SINGULARITY BEHAVIOR OF THE MEAN CURVATURE FLOW

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Abstract. It was proved that a blow-up solution to the mean curvature flow with positive mean curvature is an ancient convex solution, that is a convex solution which exists for time $t$ from $-\infty$. In this paper we study the geometry of ancient convex solutions. Our main results are contained in Theorems 1-3 below. Theorem 1 asserts that after normalization, the solution converges to a sphere or cylinder as $t \to -\infty$. Theorem 2 shows that in any dimension $n \geq 3$, there exists ancient convex solutions which are not rotationally symmetric. But Theorem 3 shows that a translating convex solution in $\mathbb{R}^3$ must be rotationally symmetric if it is a blow-up solution. These results are contained in paper [W].

1. Introduction

Let $\mathcal{M}_0$ be a smooth, compact hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Let $\mathcal{M} = \{\mathcal{M}_t\}$ be the solution to the mean curvature flow (MCF)

$$\partial_t x = h \gamma$$

with initial condition $\mathcal{M}_0$, where $x$ is a point on $\mathcal{M}_t$, $h$ is the mean curvature of $\mathcal{M}_t$ at $x$, and $\gamma$ is the unit normal. As $\mathcal{M}_0$ is compact, the solution $\mathcal{M}$ develops singularity in finite time. An important issue is to investigate the behaviour of the solution at singularity.

Suppose $\mathcal{M}$ is smooth when $t < t_0$ and develops singularity at $(x_0, t_0)$. Let $(x_k, t_k)$ be an arbitrary sequence of points converging to $(x_0, t_0)$ and $\lambda_k$ be a sequence of positive numbers tending to $\infty$. We first make the translation in space-time $(x, t) \to (x - x_k, t - t_k)$, then the parabolic dilation $(x, t) \to (\lambda_k x, \lambda_k^2 t)$. Then we get a blow-up sequence $\mathcal{M}_k$, which is also a solution to MCF. If there is a subsequence of $\mathcal{M}_k$ which converges to a solution $\mathcal{M}'$, then $\mathcal{M}'$ is called a blow-up solution (or limit flow) of $\mathcal{M}$ at $(x_0, t_0)$.

To study the singularity behavior, one has the following three steps.

(I) Prove that any blow-up sequence sub-converges to a limit flow.

(II) Prove the limit flow enjoys some special properties, such as nonnegative curvatures.

(III) Classify all limit flows with the special properties.

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If all the three steps could be satisfactorily achieved, one would have a clear understanding of the singularity behaviour. For the Ricci curvature flow of 3-manifolds, this is indeed the case.

(I) First Hamilton [Ha1] proved that if the solution $\mathcal{M}$ satisfies a proper injectivity radius estimate, then there is a sequence $(x_k, t_k)$ such that the blow-up sequence at $(x_k, t_k)$ converges to a shrinking $S^3$ or $S^2 \times R^1$ or their quotients. Recently Perelman [P] established the injectivity radius estimate and proved furthermore that any blow-up sequence sub-converges locally uniformly.

(II) Hamilton and Ivey (see [Ha1]) independently proved a curvature pinching estimate, that is the minimal sectional curvature satisfies $K_{\text{min}} \geq -C \frac{R}{\log R}$ whenever the scalar curvature $R$ is sufficiently large. This estimate implies that a limit flow must have nonnegative sectional curvatures.

(III) Perelman [P] proved that if $\mathcal{M}' = \{\mathcal{M}'_t\}$ is a non-compact blow-up solution, then for any $\varepsilon > 0$, there exists compact set $G_{\varepsilon} \subset \mathcal{M}_0$ (with $\text{diam}(G_{\varepsilon}) < C / \sup_{G_{\varepsilon}} R$) such that at any point $p \in \mathcal{M}_0 - G_{\varepsilon}$, $\mathcal{M}_0$ is $\varepsilon$-close to $S^2 \times R^1$ after normalization.

The curvature pinching estimate in step II is striking. Without this estimate, it would be impossible to classify all the limit flows. For the MCF, step I was proved by Brakke [B] in a weak sense. However at step II we are not as lucky as in the Ricci flow. The situation becomes extremely complicated, even for MCF in $R^3$, as illustrated below.

If we allow that $\mathcal{M}$ is immersed, then there are many rotationally symmetric self-similar solutions to the MCF. Each of them is a model of singularity of the MCF. If we consider embedded surface in $R^3$, then there exists a self-similar, rotationally symmetric toric solution to the MCF. A conjecture is that for any $k \geq 2$, there exists at least one self-similar shrinking toric solution of genus $k$ to the MCF. Therefore there are possibly infinitely many models for the singularity behaviour of the MCF.

Therefore to study the singularity behaviour of the MCF, one has to make some assumption. A reasonable one is to assume that the mean curvature of the initial hypersurface is positive. The positivity of the mean curvature is preserved under the MCF [Hu1]. A MCF with positive mean curvature is referred to as mean convex flow. For the mean convex flow, it is proved [HS] that a blow-up solution $\mathcal{M}'$ must be an ancient convex solution, namely $\mathcal{M}'$ exists for time from $t = -\infty$ to $t = 0$, and at each time $t$, $\mathcal{M}'_t$ is a convex hypersurface. Furthermore, the blow-up sequence $\mathcal{M}_k$ converges locally uniformly to the blow-up solution $\mathcal{M}'$. If $\mathcal{M}_0$ is embedded, these results were also proved in [Wh], which proved furthermore that if $n < 7$, these results hold for blow-up sequences beyond the first time singularity.

Therefore for the mean convex flow, one has similar results as the Ricci flow of 3-manifolds in steps I and II. The open problem is step III, namely the classification of ancient convex solutions, which we are going to address below.
2. Ancient convex solutions

It is known that there are two types of ancient convex solutions to the MCF. One is the shrinking sphere or cylinder

$$\mathcal{M}_t = \sqrt{-2kt} S^k \times \mathbb{R}^{n-k}, \quad 1 \leq k \leq n, \quad t \in (-\infty, 0)$$

which are self-similar solutions. The other one is the rotationally symmetric translating solutions (soliton). When $n = 1$, the grim reaper $x_2 = \log \sec x_1$ is the unique translating solution.

A translating solution to the MCF is a solution which can be represented (in an appropriate coordinate system) as

$$\mathcal{M}_t = \{ x_{n+1} = u(x) + t \mid x \in \mathbb{R}^n \}, \quad t \in \mathbb{R}^1,$$

where $u$ satisfies the elliptic equation

$$\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{1}{\sqrt{1 + |Du|^2}} \quad \text{in} \quad \mathbb{R}^n.$$  

A translating solution is an eternal solution, namely it exists for time $t \in (-\infty, \infty)$.

Singularities are roughly divided into two types. Following [Hu2], a singular point $(x_0, t_0)$ is called type I if $\sup_{\mathcal{M}_t} |A| \leq C(t_0 - t)^{-1/2}$ as $t \nearrow t_0$, and type II otherwise, where $|A|$ is the norm of the second fundamental form. There are several results about singularities and ancient convex solutions, which we collect below.

(i) At type I singularity there exists a blow-up solution which is a self-similar ancient convex solution [Hu2].

(ii) At type II singularity there exists a blow-up solution which is a convex translating solution [HS].

(iii) A self-similar ancient solution with positive mean curvature must be a shrinking sphere or cylinder [Hu2].

(iv) An eternal convex solution of which the mean curvature attains its maximum at an interior point in space time must be a translating solution [Ha2].

(v) A blow-up solution to the mean convex flow cannot lie in a strip $\{|x_1| < C\}$ for any constant $C > 0$ [Wh].

We remark that self-similar solutions can also arise at type II singularity, and translating solution can also occur at type I singularity if one chooses a blow-up sequence $(x_k, t_k)$ with $t_k \searrow t_0$.

To study the singularity behaviour, we need to know the geometry of all blow-up solutions. The best we can wish is that a blow-up solution to the mean convex flow is either a
shrinking sphere or cylinder, or a translating solution which can be split as $\mathcal{M} = \Sigma_k \times \mathbb{R}^{n-k}$, where $\Sigma_k$ is a rotationally symmetric translating solution of $k$-dimension. Our results in the next section show that this is true in the asymptotic sense (Theorem 1), but is not true in rigorous sense (Theorem 2).

3. Geometric properties of ancient convex solutions

Let $\mathcal{M}$ be an ancient convex solution in $\mathbb{R}^n$ (instead of $\mathbb{R}^{n+1}$ before). Regard $\mathcal{M}$ as a hypersurface in the space-time $(x, t) \in \mathbb{R}^n \times \mathbb{R}$. Then it is a graph of a function $u$, namely $\mathcal{M} = \{x_{n+1} = u(x) \mid x \in \mathbb{R}^n\}$, where $x_{n+1} = -t$, such that $\mathcal{M}_t = \{u = -t\}$, and $u$ satisfies the equation

$$\text{(2)} \quad \text{div} \left( \frac{Du}{|Du|} \right) = \frac{1}{|Du|} \quad \text{in} \ \mathbb{R}^n.$$ 

This is indeed the so-called level set flow. Conversely, if $u$ is a solution of (2), then the level set $\mathcal{M}_t = \{u = -t\}$ (time $t$) is a solution to the MCF in $\mathbb{R}^n$.

Let $u$ be a solution of (2). Huisken [Hu1] proved that if the level set $\{u = h_0\}$ is convex, then the level set $\{u = h\}$ is convex for $h < h_0$. For ancient convex solutions we have

**Lemma 1.** Suppose the graph of $u$ is a complete hypersurface. If the level sets $\{u = h\}$ are convex for all $h > 0$, then $u$ itself is a convex function.

Lemma 1 is a consequence of the concavity of $\log(c - u)$ for any constant $c$, see [K].

In Lemma 1 we automatically assume that the graph of $u$ is a complete hypersurface. In the following we will always assume this condition without further mentioning.

Translating solutions are of independent interest to the MCF. For our purpose we need only to study convex translating solutions. Therefore by Lemma 1, we can unify equations (1) and (2) to the following

$$\text{(3)} \quad \text{div} \left( \frac{Du}{\sqrt{\theta + |Du|^2}} \right) = \frac{1}{\sqrt{\theta + |Du|^2}} \quad \text{in} \ \mathbb{R}^n,$$

where $\theta \geq 0$ a constant. Note that when $\theta > 0$, equation (3) is equivalent to (1) by a dilation.

From (v) in Section 2, a blow-up solution cannot lie in a strip. We claim that a blow-up solution must be an entire solution.

**Lemma 2.** Let $u$ be a convex solution of (3). If it is not defined in a strip, it must be defined on the whole space.

Our main results are contained in the following theorems.
**Theorem 1.** Let \( u \) be an entire convex solution of (3). Let

\[
 u_h(x) = \frac{1}{h} u(h^{1/2}x).
\]

Then there exists \( 2 \leq k \leq n \), such that after a rotation of axes for each \( h \),

\[
 u_h(x) \to \frac{1}{2(k-1)} \sum_{i=1}^{k} x_i^2.
\]

The proof of Theorem 1 involves many technical but elementary estimates. A key one is the compactness result: If \( u \) is an entire solution of (3) with

\[
 u(0) = 0, \quad |Du(0)| \leq 1,
\]

then

\[
 u(x) \leq C(1 + |x|^2).
\]

Theorem 1 implies that if \( \mathcal{M} \) is an ancient convex solution, then after normalization \( \mathcal{M}_t \) converges to a sphere or cylinder as \( t \to -\infty \). However this does not imply \( \mathcal{M}_t \) itself is a sphere or cylinder. Indeed we have the following result.

**Theorem 2.** When \( n \geq 3 \), there exists an entire convex solution \( u \) to (3) such that the level set \( \{u = \text{constant}\} \) is not a sphere nor a cylinder.

To prove Theorem 2, one chooses a sequence of domains \( \Omega_k \), and constants \( h_k \to \infty \), such that the solution \( u_k \) to equation (3) with the Dirichlet boundary condition \( u = h_k \) on \( \partial \Omega_k \) satisfies

\[
 u_k \geq u_k(0) = 0
\]

and the level set \( \{u_k = 1\} \) is not round. Moreover, by proper choice of \( \Omega_k \), the level set \( \{u = 1\} \) of the limit function \( u = \lim u_k \) is not a sphere and \( u \) is defined on the whole space \( \mathbb{R}^n \).

However in dimension \( n = 2 \), we have

**Theorem 3.** Let \( u \) an entire convex solution of (3). If \( n = 2 \), then \( u \) is rotationally symmetric.

Note that when \( n = 2 \) and \( \theta = 0 \), equation (3) corresponds to the curve shortening flow. For the proof of Theorem 3, first note that by Theorem 1,

\[
 u(x) = \frac{1}{2} |x|^2 + o(|x|^2).
\]
Using an iteration for the level set \( \{ u = \beta^k \} \) \((\beta > 1)\) from \( k = \infty \), and using some estimates from [GH], we improve the estimate to

\[
u(x) = \frac{1}{2} |x|^2 + o(|x|)
\]

if the origin is properly chosen. Let \( u_r \) be the unique radial solution. Let \( u^* \) and \( u^*_r \) be the Legendre transforms of \( u \) and \( u_r \) respectively. Then \( u^* - u^*_r \) satisfies an (degenerate) elliptic equation of the form

\[
\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} = 0
\]

with \( u^* - u^*_r = o(|x|) \) at \( \infty \). By the classical Bernstein theorem, we have \( u^* - u^*_r = \text{const.} \). Hence \( u - u_r = \text{constant} \).

The Legendre transform of a convex function \( u \) is a convex function \( u^* \) defined on \( Du(R^n) \), given by

\[
u^*(y) = \sup \{ x \cdot y - u(x) \mid x \in R^n \}.
\]

The sup is attained at \( x \) such that \( Du(x) = y \). The purpose to use the Legendre transform is to get the equation (4), which does not involve the gradient term. The Bernstein theorem we used above is the following (see [S]).

**Bernstein Theorem.** Suppose \( u \in C^\infty(R^2) \) and \( u(x) = o(|x|) \) as \( |x| \to \infty \). If \( \det D^2 u \leq 0, \neq 0 \), then \( u \equiv \text{constant} \).

We note that Theorems 1-3 were reported in December 2002 at the Pacific Rim Geometric conference at CUHK. Some of these results were first announced at a symposium in September 2002 at ANU.

4. Applications to the mean curvature flow

From Theorem 1, we have

**Corollary 1.** Let \( \mathcal{M} = \{ \mathcal{M}_t \} \) be an ancient convex solution to the MCF in \( R^{n+1} \). Let

\[
\mathcal{M}_t' = \{ x \in R^{n+1} \mid (-t)^{1/2} x \in \mathcal{M}_t \}.
\]

Then \( \mathcal{M}_t' \), after a proper rotation of axes, converges as \( t \to -\infty \) to one of the following

(i) an \( n \)-sphere of radius \( \sqrt{2n} \);

(ii) a cylinder \( S^k \times R^{n-k} \), where \( S^k \) is a \( k \)-sphere of radius \( \sqrt{2k} \);

(iii) the plane \( R^n \) of multiplicity two.

From Lemma 2 we see that a blow-up solution must sweep the whole space \( R^{n+1} \). By Theorem 3, a convex solution to the curve shortening flow (namely the MCF of \( n = 1 \)) which sweeps the whole space \( R^2 \) must be a shrinking circle.

By Theorems 1 and 3, we have the following result, which corresponds to Perelman’s classification of limit flows to the Ricci flow of 3-manifolds, see Section 1 above.
**Corollary 2.** Let $\mathcal{M}$ be a non-compact blow-up solution to the MCF in $\mathbb{R}^3$. Then for any $\varepsilon > 0$, there is a compact set $G_\varepsilon \subset \mathcal{M}_0$ such that any point in $\mathcal{M}_0 \setminus G_\varepsilon$ has a neighborhood which is, after normalization, in the $\varepsilon$-neighborhood of the cylinder $S^1 \times \mathbb{R}^1$.

Note that if $\mathcal{M}$ is a blow-up solution, then $\{\mathcal{M}_{t+a}\}$, a translation of $\mathcal{M}$ in time, is also a blow-up solution. Hence Corollary 2 mean not only at time $t = 0$, but also at any time $t$. By a compactness argument, we have $\text{diam}(G_\varepsilon) \leq C/\sup_{G_\varepsilon} h$, where $h$ is the mean curvature.

From Corollary 2, we conclude as with the Ricci flow that if $\mathcal{M}$ is a mean convex flow in $\mathbb{R}^3$, then at any point with large curvature, $\mathcal{M}_t$ satisfies a canonical neighborhood condition, namely there is a neighbourhood which behaves like a cap, or cylinder, or a sphere.

This result is not true for MCF in higher dimension $n > 3$. In high dimension, from Theorem 1 we have a weak result.

**Corollary 2’.** Let $\mathcal{M}$ be a non-compact blow-up solution to the MCF in $\mathbb{R}^{n+1}$. Then for any $\varepsilon > 0$, there exists a sufficiently large $T$ such that for any $t < -T$, there is a compact set $G_{\varepsilon,t} \subset \mathcal{M}_t$ such that any point in $\mathcal{M}_t \setminus G_{\varepsilon,t}$ has a neighborhood which is, after normalization, in the $\varepsilon$-neighborhood of the cylinder $S^k \times \mathbb{R}^{n-k}$.

From Theorem 1 we also have the following

**Corollary 3.** Let $\mathcal{M} = \{\mathcal{M}_t\}$ be a solution to the mean convex flow in $\mathbb{R}^{n+1}$. If the mean curvature at $(x_k, t_k) \in \mathcal{M}$ converges to infinity, then there exists a sequence of positive constant $\lambda_k \to \infty$ such that the blow-up sequence

$$\mathcal{M}_k = \{(\lambda_k(x - x_k), \lambda_k^2(t - t_k)) \mid (x, t) \in \mathcal{M}\}$$

converges by a subsequence to a shrinking sphere or cylinder.

5. Open problems

Though our theorems give much information about ancient convex solutions to the MCF, many problems remain open. We would like to mention a few which are related to our results above.

The first one is the Bernstein problem for equation (1), or the uniqueness of entire solutions to (1). The question is, if $u$ is a solution of (1) defined on the whole space $\mathbb{R}^2$, is $u$ rotationally symmetric? By Theorem 3, this question is equivalent to whether an entire solution is convex. A related question is the convexity of level sets of solutions. That is if $u$ is a solution of (1), and if the level set $\{u = h_0\}$ is convex for some $h_0$, is the level set $\{u = h\}$ convex for $h < h_0$? This is an open problem even for the mean curvature equation

\[
\text{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = 1.
\]
Note that when \( n \geq 3 \), by Theorem 2 an entire solution to (1) is not necessarily rotationally symmetric. Hence the Bernstein theorem does not hold for equation (1) when \( n \geq 3 \). This is not at all a surprise, as in many cases the Bernstein theorem holds in low dimensions but not in high dimensions.

Theorem 2 asserts that when \( n \geq 2 \), there exists an ancient convex solution to the MCF in \( \mathbb{R}^{n+1} \) which is not rotationally symmetric. The solution constructed in Theorem 2 is not an eternal solution, as its level set \( \{ u = \text{const} \} \) is compact. When \( n \geq 3 \), such solution may occur as a blow-up solution to the mean convex flow, as was indicated in [W]. But for MCF in \( \mathbb{R}^3 \), by Theorem 3, we may hope for better results. In particular we may ask whether a limit flow of the MCF in \( \mathbb{R}^3 \) is either a shrinking sphere, or a shrinking cylinder, or a rotationally symmetric translating solution.

The geometry of the singularity set is another interesting problem. For mean convex flow in \( \mathbb{R}^3 \), we expect that the singularity set consists of finitely many isolated points and \( C^1 \) curves. For mean convex flow in higher dimension space, the singularity set is more complicated. We conjecture that

(i) the singularity set consists of finitely many connected components;
(ii) each component is contained in a \( C^1 \)-smooth \( (n-1) \)-submanifold in a time slice.

REFERENCES


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