpha=1}^{n+1}\frac{\partial g}{\partial X_\alpha}\nabla_1\nabla_1 X_\alpha + \sum_{\alpha,\beta=1}^{n+1}\frac{\partial^2 g}{\partial X_\alpha\partial X_\beta}\nabla_1 X_\alpha\nabla_1 X_\beta \\
&= \sum_{\alpha=1}^{n+1}\frac{\partial g}{\partial X_\alpha}\nu_\alpha h_{11} + \sum_{\alpha,\beta=1}^{n+1}\frac{\partial^2 g}{\partial X_\alpha\partial X_\beta}\nabla_1 X_\alpha\nabla_1 X_\beta.
\end{aligned}
$$

Consequently,

$$\frac{\nabla_1\nabla_1 g}{h_{11}} \geq -C \tag{2.8}$$

and

$$-\Phi'\langle\nabla g, \mathbf{e}_{n+1}\rangle \geq -C|\Phi'|. \tag{2.9}$$

Using (1.4), we have

$$F_{ij}h_{ij}h_{11} \geq \sigma_1 h_{11}. \tag{2.10}$$

Next we assume that ϕ has been extended to be constant in the $\mathbf{e}_{n+1}$ direction. We compute

$$
\begin{aligned}
\nabla_i\nabla_j\eta &= \sum_{\alpha,\beta=1}^{n}\frac{\partial^2\phi}{\partial X_\alpha\partial X_\beta}\nabla_i X_\alpha\nabla_j X_\beta + \sum_{\alpha=1}^{n}\frac{\partial\phi}{\partial X_\alpha}\nabla_i\nabla_j X_\alpha - \nabla_i\nabla_j X_{n+1} \\
&= \sum_{\alpha=1}^{n}\frac{\partial\phi}{\partial X_\alpha}\nu_\alpha h_{ij} - h_{ij}\nu_{n+1},
\end{aligned}
$$

where we have again used Gauss's formula (2.7) and the assumption that ϕ is affine. Consequently,

$$F_{ij}\nabla_i\nabla_j\eta = \Big(\sum_{\alpha=1}^{n}\frac{\partial\phi}{\partial X_\alpha}\nu_\alpha - \nu_{n+1}\Big)F_{ij}h_{ij} \geq -C. \tag{2.11}$$

Here we have used the fact that

$$0 \leq \sum f_i \lambda_i \leq C;$$

the second inequality follows immediately from the concavity of f and the fact that $f(0) = 0$.

Remarks 2.1

(i) It is in the derivation of (2.11) that the special form of η is used. For the remainder of the proof, all we need is that $|\nabla \eta|$ be bounded. In particular, under the additional structure condition

$$\sum_{i=1}^{n} f_i \leq \sigma_2 \quad \text{on } \Gamma_{\mu_1,\mu_2} \tag{2.12}$$

for some positive constant σ_2 depending on $\mu_2 \geq \mu_1 > 0$, we may use a standard cutoff function $\eta \in C_0^2(\mathscr{M})$. The main examples of functions satisfying (2.12) are the quotients $f = S_k/S_{k-1}, k = 1, \ldots, n$. For these examples, (2.12) follows immediately from [LT, equation (2.7)].

(ii) If ϕ is convex, not necessarily affine, the term

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \phi}{\partial X_\alpha \partial X_\beta} F_{ij} \nabla_i X_\alpha \nabla_j X_\beta$$

is nonnegative and can be discarded. This observation is used in Section 4 to extend the curvature bound to $C^{1,1}$ boundary data in the case $\Gamma = \Gamma_+$.

Using the above estimates in (2.6) and using (1.4), we have, at X_0,

$$\begin{aligned} 0 \geq & -\frac{C\beta}{\eta} - \beta F_{ij} \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} + \Phi'' F_{ij} \nabla_i \nu_{n+1} \nabla_j \nu_{n+1} \\ & - (\Phi' \nu_{n+1} + 1) F_{ij} h_{ik} h_{jk} + \sigma_1 h_{11} - C(1 + |\Phi'|) \\ & - \frac{1}{h_{11}} F_{ij,kl} \nabla_1 h_{ij} \nabla_1 h_{kl} - F_{ij} \frac{\nabla_i h_{11} \nabla_j h_{11}}{h_{11}^2}. \end{aligned} \tag{2.13}$$

We now consider two cases.

Case 1

There is a positive constant $\theta > 0$ to be chosen ($\theta = 1/5$ is our eventual choice) such that

$$\lambda_n < -\theta \lambda_1. \tag{2.14}$$

In this case we use the concavity of F to discard the second to last term in (2.13). Next, using (2.4), we have

$$\begin{aligned}
F_{ij}\frac{\nabla_i h_{11}\nabla_j h_{11}}{h_{11}^2} &= F_{ij}\Big(\beta\frac{\nabla_i\eta}{\eta}+\Phi'\nabla_i\nu_{n+1}\Big)\Big(\beta\frac{\nabla_j\eta}{\eta}+\Phi'\nabla_j\nu_{n+1}\Big)\\
&\le (1+\gamma^{-1})\beta^2F_{ij}\frac{\nabla_i\eta\nabla_j\eta}{\eta^2}\\
&\quad+(1+\gamma)(\Phi')^2F_{ij}\nabla_i\nu_{n+1}\nabla_j\nu_{n+1}
\end{aligned}$$

for any $\gamma > 0$. Therefore at X_0 we have, since $|\nabla\eta|\le C$,

$$\begin{aligned}
0\ge{}& -\frac{C\beta}{\eta}-C\big[\beta+(1+\gamma^{-1})\beta^2\big]\frac{\mathscr{T}}{\eta^2}\\
&+\big[\Phi''-(1+\gamma)(\Phi')^2\big]F_{ij}\nabla_i\nu_{n+1}\nabla_j\nu_{n+1}\\
&-[\Phi'\nu_{n+1}+1]F_{ij}h_{ik}h_{jk}+\sigma_1h_{11}-C(1+|\Phi'|).
\end{aligned}\tag{2.15}$$

We now write

$$1+\gamma=(1+\epsilon)(1+\beta^{-1})$$

where $\epsilon > 0$ and $\beta > 0$ are still to be fixed. The reason for this becomes apparent below. We then choose Φ as in the global curvature bound of [CNS1]. For a controlled positive constant a, depending only on $\sup_\Omega|Du|$, we have

$$2a\le\nu_{n+1}$$

and therefore

$$\frac{1}{\nu_{n+1}-a}\le\frac{1}{a}\le C.$$

We now choose

$$\Phi(t)=-\log(t-a).$$

Then

$$\Phi'(t)=\frac{-1}{t-a},\qquad \Phi''(t)=\frac{1}{(t-a)^2},$$

and

$$\begin{aligned}
-(\Phi't+1)&=\frac{a}{t-a},\\
\Phi''-(1+\epsilon)(1+\beta^{-1})(\Phi')^2&=-\frac{\epsilon+\beta^{-1}+\epsilon\beta^{-1}}{(t-a)^2}.
\end{aligned}$$

By direct computation (see [U3]), we have $\nabla_i\nu_{n+1}=-h_{ik}\langle\hat{\mathbf{e}}_k,\mathbf{e}_{n+1}\rangle$, and therefore

$$F_{ij}\nabla_i\nu_{n+1}\nabla_j\nu_{n+1}=F_{ij}h_{ik}h_{jl}\langle\hat{\mathbf{e}}_k,\mathbf{e}_{n+1}\rangle\langle\hat{\mathbf{e}}_l,\mathbf{e}_{n+1}\rangle\le F_{ij}h_{ik}h_{jk}.$$

Fixing $\epsilon = a^2/8$ and assuming henceforth that

$$\beta \geq \beta_0 := \frac{4}{a^2},$$

we have

$$\begin{aligned} -(\Phi' t + 1) + &\left[\Phi'' - (1+\epsilon)(1+\beta^{-1})(\Phi')^2\right] \\ &= \frac{a}{t-a} - \frac{(\epsilon + \beta^{-1} + \epsilon\beta^{-1})}{(t-a)^2} \\ &\geq \frac{1}{2}a^2. \end{aligned}$$

Using this in (2.15), together with

$$F_{ij}h_{ik}h_{jk} = \sum f_i \lambda_i^2 \geq f_n \lambda_n^2 \geq \frac{\theta^2}{n}\lambda_1^2 \mathscr{T},$$

which follows from (2.14) and the fact that $f_n \geq \frac{1}{n}\mathscr{T}$, we have, at X_0,

$$0 \geq -\frac{C(\beta)}{\eta} - C(\beta)\frac{\mathscr{T}}{\eta^2} + \frac{1}{2n}a^2\theta^2\lambda_1^2\mathscr{T} + \sigma_1\lambda_1 - C(\beta). \tag{2.16}$$

An upper bound

$$\eta\lambda_1 \leq \frac{C(\beta)}{\theta} \quad \text{at } X_0$$

follows from this and (1.3). Consequently $W(X_0)$ satisfies a similar bound.

Remark 2.2
The fact that $\sigma_1 > 0$ is not needed at this point.

Case 2
We now assume that

$$\lambda_n \geq -\theta\lambda_1.$$

Since $\lambda_1 \geq \cdots \geq \lambda_n$, this implies

$$\lambda_i \geq -\theta\lambda_1 \quad \text{for all } i = 1, \ldots, n. \tag{2.17}$$

We partition $\{1, \ldots, n\}$ into

$$I = \{j : f_j \leq 4f_1\}, \qquad J = \{j : f_j > 4f_1\},$$

where f_i is evaluated at $\lambda(X_0)$. Then for each $j \in I$ we have, by (2.4),

$$\begin{aligned} f_j\frac{|\nabla_j h_{11}|^2}{h_{11}^2} &= f_j\left(\Phi'\nabla_j \nu_{n+1} + \beta\frac{\nabla_j\eta}{\eta}\right)^2 \\ &\leq (1+\epsilon)(\Phi')^2 f_j|\nabla_j\nu_{n+1}|^2 + (1+\epsilon^{-1})\beta^2 f_j\frac{|\nabla_j\eta|^2}{\eta^2}, \end{aligned} \tag{2.18}$$

for any $\epsilon > 0$. Again by (2.4), for each $j \in J$ we have

$$\begin{aligned}\beta f_j \frac{|\nabla_j \eta|^2}{\eta^2} &= \beta^{-1} f_j \Big(\Phi' \nabla_j \nu_{n+1} + \frac{\nabla_j h_{11}}{h_{11}}\Big)^2 \\ &\le \frac{1+\epsilon}{\beta} (\Phi')^2 f_j |\nabla_j \nu_{n+1}|^2 + \frac{(1+\epsilon^{-1})}{\beta} f_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2}\end{aligned} \tag{2.19}$$

for any $\epsilon > 0$. Consequently,

$$\begin{aligned}&\beta \sum_{j=1}^n f_j \frac{|\nabla_j \eta|^2}{\eta^2} + \sum_{j=1}^n f_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2} \\ &\quad \le \big[\beta + (1+\epsilon^{-1})\beta^2\big] \sum_{j\in I} f_j \frac{|\nabla_j \eta|^2}{\eta^2} + (1+\epsilon)(\Phi')^2 \sum_{j\in I} f_j |\nabla_j \nu_{n+1}|^2 \\ &\qquad + \frac{(1+\epsilon)}{\beta} (\Phi')^2 \sum_{j\in J} f_j |\nabla_j \nu_{n+1}|^2 + \big[1 + (1+\epsilon^{-1})\beta^{-1}\big] \sum_{j\in J} f_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2} \\ &\quad \le 4n\big[\beta + (1+\epsilon^{-1})\beta^2\big] f_1 \frac{|\nabla \eta|^2}{\eta^2} + (1+\epsilon)(1+\beta^{-1})(\Phi')^2 \sum_{j=1}^n f_j |\nabla_j \nu_{n+1}|^2 \\ &\qquad + \big[1 + (1+\epsilon^{-1})\beta^{-1}\big] \sum_{j\in J} f_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2}.\end{aligned}$$

Using the above estimates in (2.13), we see that at X_0 we have

$$\begin{aligned}0 \ge &-\frac{C\beta}{\eta} - 4n\big[\beta + (1+\epsilon^{-1})\beta^2\big] f_1 \frac{|\nabla \eta|^2}{\eta^2} \\ &+ \big[\Phi'' - (1+\epsilon)(1+\beta^{-1})(\Phi')^2\big] F_{ij} \nabla_i \nu_{n+1} \nabla_j \nu_{n+1} \\ &- [\Phi' \nu_{n+1} + 1] F_{ij} h_{ik} h_{jk} + \sigma_1 h_{11} - C(1 + |\Phi'|) \\ &- \frac{1}{h_{11}} F_{ij,kl} \nabla_1 h_{ij} \nabla_1 h_{kl} - \big[1 + (1+\epsilon^{-1})\beta^{-1}\big] \sum_{j\in J} f_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2}\end{aligned} \tag{2.20}$$

For $\epsilon = a^2/8$ and $\beta \ge \beta_0 = 4/a^2$ exactly as above, we therefore have, at X_0,

$$\begin{aligned}0 \ge &-\frac{C\beta}{\eta} - C(\beta + \beta^2) \frac{f_1}{\eta^2} + \frac{1}{2} a^2 f_1 \lambda_1^2 + \sigma_1 \lambda_1 - C \\ &- \frac{1}{h_{11}} F_{ij,kl} \nabla_1 h_{ij} \nabla_1 h_{kl} - (1 + C_0 \beta^{-1}) \sum_{j\in J} f_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2},\end{aligned} \tag{2.21}$$

where $C_0 = 1 + 2a^{-2}$.

Next we estimate the last two terms in (2.21). Using the concavity of f, Lemma 2.3, and the Codazzi equations, which tell us that $\nabla_i h_{jk}$ is symmetric in all indices, we see that

$$-\frac{1}{h_{11}} F_{ij,kl} \nabla_1 h_{ij} \nabla_1 h_{kl} \geq -\frac{2}{\lambda_1} \sum_{j \in J} \frac{f_1 - f_j}{\lambda_1 - \lambda_j} |\nabla_j h_{11}|^2.$$

We therefore need to show that

$$-\frac{2(f_1 - f_j)}{\lambda_1(\lambda_1 - \lambda_j)} \geq (1 + C_0 \beta^{-1}) \frac{f_j}{\lambda_1^2} \quad \text{for } j \in J,$$

provided β is sufficiently large. Let us set $\delta = C_0/\beta$. Then we need to show

$$(1-\delta) f_j \lambda_1 \geq 2 f_1 \lambda_1 - (1+\delta) f_j \lambda_j \quad \text{for} \quad j \in J, \tag{2.22}$$

provided $\delta > 0$ is sufficiently small. We show this if either $\lambda_j \geq 0$ or $\lambda_j \leq 0$ and $|\lambda_j| \leq \theta \lambda_1$ for a sufficiently small positive constant θ.

Since $j \in J$, we have $f_j > 4 f_1$. Therefore, if $\lambda_j \geq 0$, then (2.22) is satisfied if $\delta = 1/4$. On the other hand, if $\lambda_j \leq 0$, then $|\lambda_j| \leq \theta \lambda_1$ by (2.17), and therefore (2.22) is again satisfied if $\delta = 1/4$ and $\theta = 1/5$. Notice that with this choice of δ we have $\beta = 4(1 + 2a^{-2})$, so the previous restriction $\beta \geq \beta_0$ is automatically satisfied. Notice also that β depends only on $\sup_\Omega |Du|$.

Having fixed δ and θ in this way, we see from (2.21) that at X_0 we have

$$\sigma_1 \lambda_1 + \frac{1}{2} a^2 f_1 \lambda_1^2 \leq C\Big(1 + \frac{1}{\eta} + \frac{f_1}{\eta^2}\Big), \tag{2.23}$$

from which we again conclude a bound for $\eta\lambda_1$ at X_0, and hence also for $W(X_0)$. The curvature bound of Theorem 1.1 then follows. $\square$

Remarks 2.3
(i) At this point we have used the fact that $\sigma_1 > 0$. If the term $\sigma_1 \lambda_1$ were absent from (2.23), we could still conclude a curvature estimate if we assumed in addition that

$$f_1 \geq c \lambda_1^{-1} \tag{2.24}$$

for a controlled positive constant c. This structure condition is satisfied for f given by (1.8), but not for f given by (1.9).

(ii) We can deduce the following result by taking into account Remark 2.1(i). This has been proved independently by Trudinger [T4].

THEOREM 2.1
Suppose that $f \in C^2(\Gamma) \cap C^0(\overline{\Gamma})$ satisfies (1.1) to (1.4) and (2.12). Let Ω be a bounded

domain in $\mathbf{R}^n$*, let* $g \in C^{1,1}(\overline{\Omega}\times\mathbf{R})$ *be a positive function, and let* $u \in C^4(\Omega)\cap C^{0,1}(\overline{\Omega})$ *be an admissible solution of*

$$F[u] = g(x,u) \quad \text{in } \Omega. \tag{2.25}$$

Then for any $\Omega' \subset\subset \Omega$ *there is a positive constant* $C(\Omega')$*, such that the second fundamental form* $\mathbf{A}$ *of graph* u *satisfies*

$$\sup_{\Omega'} |\mathbf{A}| \le C(\Omega'). \tag{2.26}$$

The positive constant $C(\Omega')$ *depends only on* n*,* $\operatorname{dist}(\Omega', \partial\Omega)$*,* $\|u\|_{C^1(\overline{\Omega})}$*,* g *and its first and second derivatives,* $\mu_1 = \inf_\Omega g(x,u)$*,* $\mu_2 = \sup_\Omega g(x,u)$*, and on the structure constants* $\sigma_0, \sigma_1, \sigma_2$ *in (1.3), (1.4), and (2.12).*

(iii) It is also evident that (2.12) could be weakened to

$$\sum_{i=1}^n f_i \le \sigma_2 |\lambda|^{1-\epsilon} \quad \text{on } \Gamma_{\mu_1,\mu_2}, \tag*{(2.12)$'$}$$

for any $\mu_2 \ge \mu_1 > 0$, where $\sigma_2 > 0$ and $\epsilon \in (0,1)$ are constants (the scalar curvature case $k = 2$ in (1.8) lies just outside the scope of this refinement of Theorem 2.1). Some additional modifications in the proof are necessary. First, in place of (2.11) we have

$$F_{ij}\nabla_i\nabla_j\eta \ge -C\lambda_1^{1-\epsilon}. \tag*{(2.11)$'$}$$

Second, the proof of (2.22) is valid provided β is sufficiently large; it does not need to be fixed at that point. In place of (2.23) we then obtain

$$\sigma_1\lambda_1 + \frac{1}{2}a^2 f_1\lambda_1^2 \le C(\beta)\Big(1 + \frac{\lambda_1^{1-\epsilon}}{\eta} + \frac{f_1}{\eta^2}\Big). \tag*{(2.23)$'$}$$

The proof then proceeds similarly to before, provided $\epsilon \in (0,1)$ and β is fixed sufficiently large, depending on ϵ, and also so that all the previous requirements are satisfied.

(iv) The techniques of this paper can be used to obtain analogous results for admissible solutions of Hessian equations

$$F(D^2u) = g(x,u),$$

where now $F(D^2u) = f(\lambda_1,\dots,\lambda_n)$ and $\lambda_1,\dots,\lambda_n$ are the eigenvalues of D^2u. If we make the analogous computations for Hessian equations, then in place of Lemmas 2.1 and 2.2 we use

$$F_{ij}D_{ij\alpha\alpha}u = -F_{ij,kl}D_{ij\alpha}uD_{kl\alpha}u + D_{\alpha\alpha}g$$

and

$$F_{ij}D_{ij}|Du|^2 = 2F_{ij}D_{ik}uD_{jk}u + 2D_kgD_ku.$$

We consider a function

$$\widetilde{W}(x,\xi) = \eta^\beta\big[\exp\Phi(|Du|^2/2)\big]D_{\xi\xi}u$$

where η is as above, but now

$$\Phi(t) = -\alpha\log\left(1-\frac{t}{N}\right)$$

where α is a sufficiently small positive constant, and $N = \sup_\Omega |Du|^2$. In place of (2.13), we obtain

$$\begin{aligned}0 \geq &-\frac{C\beta}{\eta} - \beta F_{ij}\frac{D_i\eta D_j\eta}{\eta^2}\\ &+\Phi' F_{ij}D_{ik}uD_{jl}uD_kuD_lu + \Phi'' F_{ij}D_{ik}uD_{jk}u - C(1+|\Phi'|)\\ &-\frac{1}{D_{11}u}F_{ij,kl}D_{1ij}uD_{1kl}u - F_{ij}\frac{D_{i11}uD_{j11}u}{(D_{11}u)^2}.\end{aligned}$$

We then proceed as above with obvious modifications and simplifications. It is clear that in the Hessian case the term $\sigma_1\lambda_1$ is missing in (2.23). To compensate for this, we can impose the additional structure condition

$$f_i \geq \frac{\sigma_3}{\max\{\lambda_1,\ldots,\lambda_n\}} \quad \text{on } \Gamma_{\mu_1,\mu_2}, \qquad i = 1,\ldots,n,$$

for any $\mu_2 \geq \mu_1 > 0$, where σ_3 is a positive constant. This condition is satisfied by $f = S_k^{1/k}$, $k = 1,\ldots,n$, but not by the quotients $(S_k/S_l)^{1/(k-l)}$, $0 < l < k \leq n$. The derivation of interior second derivative bounds for solutions of the Hessian quotient equations remains an interesting open problem.

3. The Dirichlet problem

In this section we prove Theorem 1.2 and indicate some straightforward extensions and modifications.

Proof of Theorem 1.2

This can be proved by solving uniformly elliptic approximating problems

$$F_\epsilon[u_\epsilon] = g_\epsilon(x,u_\epsilon) \quad \text{in } \Omega, \qquad u_\epsilon = 0 \quad \text{on } \partial\Omega$$

for $\epsilon > 0$, as in [T1], such that $\underline{u}$ is an admissible subsolution for each of the approximating problems (in [T1] Γ is assumed to be a cone, but this is not necessary). The comparison principle then implies that $\underline{u} \leq u_\epsilon \leq 0$ in Ω. Uniform bounds for

$\|Du_\epsilon\|_{L^\infty(\Omega)}$ follow with the aid of the global gradient bound proved in [T1], while the interior curvature bound, Theorem 1.1, implies uniform bounds for $\|D^2u_\epsilon\|_{L^\infty(\Omega')}$ for any $\Omega' \subset\subset \Omega$. The interior second derivative Hölder estimates of Evans and Krylov, together with Schauder theory (see [GT]), then imply uniform estimates for $\|u_\epsilon\|_{C^{3,\alpha}(\Omega')}$ for any $\Omega' \subset\subset \Omega$. Theorem 1.2 then follows by extracting a suitable subsequence as $\epsilon \to 0$. □

Remarks 3.1

(i) The regularity assumption on $\underline{u}$ in Theorem 1.2 can be weakened to $\underline{u} \in C^{0,1}(\overline{\Omega})$, provided $\underline{u}$ is a *strict* viscosity subsolution of (1.16), that is, there is a $\delta > 0$ such that

$$F[\underline{u}] \geq g(x, \underline{u}) + \delta \quad \text{in } \Omega, \qquad \underline{u} = 0 \quad \text{on } \partial\Omega, \tag{1.15$'$}$$

in the viscosity sense. The proof proceeds as above, with the usual comparison principle replaced by the comparison principle for viscosity solutions; this holds if (1.15) is strengthened to (1.15)$'$ (see [T1, Section 2]).

(ii) If $\underline{u} \in C^2(\Omega) \cap C^0(\overline{\Omega})$ (or even if $\underline{u} \in C^0(\overline{\Omega})$ satisfies (1.15)$'$ in the viscosity sense for some $\delta > 0$), we obtain an admissible solution $u \in C^{3,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ by invoking the interior gradient bounds proved in [T1] (see also [K], [L], and [W]) in place of the global gradient bounds. If $u_\epsilon \in C^4(\Omega) \cap C^0(\overline{\Omega})$ are the approximating solutions, by applying Remark 1.2 to each u_ϵ, we get

$$\sup_{\Omega'} u_\epsilon \leq -c_0(\Omega')$$

for a positive constant $c_0(\Omega')$ independent of $\epsilon > 0$. Consequently, for any $\Omega' \subset\subset \Omega$ we can choose $\tau > 0$ such that

$$\Omega' \subset\subset \Omega^\epsilon_{3\tau} := \{x \in \Omega : u_\epsilon(x) < -3\tau\}.$$

By the comparison principle (for either classical or viscosity solutions, as appropriate), we have $\underline{u} \leq u_\epsilon$ in Ω. Therefore

$$\operatorname{dist}(\Omega^\epsilon_\tau, \partial\Omega) \geq c_1(\tau)$$

for a positive constant $c_1(\tau)$ depending on τ and the modulus of continuity of $\underline{u}$, but not on ϵ. By applying the interior gradient bounds of [T1], we therefore have bounds independent of ϵ (but depending on τ) for $\|Du_\epsilon\|_{L^\infty(\Omega^\epsilon_\tau)}$. By Theorem 1.1, the Evans-Krylov estimates, and the Schauder theory, we then have bounds independent of ϵ (but depending on τ) for $\|D^2u_\epsilon\|_{L^\infty(\Omega^\epsilon_{2\tau})}$ and $\|u_\epsilon\|_{C^{3,\alpha}(\Omega^\epsilon_{3\tau})}$ for all $\alpha \in (0, 1)$. We then obtain an admissible solution $u \in C^{3,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ of (1.16) by extracting a suitable sequence as $\epsilon \to 0$ and using the estimates for u_ϵ on a sequence of subdomains increasing to Ω.

(iii) For the prescribed kth mean curvature equations and curvature quotient equations, the existence of an admissible subsolution can be replaced by appropriate assumptions on g and Ω guaranteeing uniform lower bounds, and uniform boundary gradient estimates, for the approximating solutions u_ϵ (see [T2], [T3]).

(iv) We could also impose affine Dirichlet data rather than just constant data.

(v) If f satisfies (2.12) (or more generally (2.12)$'$; see Remark 2.3(iii)), and there exists an admissible subsolution $\underline{u} \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega})$ (respectively $\underline{u} \in C^2(\Omega) \cap C^0(\overline{\Omega})$) of the equation $F[u] = g(x, u)$ in Ω, then there exists an admissible solution $u \in C^{3,\alpha}(\Omega) \cap C^{0,1}(\overline{\Omega})$ (respectively $u \in C^{3,\alpha}(\Omega) \cap C^0(\overline{\Omega})$) with $u = \underline{u}$ on $\partial\Omega$. To prove this we argue as above, using Theorem 2.1 in place of Theorem 1.1.

(vi) We may also obtain analogous existence results if f satisfies all the required structure conditions except for smoothness, or if f does not satisfy the strict ellipticity condition (1.13). In these cases the resulting solution belongs to $C^{2,\alpha}(\Omega) \cap C^{0,1}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ in the first case, and to $C^{1,1}(\Omega) \cap C^{0,1}(\overline{\Omega})$ in the degenerate case (with the obvious modification if $\underline{u} \in C^0(\overline{\Omega})$ rather than $\underline{u} \in C^{0,1}(\overline{\Omega})$). These results can be proved by solving suitable approximating problems using the above existence assertions. A particularly interesting example (which also satisfies (2.12)) is

$$f_\infty(\lambda) = \min_{1 \le i \le n} \lambda_i \quad \text{on } \Gamma_+. \tag{3.1}$$

We leave the precise formulation of these results to the reader.

(vii) It is known that if $k \ge 3$, there are no interior curvature bounds for graphs of prescribed kth mean curvature unless we make some additional assumptions (see [U2]). Purely interior curvature bounds (i.e., independent of any boundary data) for hypersurfaces of prescribed kth mean curvature have been proved under certain integrability assumptions on the second fundamental form (see [U3], [U4]).

(viii) Purely interior curvature bounds have been proved in [CNS2] for curvature equations that are uniformly elliptic once the gradient of the solution is bounded. Some weakening of the uniform ellipticity is permitted in [N] and [NS]. In particular, for admissible graphs of prescribed scalar curvature ($k = 2$ in (1.8)), an interior curvature bound is derived under the unnatural strict ellipticity assumption

$$\min_{1 \le i \le n} f_i \ge c(\mu_1, \mu_2) > 0 \quad \text{on } \Gamma_{\mu_1,\mu_2}, \tag{3.2}$$

for any $\mu_2 \ge \mu_1 > 0$. The main application of the curvature bound in [N] and [NS] is the derivation of some structure and compactness theorems for hypersurfaces of constant positive scalar curvature. It is apparent from the proofs in [NS] that an interior curvature for graphs of constant positive scalar curvature with constant or affine Dirichlet data suffices for this. Thus condition (3.2) can be eliminated from these results by using Theorem 1.1.

4. Convex solutions

In this section we provide the additional observations necessary to extend the curvature bound to solutions with $C^{1,1}$ boundary data, in the case $\Gamma = \Gamma_+$. We have already observed in Remark 2.1 that if ϕ is convex, the additional term that arises in the computation of $F_{ij}\nabla_i\nabla_j\eta$ is nonnegative and can be discarded. If the domain Ω is uniformly convex with $\partial\Omega \in C^{1,1}$, then any $\phi \in C^{1,1}(\partial\Omega)$ has a convex extension belonging to $C^{1,1}(\overline{\Omega})$. For the proof of Theorem 4.1 below it is sufficient to show that there is a convex extension $v \in C^{1,1}(\Omega) \cap C^{0,1}(\overline{\Omega})$ of ϕ such that

$$\inf_{\Omega'}(v - u) \geq c(\Omega') > 0 \tag{4.1}$$

for any $\Omega' \subset\subset \Omega$. We then choose $\eta = v - u$. The key to this is the following result of [TU2] for the homogeneous Monge-Ampère equation.

LEMMA 4.1
Let Ω be a $C^{1,1}$ bounded uniformly convex domain in $\mathbf{R}^n$. Then for any $\phi \in C^{1,1}(\partial\Omega)$ the problem

$$\det D^2 v = 0 \quad \text{in } \Omega, \qquad v = \phi \quad \text{on } \partial\Omega, \tag{4.2}$$

has a unique convex solution $v \in C^{1,1}(\Omega) \cap C^{0,1}(\overline{\Omega})$.

Now let f be as in Section 1, with $\Gamma = \Gamma_+$, the positive cone in $\mathbf{R}^n$. Then, since $v \in C^{1,1}(\Omega)$, the principal curvatures $\lambda_1, \ldots, \lambda_n$ of graph v are defined for almost all points of Ω, and $\lambda = (\lambda_1, \ldots, \lambda_n)$ belongs to $\partial\Gamma_+$. It follows that v satisfies

$$F[v] = 0 \quad \text{in } \Omega, \qquad v = \phi \quad \text{on } \partial\Omega. \tag{4.3}$$

We now verify (4.1). Let $x_0 \in \Omega$; for convenience let us assume that $x_0 = 0$. For $\rho > 0$ fixed so small that $B_\rho = B_\rho(0) \subset\subset \Omega$ and $\epsilon > 0$ to be chosen, we consider

$$v_\epsilon = v + \epsilon(|x|^2 - \rho^2) \quad \text{in } B_\rho.$$

The principal curvatures $\lambda_1^\epsilon, \ldots, \lambda_n^\epsilon$ of graph v_ϵ are the eigenvalues of

$$h_{ij}^\epsilon = \frac{D_{ij}v_\epsilon}{\sqrt{1 + |Dv_\epsilon|^2}} = \frac{D_{ij}v}{\sqrt{1 + |Dv|^2}} + O(\epsilon)$$

relative to the metric

$$g_{ij}^\epsilon = \delta_{ij} + D_i v_\epsilon D_j v_\epsilon = \delta_{ij} + D_i v D_j v + O(\epsilon).$$

Since Dv and D^2v are bounded on B_ρ by a controlled constant, and the eigenvalues of a matrix are locally Lipschitz functions of the matrix entries (see [A, Lemma 1]), we see that for $\epsilon > 0$ sufficiently small,

$$|\lambda - \lambda^\epsilon| \leq C\epsilon,$$

where C depends on bounds for $|Dv|$ and $|D^2v|$ on B_ρ. Therefore

$$F[v_\epsilon] \le F[v] + \omega(\epsilon) = \omega(\epsilon)$$

where $\omega(\epsilon) \to 0$ as $\epsilon \to 0$. Consequently, for $\epsilon > 0$ fixed sufficiently small we have

$$F[v_\epsilon] < F[u] \quad \text{in } B_\rho.$$

Since v is characterized by

$$v = \sup\{w : w \text{ is convex on } \Omega,\, w \le \phi \text{ on } \partial\Omega\},$$

we have $u \le v = v_\epsilon$ on ∂B_ρ. Therefore $v_\epsilon \ge u$ in B_ρ by the comparison principle, and hence $(v-u)(0) \ge \epsilon\rho^2$. This proves (4.1). Therefore we have proved the following result.

THEOREM 4.1
Suppose that f satisfies (1.1) to (1.4) with $\Gamma = \Gamma_+$. Let Ω be a $C^{1,1}$ bounded uniformly convex domain in $\mathbf{R}^n$, and let $u \in C^4(\Omega) \cap C^{0,1}(\overline{\Omega})$ be an admissible solution of

$$F[u] = g(x,u) \quad \text{in } \Omega, \qquad u = \phi \quad \text{on } \partial\Omega, \tag{4.4}$$

where $g \in C^{1,1}(\overline{\Omega}\times\mathbf{R})$ is a positive function and $\phi \in C^{1,1}(\partial\Omega)$. Then there exist positive constants β, depending only on $\sup_\Omega |Du|$, and C, depending only on n, $\|u\|_{C^1(\overline{\Omega})}$, g and its first and second derivatives, $\mu_1 = \inf_\Omega g(x,u)$, $\mu_2 = \sup_\Omega g(x,u)$, and the structure constants σ_0, σ_1 in (1.3) and (1.4), such that the second fundamental form $\mathbf{A}$ of graph u satisfies

$$|\mathbf{A}| \le \frac{C}{(v-u)^\beta}, \tag{4.5}$$

where $v \in C^{1,1}(\Omega) \cap C^{0,1}(\overline{\Omega})$ is the unique convex solution of

$$F[v] = 0 \quad \text{in } \Omega, \qquad v = \phi \quad \text{on } \partial\Omega. \tag{4.6}$$

The constant C in Theorem 4.1 is independent of bounds for v and Dv because these quantities are controlled by $\|u\|_{C^1(\overline{\Omega})}$.

By an argument similar to that used in Section 3, we can conclude the following theorem, which can be viewed as a generalization of a result of [TU1] for the equation of prescribed Gauss curvature.

THEOREM 4.2
Let f and Ω satisfy the hypotheses of Theorem 4.1. Suppose that $\phi \in C^{1,1}(\overline{\Omega})$, and let

$g \in C^{1,1}(\overline{\Omega} \times \mathbf{R})$ be a positive function satisfying $g_z \geq 0$. Suppose there is a convex function $\underline{u} \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega})$ satisfying

$$F[\underline{u}] \geq g(x, \underline{u}) \quad \textit{in } \Omega, \qquad \underline{u} = \phi \quad \textit{on } \partial). \tag{4.7}$$

Then the problem

$$F[u] = g(x, u) \quad \textit{in } \Omega, \qquad u = \phi \quad \textit{on } \partial\Omega, \tag{4.8}$$

has a unique convex solution $u \in C^{3,\alpha}(\Omega) \cap C^{0,1}(\overline{\Omega})$ for all $\alpha \in (0, 1)$.

Proof

We solve uniformly elliptic approximating problems having smooth solutions u_ϵ, exactly as in the proof of Theorem 1.2. Note, however, that the elliptic regularization used in [T1] enlarges the cone Γ_+, so the solutions u_ϵ need not be convex, even though the limit solution u is convex. Consequently, Theorem 4.1 is not applicable to the approximations u_ϵ. Instead, we first pass to a limit via a suitable sequence $\epsilon_i \to 0$ to obtain a convex viscosity solution $u \in C^{0,1}(\overline{\Omega})$ of (4.8). By the above considerations involving v (which are valid even if u is a viscosity rather than classical solution), we then see that for any $x_0 \in \Omega$ there is a neighbourhood $U \subset\subset \Omega$ of x_0 such that u is equal to an affine function l on ∂U. But then, for large enough i, the set $U_{\epsilon_i} = \{x \in \Omega : u_{\epsilon_i}(x) < l(x)\} \subset\subset \Omega$, and $x_0 \in U_{\epsilon_i}$. Furthermore, $U_{\epsilon_i} \to U$. By applying the curvature bound of Theorem 1.1 to each u_{ϵ_i} on U_{ϵ_i} and then using the Evans-Krylov estimates and Schauder theory and passing to the limit, we conclude that $u \in C^{3,\alpha}$ in a neighbourhood of x_0. Since $x_0 \in \Omega$ is arbitrary, $u \in C^{3,\alpha}(\Omega)$. □

Remark 4.1

The main examples covered by Theorem 4.2 are the quotients $f = (S_n/S_l)^{1/(n-l)}$ with $l = 0, \ldots, n-1$. In the Gauss curvature case $l = 0$ the regularity assumptions on $\partial\Omega$ and ϕ can be weakened to $\partial\Omega$, $\phi \in C^{1,\alpha}$ for some $\alpha > 1 - 2/n$ (see [C], [U1]).

5. The Plateau problem

In this section we prove Theorem 1.3. The notion of locally convex hypersurface we use is the same as that used in [TW]. To state precisely some of the results we use we need to recall this.

Definition 5.1

A compact, connected, locally convex hypersurface $\mathscr{M}$ in $\mathbf{R}^{n+1}$ (possibly with boundary) is an immersion of an n-dimensional, compact, oriented, and connected manifold $\mathscr{N}$ (possibly with boundary) in $\mathbf{R}^{n+1}$, that is, a mapping $T : \mathscr{N} \to \mathscr{M} \subset \mathbf{R}^{n+1}$, such that for any $p \in \mathscr{N}$ there is a neighbourhood $\omega_p \subset \mathscr{N}$ such that

(i) T is a homeomorphism from ω_p to $T(\omega_p)$;
(ii) $T(\omega_p)$ is a convex graph;
(iii) the convexity of $T(\omega_p)$ agrees with the orientation.

Since $\mathscr{M}$ is immersed, a point $x \in \mathscr{M}$ may be the image of several points in $\mathscr{N}$ (since $\mathscr{N}$ is compact, $\mathscr{M}$ is also compact, and $T^{-1}(x)$ consists of only finitely many points). Let $r > 0$ and $x \in \mathscr{M}$. Then for small enough r, $T^{-1}(\mathscr{M} \cap B_r^{n+1}(x))$ consists of several disjoint open subsets $U_1, \ldots, U_s$ of $\mathscr{N}$ such that $T|_{U_i}$ is a homeomorphism of U_i onto $T(U_i)$ for each $i = 1, \ldots, s$. By an *r-neighbourhood* $\omega_r(x)$ of x in $\mathscr{M}$ we mean any one of the sets $T(U_i)$. We say that $\omega_r(x)$ is *convex* if $\omega_r(x)$ lies on the boundary of its convex hull.

The key ingredient in proving Theorem 1.3 is the following lemma (see [TW, Theorem A]).

LEMMA 5.1
Let $\mathscr{M}_0 \subset B_R(0)$ be a locally convex hypersurface with C^2 boundary $\partial\mathscr{M}_0$. Suppose that on $\partial\mathscr{M}_0$, the principal curvatures $\lambda_1^0, \ldots, \lambda_n^0$ of $\mathscr{M}_0$ satisfy

$$C_0^{-1} \le \lambda_i^0 \le C_0 \qquad i = 1, \ldots, n \tag{5.1}$$

for some $C_0 > 0$. Then there exist positive constants r and α, depending only on n, C_0, R and $\partial\mathscr{M}_0$, such that for any point $p \in \mathscr{M}_0$, each r-neighbourhood $\omega_r(p)$ of p is convex, and there is a closed cone $C_{p,\alpha}$ with vertex p and angle α such that $\omega_r(p) \cap C_{p,\alpha} = \{p\}$.

We make two observations related to Lemma 5.1. The first is that for any point $p \in \mathscr{M}_0$, if one chooses the axial direction of the cone $C_{p,\alpha}$ as the x_{n+1}-axis, then each δ-neighbourhood of p can be represented as a graph,

$$x_{n+1} = u(x), \qquad |x| \le \delta$$

for any $\delta < r\sin(\alpha/2)$. Moreover, the cone condition implies

$$|Du(x)| \le C, \qquad |x| < \delta,$$

where $C > 0$ depends only on α.

The second observation is that Lemma 5.1 holds not just for $\mathscr{M}_0$, but also for a family of locally convex hypersurfaces, with uniform r and α. Indeed, by extending $\mathscr{M}_0$ to a larger locally convex hypersurface $\mathscr{M}_1$ such that $\partial\mathscr{M}_0$ lies in the interior of $\mathscr{M}_1$ (see [TW]), and applying Lemma 5.1 to $\mathscr{M}_1$, we see that Lemma 5.1 holds for any locally convex hypersurface $\mathscr{M}$ such that $(\mathscr{M}_1 - \mathscr{M}_0) \cup \mathscr{M}$ is locally convex, with uniform r and α.

With the aid of Lemma 5.1 we can use the Perron method to obtain a viscosity solution of the Plateau problem for the curvature function f, as was done for the Gauss curvature case in [TW]. The only change is to replace the notion of a generalized solution for the prescribed Gauss curvature equation by that of viscosity solution for more general curvature equations, using the following lemma.

LEMMA 5.2
Let Ω be a bounded domain in $\mathbf{R}^n$ with Lipschitz boundary. Let $\phi \in C^{0,1}(\overline{\Omega})$ be a convex viscosity subsolution of

$$f(\lambda) = k \quad \text{in } \Omega \tag{5.2}$$

Then there is a viscosity solution u of (5.2) such that $u = \phi$ on $\partial\Omega$.

Proof
The proof uses the well-known Perron method. Let Ψ denote the set of convex subsolutions v of (5.2) with $v = \phi$ on $\partial\Omega$. Then Ψ is nonempty and the required solution u is given by

$$u(x) = \sup\{v(x) : v \in \Psi\}.$$

The proof of this uses standard arguments. The only point that needs to be mentioned is the solvability of the Dirichlet problem

$$f(\lambda) = k \quad \text{in } B_r, \qquad u = u_0 \quad \text{on } \partial B_r, \tag{5.3}$$

in small enough balls $B_r \subset \mathbf{R}^n$, if u_0 is any Lipschitz viscosity subsolution of (5.3). This is a consequence of [T1, Theorem 6.2]. □

Using Lemma 5.2 and the argument of [TW], we conclude that there is a locally convex hypersurface $\mathscr{M}$ with boundary Σ, which satisfies the equation $f(\lambda) = k$ in the viscosity sense; that is, for any point $p \in \mathscr{M}$, if $\mathscr{M}$ is locally represented as the graph of a convex function u (by Lemma 5.1), then u is a viscosity solution of $f(\lambda) = k$.

The remaining question is the regularity of $\mathscr{M}$.

Interior regularity
We use Theorem 1.2 to prove the interior regularity of $\mathscr{M}$. For any point $p \in \mathscr{M}$, if $\mathscr{M}$ is strictly convex at p (i.e., if there is a tangent plane $\mathscr{L}$ of $\mathscr{M}$ at p such that $\mathscr{L} \cap \omega_r(p) = \{p\}$ for some $r > 0$, where $\omega_r(p)$ is any r-neighbourhood of p), then $\mathscr{M}$ is smooth and uniformly convex near p. This is because we can choose the coordinate system so that p is the origin, and near p, $\mathscr{M}$ is represented as the graph of a nonnegative convex function u. Then by the strict convexity of $\mathscr{M}$ at p, the set

$\Omega_\epsilon := \{u < \epsilon\}$ is a convex set that shrinks to the point $\{p\}$ as $\epsilon \to 0$. Therefore, by Theorem 1.2 and the uniqueness of viscosity solutions of the Dirichlet problem, we conclude that u is smooth and uniformly convex in Ω_ϵ when $\epsilon > 0$ is sufficiently small.

Next we prove that $\mathscr{M}$ is strictly convex. Suppose to the contrary that $\mathscr{M}$ contains a line segment ℓ. Choose an arbitrary point $p \in \ell$, and let $\mathscr{L}$ be a tangent plane of $\mathscr{M}$ at p such that $\ell \subset \mathscr{L}$. Let $\mathscr{C}$ be a component of the contact set $\mathscr{L} \cap \mathscr{M}$, by which we mean that $\mathscr{C}$ is the image under T of one of the components of $T^{-1}(\mathscr{L} \cap \mathscr{M}) \subset \mathscr{N}$ (see Definition 5.1). Then $\mathscr{C}$ is a closed convex set.

Let $p_0 \in \mathscr{C}$ be the farthest point from p (if there is more than one such point, choose any one). If p_0 is an interior point of $\mathscr{M}$, we choose p_0 as the origin and suppose $p = (t, 0, \cdots, 0)$ for some $t > 0$. Then $\mathscr{C} \subset \{x_1 \geq 0\}$ and $\mathscr{C} \cap \{x_1 = 0\} = \{p_0\}$.

By Lemma 5.1, $\omega_r(p_0)$ is convex for small enough $r > 0$. Hence the point $p_1 = (\frac{r}{2}\cos\theta, 0, \ldots, 0, \frac{r}{2}\sin\theta)$ is an interior point of the convex hull of $\omega_r(p_0)$ if $\theta > 0$ is sufficiently small. We introduce a new coordinate system $(y_1, \ldots, y_{n+1})$ such that $\overline{p_0 p_1}$ lies in the y_{n+1}-axis, and

$$\begin{aligned} y_1 &= x_1 \sin\theta - x_{n+1}\cos\theta, \\ y_i &= x_i, \quad i = 2, \cdots, n. \end{aligned}$$

As explained after Lemma 5.1, $\mathscr{M}$ can be locally represented as a graph

$$y_{n+1} = v(y), \qquad y = (y_1, \cdots, y_n), \qquad |y| < 2\delta,$$

for small enough $\delta > 0$. In the new coordinates

$$\mathscr{C} \subset \{y_{n+1} = y_1 \cot\theta\} \cap \{y_1, y_{n+1} > 0\}.$$

Let $\Omega_\epsilon = \{v(y) < -\epsilon(y_1 - \delta) + y_1 \cot\theta\}$. Then $\Omega_\epsilon \subset \Omega_{\epsilon'}$ for any $0 < \epsilon < \epsilon'$. Since $\mathscr{C} \cap \{y_1 = 0\} = \{p_0\}$, we see that $\Omega_\epsilon \searrow \overline{p_0 p_2}$, where $p_2 = (\delta, 0, \ldots, 0, \delta\cot\theta)$ in the y-coordinates. Therefore, when $\epsilon > 0$ is sufficiently small, v is equal to an affine function on $\partial\Omega_\epsilon$. Applying Theorem 1.2 and the uniqueness of viscosity solutions of the Dirichlet problem to (Ω_ϵ, v), we conclude that v is smooth and uniformly convex in Ω_ϵ. That is, $\mathscr{M}$ is uniformly convex near p_0, which is a contradiction.

If p_0 is a boundary point, let $\mathscr{M}_1$ be an extension of $\mathscr{M}_0$ as mentioned after Lemma 5.1, such that $\mathscr{M}_1 - \mathscr{M}_0$ is locally uniformly convex. Then $\widetilde{\mathscr{M}} = \mathscr{M} \cup \{\mathscr{M}_1 - \mathscr{M}_0\}$ is a locally convex extension of $\mathscr{M}$. Take p_0 as the origin, and choose a point $p_1 \in \mathscr{M}_1 - \mathscr{M}_0$ sufficiently close to p_0 such that $\overline{p_0 p_1}$ is perpendicular to Σ at p_0. Take $\overline{p_0 p_1}$ as the x_{n+1}-axis direction. Then Σ is tangent to the plane $\{x_{n+1} = 0\}$ at p_0. By Lemma 5.1 above, for small enough $r, \delta > 0$, the r-neighbourhood of p_0 in

$\widetilde{\mathscr{M}}$, $\omega_r(p_0)$, is convex, and the line segment $\{te_{n+1}, 0 \leq t \leq \delta\}$ lies in the interior of the convex hull of $\omega_r(p_0)$. Hence, by the observations following Lemma 5.1, near the origin $\widetilde{\mathscr{M}}$ can be represented as a graph,

$$x_{n+1} = u(x), \qquad x \in B_\delta(0),$$

where $B_\delta(0)$ is the ball in $\mathbf{R}^n$. Similarly, $\widetilde{\mathscr{M}_0} = \mathscr{M}_0 \cup \{\mathscr{M}_1 - \mathscr{M}_0\}$ can be represented near p_0 as a graph

$$x_{n+1} = u_0(x), \qquad x \in B_\delta(0),$$

in the *same coordinate system*. Since $\mathscr{M}$ is obtained from $\mathscr{M}_0$ by a sequence of Perron liftings, we have $u \geq u_0$ near the origin.

Let ℓ be the line segment in $\mathscr{C}$ connecting p_0 to p (indeed, p can be any fixed point in $\mathscr{C}$). Then ℓ cannot be tangent to Σ at p_0. For, if ℓ is tangent to Σ, then $\partial_\xi \partial_\xi u_0 = 0$ since $u \geq u_0$, where ξ is a unit vector in the direction of ℓ. This is a contradiction.

It follows that ℓ is transversal to Σ at p_0. Let $\{x_{n+1} = \xi \cdot x\}$ be a tangent plane of $\mathscr{M}$ at p_0 containing the line segment ℓ. By a rotation of coordinates, we may suppose ℓ', the projection of ℓ onto $\{x_{n+1} = 0\}$, lies on the x_n-axis. By the smoothness of Σ and the convexity of u, we then have

$$0 \leq u(x) - \xi \cdot x \leq C \sum_{i=1}^{n-1} x_i^2 \tag{5.4}$$

for x near ℓ'. But since u is a viscosity solution of $f(\lambda) = k$ for some $k > 0$, one can easily construct a supersolution to show that (5.4) is impossible. Indeed, u is a viscosity solution of $\det D^2 u \geq \tilde{k}$ near $\tilde{\ell}$ for some positive constant $\tilde{k}$ depending only on k, l, n, and $\sup_{\omega_r(p)} |Du|$, so one can appeal directly to [CY, Theorem 4]. Therefore $\mathscr{M}$ must be locally strictly convex, and therefore it is a smooth, locally uniformly convex hypersurface.

Boundary regularity

The boundary regularity of $\mathscr{M}$ is a local property. The boundary estimates we need are contained in the work of Ivochkina and Tomi [IT] (see also [ILT]). However, they cannot be applied directly to $\mathscr{M}$ because their proof requires somewhat more regularity of $\mathscr{M}$ up to $\partial\mathscr{M}$ than we have established so far. We need to apply the estimates to suitable approximating solutions.

As observed above, since we are working in a neighbourhood of a boundary point $p \in \partial\mathscr{M}$, which we may take to be the origin, we may assume that for a smooth bounded domain $\Omega \subset \mathbf{R}^n$ with $0 \in \partial\Omega$ and small enough $\rho > 0$ we have

$$\mathscr{M} \cap (B_\rho \times \mathbf{R}) = \operatorname{graph} u, \qquad \mathscr{M}_0 \cap (B_\rho \times \mathbf{R}) = \operatorname{graph} u_0,$$

where $u \in C^\infty(\Omega_\rho) \cap C^{0,1}(\overline{\Omega}_\rho)$, and $u_0 \in C^\infty(\overline{\Omega}_\rho)$ are convex solutions of

$$F[u] = k \quad \text{in } \Omega_\rho, \qquad F[u_0] \geq k \quad \text{in } \Omega_\rho,$$

with

$$u \geq u_0 \quad \text{in } \Omega_\rho, \qquad u = u_0 \quad \text{on } \partial\Omega \cap B_\rho.$$

Here $\Omega_\rho = \Omega \cap B_\rho$ where B_ρ denotes the ball in $\mathbf{R}^n$ centered at the origin.

Next we observe that by the argument of [IT, Section 1], by making ρ smaller if necessary, we may choose the coordinate system in $\mathbf{R}^{n+1}$ in such a way that Ω is uniformly convex, and moreover, so that for some $\epsilon > 0$ we have

$$k \leq (1-\epsilon)\Big(\frac{S_{n-1}(\kappa')}{S_{l-1}(\kappa')}\Big)^{1/(n-l)} \tag{5.5}$$

on $\partial\Omega \cap B_\rho$, where $\kappa' = (\kappa'_1, \ldots, \kappa'_{n-1})$ denotes the vector of principal curvatures of $\partial\Omega$. The choice of coordinate system (and ρ, of course) depends on $\mathscr{M}_0$ and $\partial\mathscr{M}_0$, but not on $\mathscr{M}$.

By making ρ smaller if necessary, we may assume that (5.5) also holds at all points of ∂B_ρ. It is clear then that by suitably smoothing the corners of Ω_ρ in $B_\rho - B_{7\rho/8}$ we can find a uniformly convex domain $\Omega' \subset \Omega_\rho$ such that $\Omega \cap B_{3\rho/4} = \Omega' \cap B_{3\rho/4}$ and such that (5.5) holds at each point of $\partial\Omega'$.

For each positive integer i, we divide $\partial\Omega'$ into three pieces as follows:

$$\begin{aligned} \Gamma_0 &= \partial\Omega' \cap B_{3\rho/4}, \\ \Gamma_i &= \partial\Omega' \cap B_\rho \cap \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > i^{-1}\}, \\ \widetilde{\Gamma}_i &= \partial\Omega' - (\Gamma_0 \cup \Gamma_i). \end{aligned}$$

Let $\{\phi_i\} \subset C^\infty(\overline{\Omega}')$ be a sequence of functions such that $\phi_i \to u$ in $C^{0,\alpha}(\overline{\Omega}')$ for $\alpha \in (0,1)$, $\|\phi_i\|_{C^1}$ is uniformly bounded, and

$$\begin{aligned} \phi_i &= u_0 = u \quad \text{on } \Gamma_0, \\ \phi_i &= u \quad \text{on } \Gamma_i, \\ \phi_i &\geq u \quad \text{on } \widetilde{\Gamma}_i. \end{aligned}$$

Let $\{\eta_i\} \subset C^\infty(\overline{\Omega}')$ be a nondecreasing sequence of positive functions such that $\eta_i \to 1$ in Ω',

$$\begin{aligned} \eta_i &= 1 \quad \text{in} \quad \overline{\Omega}' - \{x \in \Omega \cap (B_\rho - B_{\rho/4}) : \operatorname{dist}(x, \partial\Omega) \leq 4i^{-1}\}, \\ \eta_i &= \delta_i \quad \text{in} \quad \overline{\Omega}' \cap (B_\rho - B_{\rho/2}) \cap \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < 2i^{-1}\}, \\ \delta_i &\leq \eta_i \leq 1 \quad \text{in } \Omega', \end{aligned}$$

where $\{\delta_i\}$ is a sequence of positive constants to be chosen.

We claim that for a suitable choice of $\{\delta_i\}$ the problems

$$F[u_i] = \eta_i k \quad \text{in } \Omega', \qquad u_i = \phi_i \quad \text{on } \partial\Omega', \tag{5.6}$$

are solvable, with convex solutions $u_i \in C^\infty(\overline{\Omega}')$. The proof of this uses the well known method of continuity, which relies on establishing a priori estimates in $C^{2,\alpha}(\overline{\Omega}')$ for solutions of (5.6), or more precisely, for solutions of a suitable family of problems containing (5.6), for example,

$$\begin{aligned} F[u_i] &= t\eta_i k + (1-t)\sigma_i \eta_i F[\phi_i] \quad \text{in } \Omega', \\ u &= \phi_i \quad \text{on } \partial\Omega', \end{aligned} \tag{5.7}$$

where $t \in [0, 1]$, ϕ_i is assumed to be uniformly convex, and $\sigma_i \in (0, 1]$ is a positive constant so small that $\sigma_i F[\phi_i] \le k$ in Ω'. We describe the estimates only for (5.6), that is, for (5.7) with $t = 1$, because they are similar for other $t \in [0, 1]$. We observe also that the solvability of (5.7) when $t = 0$ follows from [ILT] or [IT], since ϕ_i is a convex subsolution of the problem in this case.

First, since u_i is convex, and u_0 is a subsolution of the equation with $u_0 \le u_i$ on $\partial\Omega'$, we have

$$u_0 \le u_i \le h_i \le \sup_{\Omega'} \phi_i \quad \text{in } \Omega',$$

where h_i is the harmonic function in Ω' with $h_i = \phi_i$ on $\partial\Omega'$. It follows then that

$$D_\gamma u_i \le D_\gamma h_i \le C(i) \quad \text{on } \partial\Omega'$$

where γ denotes the inner normal vector field to $\partial\Omega'$.

To obtain a lower bound for $D_\gamma u_i$ on $\partial\Omega'$, we need to consider several cases. First, we have

$$-C \le D_\gamma u_0 \le D_\gamma u_i \quad \text{on } \Gamma_0$$

because u_0 is a lower barrier for u_i there. Second, u is a lower barrier for u_i on Γ_i, so

$$-C(i) \le D_\gamma u \le D_\gamma u_i \quad \text{on } \Gamma_i.$$

Next we need to construct a local lower barrier for u_i at each point of $\widetilde{\Gamma}_i$. Let $y \in \widetilde{\Gamma}_i$, and let $\partial B_R(z)$ be an enclosing sphere for Ω' at y; that is, $\Omega' \subset B_R(z)$ and $\partial B_R(z) \cap \partial\Omega' = \{y\}$. Then for small enough $\epsilon(i) > 0$ we have

$$\begin{aligned} &\Omega' \cap \{x : R - \epsilon(i) \le |x - z| \le R\} \\ &\subset \Omega' \cap (B_\rho - B_{\rho/2}) \cap \{x : \operatorname{dist}(x, \partial\Omega) < 2i^{-1}\}. \end{aligned}$$

Furthermore, for a sufficiently large positive constant $A(i)$,

$$\tilde{\phi}_i = \phi_i + A(i)(|x - z|^2 - R^2)$$

is uniformly convex, and $\tilde{\phi}_i \leq u_0$ on $\Omega' \cap \{x : |x - z| = R - \epsilon(i)\}$. If we now fix

$$\delta_i = \min_{\Omega'} F[\tilde{\phi}_i] > 0,$$

then $\tilde{\phi}_i$ is a local lower barrier for u_i at y, so

$$-C(i) \leq D_\gamma \tilde{\phi}_i(y) \leq D_\gamma u_i(y).$$

Since y can be any point of $\widetilde{\Gamma}_i$, and the constants $\epsilon(i)$ and $A(i)$ are uniform for $y \in \widetilde{\Gamma}_i$ for each fixed i, we conclude that

$$-C(i) \leq D_\gamma u_i \quad \text{on} \quad \widetilde{\Gamma}_i.$$

Consequently,

$$\sup_{\partial\Omega'} |Du_i| \leq C(i),$$

and therefore, by the convexity of u_i,

$$\sup_{\Omega'} |Du_i| \leq C(i).$$

Second derivative bounds for each u_i then follow from the results in [IT] (for this the curvature condition (5.5) is required at each point of $\partial\Omega'$). Finally, the second derivative Hölder estimate follows from the Evans-Krylov estimates for concave uniformly elliptic equations. Since the bounds for Du_i are not uniform with respect to i, the same is true of the bounds for $D^2 u_i$ and their Hölder seminorms.

To complete the proof of the boundary regularity, we show that uniform $C^{2,\alpha}$ estimates hold for $\{u_i\}$ on $\overline{\Omega}_{\theta\rho}$ for a sufficiently small controlled positive constant θ. It then follows that u satisfies a similar bound on $\overline{\Omega}_{\theta\rho}$. Higher regularity of u on $\overline{\Omega}_{\theta\rho/2}$ then follows by standard linear elliptic estimates if $\partial\Omega$ is smooth enough.

By [IT, Lemma 1.1], we have

$$\sup_{\Omega_{\rho/4}} |Du_i| \leq \sup_{\partial\Omega \cap B_{\rho/2}} |Du_i| + 8\rho^{-1} \sup_{\Omega'} |u_i|.$$

The first term on the right is uniformly bounded because u_0 (respectively h_i) is a lower (respectively upper) barrier for u_i on $\partial\Omega \cap B_{\rho/2}$, since $\phi_i = u_0 = h_i$ on $\partial\Omega \cap B_{\rho/2}$.

Next, by [IT, Theorem 2.1], the second derivatives of u_i are uniformly bounded on $\partial\Omega \cap B_{\rho/8}$. We claim that the second derivatives are bounded in $\Omega_{\tau\rho}$ for small enough controlled $\tau > 0$. Let $x_{n+1} = l(x)$ be the graph of the tangent hyperplane L to graph u_0 at 0. By the uniform convexity of u_0, L intersects graph u_0 and graph u_i only at 0. Let us assume that the positive x_n axis points in the direction of the inner normal to $\partial\Omega$ at 0. Then for small enough $\epsilon', \delta' > 0$, the set $U_i = \{x \in \Omega : u_i(x) \leq l(x) + \epsilon' - \delta' x_n\}$ satisfies

$$U_i \subset\subset \overline{\Omega}_{\rho/16} \qquad \text{and} \qquad U_i \supset \overline{\Omega}_{\sigma\rho}$$

for some small positive constant σ, independent of i. By the proof of Theorem 1.1 with $\eta = l(x) + \epsilon' - \delta' x_n$, we conclude that $D^2 u_i$ is uniformly bounded in $\overline{\Omega}_{2\theta\rho}$ for a sufficiently small positive constant θ. Uniform estimates for u_i in $C^{2,\alpha}(\overline{\Omega}_{\theta\rho})$, $\alpha \in (0,1)$, then follow from the Evans-Krylov estimates and the Schauder estimates. This completes the proof of the boundary regularity.

References

[A] A. D. ALEKSANDROV, *Uniqueness theorems for surfaces in the large II*, Vestnik Leningrad. Univ. **12** (1957), 5 – 44; English translation in Amer. Math. Soc. Transl. Ser. (2) **21** (1962), 345 – 388.

[An] B. ANDREWS, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. Partial Differential Equations **2** (1994), 151 – 171. MR 1385524

[C] L. A. CAFFARELLI, *Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation*, Ann. of Math. (2) **131** (1990), 135 – 150. MR 1038360

[CNS1] L. CAFFARELLI, L. NIRENBERG, and J. SPRUCK, *Nonlinear second-order elliptic equations, V, The Dirichlet problem for Weingarten hypersurfaces*, Comm. Pure Appl. Math. **41** (1988), 47 – 70. MR 0917124

[CNS2] ———, "On a form of Bernstein's theorem" in *Analyse Mathématique et Applications*, Gauthier-Villars, Paris, 1988, 55 – 66. MR 0956953

[CY] S.-Y. CHENG and S.-T. YAU, *On the regularity of the Monge-Ampère equation* $\det \partial^2 u/\partial x_i \partial x_j = F(x,u)$, Comm. Pure Appl. Math. **30** (1977), 41 – 68. MR 0437805

[CW] K.-S. CHOU and X.-J. WANG, *A variational theory of the Hessian equation*, Comm. Pure Appl. Math. **54** (2001), 1029 – 1064. MR 1835381

[G] C. GERHARDT, *Closed Weingarten hypersurfaces in Riemannian manifolds*, J. Differential Geom. **43** (1996), 612 – 641. MR 1412678

[GT] D. GILBARG and N. S. TRUDINGER, *Elliptic partial differential equations of second order*, Springer, Berlin, 2nd. edition, 1983. MR 0737190

[GS1] B. GUAN and J. SPRUCK, *Boundary-value problems on S^n for surfaces of constant Gauss curvature*, Ann. of Math. (2) **138** (1993), 601 – 624. MR 1247995

[GS2] ———, *The existence of hypersurfaces of constant Gauss curvature with prescribed boundary*, J. Differential Geom. **62** (2002), 259 – 287. MR 1988505

[HRS] D. HOFFMAN, H. ROSENBERG, and J. SPRUCK, *Boundary value problems for surfaces of constant Gauss curvature*, Comm. Pure Appl. Math. **45** (1992), 1051 – 1062. MR 1168119

[I1] N. M. IVOCHKINA, *Construction of a priori bounds for convex solutions of the Monge-Ampère equation by integral methods*, Ukranian Math. J. **30** (1978), 32 – 38.

[I2] ———, *Solution of the Dirichlet problem for equations of mth order curvature* (in Russian), Mat. Sb. **180** (1989), 867 – 887; English translation in Sb. Math. **67** (1990), 317 – 339. MR 1014618

[I3] ———, *The Dirichlet problem for the curvature equation of order m* (in Russian), Algebra i Analiz **2** (1990), 192 – 217; English translation in Leningrad Math. J. **2** (1991), 631 – 654. MR 1073214

[ILT] N. M. IVOCHKINA, M. LIN, and N. S. TRUDINGER, "The Dirichlet problem for the prescribed curvature quotient equations with general boundary values" in *Geometric analysis and the calculus of variations*, Internat. Press, Cambridge, MA, 1996, 125 – 141. MR 1449405

[IT] N. M. IVOCHKINA and F. TOMI, *Locally convex hypersurfaces of prescribed curvature and boundary*, Calc. Var. Partial Differential Equations **7** (1998), 293 – 314. MR 1660847

[K] N. J. KOREVAAR, *A priori interior gradient bounds for solutions to elliptic Weingarten equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **4** (1987), 405 – 421. MR 0921546

[L] Y.-Y. LI, *Interior gradient estimates for solutions of certain fully nonlinear elliptic equations*, J. Differential Equations **90** (1991), 172 – 185. MR 1094454

[LT] M. LIN and N. S. TRUDINGER, *The Dirichlet problem for the prescribed curvature quotient equations*, Topol. Methods Nonlinear Anal. **3** (1994), 307 – 323. MR 1281990

[N] B. NELLI, *On the structure of positive scalar curvature type graphs*, Ital. J. Pure Appl. Math. **10** (2001), 169 – 179. MR 1930324

[NS] B. NELLI and B. SEMMLER, O*n hypersurfaces embedded in Euclidean space with positive constant H_r curvature*, Indiana Univ. Math. J. **50** (2001), 989 – 1002. MR 1871396

[P] A. V. POGORELOV, *The Minkowski multidimensional problem*, Wiley, New York, 1978. MR 0478079

[R] H. ROSENBERG, *Hypersurfaces of constant curvature in space forms*, Bull. Sci. Math. **117** (1993), 211 – 239. MR 1216008

[S] J. SPRUCK, "Fully nonlinear elliptic equations and applications in geometry" in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, Birkhäuser, Basel, 1995, 1145 – 1152. MR 1404014

[T1] N. S. TRUDINGER, *The Dirichlet problem for the prescribed curvature equations*, Arch. Rational Mech. Anal. **111** (1990), 153 – 179. MR 1057653

[T2] ———, *A priori bounds and necessary conditions for solvability of prescribed curvature equations*, Manuscripta Math. **67** (1990), 99 – 112. MR 1037998

[T3] ———, *Maximum principles for curvature quotient equations*, J. Math. Sci. Univ. Tokyo **1** (1994), 551 – 565. MR 1322691

[T4] ———, private communication.

[TU1] N. S. TRUDINGER and J. I. E. URBAS, *The Dirichlet problem for the equation of prescribed Gauss curvature*, Bull. Austral. Math. Soc. **28** (1983), 217 – 231. MR 0729009

[TU2] ———, *On second derivative estimates for equations of Monge-Ampère type*, Bull. Austral. Math. Soc. **30** (1984), 321 – 334. MR 0766792

[TW] N. S. TRUDINGER and X.-J. WANG, *On locally convex hypersurfaces with boundary*, J. Reine Angew. Math. **551** (2002), 11 – 32. MR 1932171

[U1] J. I. E. URBAS, *Regularity of generalized solutions of Monge-Ampère equations*, Math.

Do you have a year for this ([T4]) reference?

Z. **197** (1988), 365 – 393. MR 0926846
[U2] ———, *On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations*, Indiana Univ. Math. J. **39** (1990), 355 – 382. MR 1089043
[U3] ———, *An interior curvature bound for hypersurfaces of prescribed k-th mean curvature*, J. Reine Angew. Math. **519** (2000), 41 – 57. MR 1739731
[U4] ———, *Monotonicity formulae and curvature equations*, J. Reine Angew. Math. **557** (2003), 199 – 218. MR 1978408
[W] X.-J. WANG, *Interior gradient estimates for mean curvature equations*, Math. Z. **228** (1998), 73 – 81. MR 1617971
[Y] Y. YU, *The Index Theorem and the Heat Equation Method*, Nankai Tracts Math. **2**, World Sci., Singapore, 2001. MR 1853830

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