CONTINUITY ESTIMATES
FOR THE MONGE-AMPÈRE EQUATION

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Abstract: In this paper, we study the regularity of solutions to the Monge-Ampère equation. We prove the log-Lipschitz continuity for the gradient under certain assumptions. We also give a unified treatment for the continuity estimates of the second derivatives. As an application we show the local existence of continuous solutions to the semi-geostrophic equation arising in meteorology.

Keywords: Monge-Ampère equation, a priori estimates, semigeostrophic equation.

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§1. Introduction

In this paper we study the regularity of solutions to the Monge-Ampère equation

\[ \det D^2 u = f \quad \text{in} \quad B_1(0), \]

where \( B_1(0) \) is the unit ball in the Euclidean space \( \mathbb{R}^n \). We are mainly concerned with
the log-Lipschitz continuity of the gradient \( Du \),

\[ |Du(x) - Du(y)| \leq C|x - y|(1 + |\log |x - y||) \quad x, y \in B_{1/2}(0), \]

which has applications in the existence of continuous solutions to the semi-geostrophic equation [E,L], or more generally in the optimal transportation problems [E,V]. We also study the continuity estimates of the second derivatives \( D^2 u \) under appropriate conditions.

For the regularity of the Monge-Ampère equation, Caffarelli [C2] established the interior \( W^{2,p} \) estimates (for any \( p > 1 \)) for strictly convex solutions when \( f \) is positive and continuous. He also obtained the \( C^{2,\alpha} \) estimate when \( f > 0, f \in C^\alpha, \alpha \in (0,1) \). In [C3] he proved the \( C^{1,\alpha} \) estimate for strictly convex solutions if \( C_1 \leq f \leq C_2 \) for some positive constants \( C_1, C_2 \). By an example in [W2], the \( C^{1,\alpha} \) regularity cannot be improved to \( W^{2,p} \) for large \( p \) if \( f \) is not continuous.

For the Laplace equation

\[ \Delta u = f, \]

the log-Lipschitz continuity of \( Du \) was established [Y] for \( f \in L^\infty \), see also Theorem 3.9 in [GT]. The log-Lipschitz continuity plays a key role in the existence and uniqueness of global solutions to the 2-dimensional Euler equation [Y]. A simple proof of the log-Lipschitz continuity was recently found by the second author [W3]. Considering applications to the semi-geostrophic equation [BB, C5, CuF, CRD, E, L], one wishes to know when a solution to the Monge-Ampère equation (1.1) satisfies the log-Lipschitz continuity. By an example in [W2], the condition \( C_1 \leq f \leq C_2 \) is not enough, stronger condition is necessary.

In this paper we first give a unified treatment for the continuity estimates of the second derivatives of solutions to the Monge-Ampère equation.

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Theorem 1. Let \( u \in C^2 \) be a strictly convex solution of (1.1). Assume that
\[
C_1 \leq f \leq C_2
\]
for some positive constants \( C_1, C_2 > 0 \). Then \( \forall \, x, y \in B_{1/2}(0) \), we have the estimate
\[
|D^2 u(x) - D^2 u(y)| \leq C\left[d + \int_0^d \frac{\omega_f(r)}{r} \, dr + d \int_d^1 \frac{\omega_f(r)}{r^2} \, dr\right],
\]
where \( d = |x - y| \), \( C > 0 \) depends only on \( n, m \) and \( C_1, C_2 \). It follows that
(i) If \( f \) is Dini continuous, then \( u \in C^2(B_1) \).
(ii) If \( f \in C^\alpha(B_1) \) and \( \alpha \in (0, 1) \), then
\[
\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\left[1 + \frac{\|f\|_{C^\alpha(B_1)}}{\alpha(1 - \alpha)}\right].
\]
(iii) If \( f \in C^{0,1}(B_1) \), then
\[
|D^2 u(x) - D^2 u(y)| \leq C d \left[1 + \frac{1}{\|f\|_{C^{0,1}} \log d}\right].
\]

In Theorem 1 we denote by \( m \) the modulus of convexity of \( u \), which is defined by
\[
m(t) = \inf\{u(x) - \ell_z(x) : |x - z| > t\},
\]
where \( t > 0 \), \( \ell_z \) is the tangent plane of \( u \) at \( z \). Obviously \( m \) is a nonnegative function. When \( u \) is strictly convex, it is a positive function. It was proved in \([C1]\) that if \( u \) is a convex solution of (1.1), vanishes in \( \partial \Omega \), then \( u \) is strictly convex. Note that in the estimate (1.5), the constant \( C \) depends also on \( \sup_{B_1} \sup_{B_1}(u - \ell_0) \), which is in turn determined by \( m, C_1 \) and \( C_2 \).

We also denote
\[
\omega_f(r) = \sup\{|f(x) - f(y)| : |x - y| < r\}.
\]
We say \( f \) is Dini continuous if
\[
\int_0^1 \frac{\omega_f(r)}{r} \, dr < \infty.
\]

The \( C^2 \) estimate in (i) and the \( C^{2,\alpha} \) estimate in (ii) were proved in \([W1]\) and \([C2]\), respectively. See also §6 of \([TW]\). Here we give a unified and shorter proof, using an idea from \([W3]\), where a short and elementary proof of (1.5) for the Laplace and heat equations was given. Our argument was also inspired by the original idea of Caffarelli \([C2]\).

The main estimate of the paper is the following log-Lipschitz continuity for the gradient \( Du \).
Theorem 2. Let $u$ be a strictly convex solution to the Monge-Ampère equation (1.1) and suppose $f$ satisfies (1.4). Then we have the estimate

$$|Du(x) - Du(y)| \leq Cd[1 + e^{-2\theta\psi(d)}]$$

for any $x, y \in B_{1/2}(0)$, where $d = |x - y|$,

$$\psi(d) = -\int_d^1 \frac{\omega \log f(r)}{r} dr,$$

$C = C(n, m, C_1, C_2)$, and $\theta$ is a constant less than $\frac{1}{2}$.

A more precise estimate for $\theta$ is given in (3.22). From Theorem 2, we see that $Du$ is log-Lipschitz continuous if for small $r > 0$,

$$\omega \log f(r) \leq \frac{1}{|\log r|}.$$

Our estimate (1.11) should be optimal, that is the log-Lipschitz continuity does not hold if $\omega \log f(r) \geq \frac{C}{|\log r|}$ for large $C$. See §4 for discussion.

Theorems 1 and 2 will be proved respectively in Sections 2 and 3. We indicate an application of Theorem 2 on the local existence of continuous solutions to the semigeostrophic equation in Section 4.

§2. Proof of Theorem 1

First we collect some basic properties.

Lemma 2.1. Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$. Then there is a unique minimum ellipsoid containing $\Omega$, which attains the minimum volume among all ellipsoids containing $\Omega$.

We refer the reader to [G] for a proof. We say a convex set $\Omega$ is normalized if its minimum ellipsoid is a ball. When $\Omega$ is normalized, one has $B_{r/n} \subset \Omega \subset B_r$ for concentrated balls $B_{r/n}$ and $B_r$ [G].

Therefore for any bounded convex domain $\Omega$, there is a unique unimodular linear transformation $T$ (namely $\det T = 1$) such that $T(\Omega)$ is normalized. Choose an appropriate coordinate system such that the minimum ellipsoid of $\Omega$ is given by $E = \{\sum x_i^2/a_i^2 < 1\}$ with $a_1 \geq \cdots \geq a_n$. Then $T$ is determined by the matrix diag$(\lambda_1, \cdots, \lambda_n)$, with

$$\lambda_i = \frac{1}{a_i} (a_1 \cdots a_n)^{1/n}, \quad i = 1, 2, \cdots, n.$$

Note that $\lambda_1$ and $\lambda_n$ are the least and largest eigenvalues of $T$. For convenience we say in this paper that $\Omega$ has a good shape if

$$\lambda_n \leq c^* \lambda_1$$

for some constant $c^*$. We then have

$$\lambda_1 = \frac{1}{a_1} (a_1 \cdots a_n)^{1/n}, \quad i = 1, 2, \cdots, n.$$
for some constant $c^*$ under control. If $\Omega$ has a good shape, then there exists two concentrated balls $B_r$ and $B_R$ with $R \leq nc^*r$ such that $B_r \subset \Omega \subset B_R$.

Let $u$ be a convex function defined in a bounded domain $\Omega$. For any $y \in \Omega$, denote

$$
S^0_{h,u}(y) = \{x \in \Omega : u(x) < \ell_y(x) + h\}
$$

the level set of $u$ and denote $S_{h,u}(y) = \partial S^0_{h,u}(y)$ its boundary, where $\ell_y$ is the tangent plane of $u$ at $y$. When no confuses arise we will drop the subscript $u$, and when $y$ is the minimum point of $u$, we will simply write the level set as $S^0_h$.

**Lemma 2.2.** Let $u_i$, $i = 1, 2$, be two convex solutions of $\det D^2u = 1$ in $B_1(0)$. Suppose $\|u_i\|_{C^4} \leq C_0$. Then if $|u_1 - u_2| \leq \delta$ in $B_1(0)$ for some constant $\delta > 0$, we have, for $1 \leq k \leq 3$,

$$
|D^k(u_1 - u_2)| \leq C\delta \quad \text{in} \quad B_{1/2}(0).
$$

**Proof.** We have

$$
\det D^2u_2 - \det D^2u_1 = \int_0^1 \frac{d}{dt} \det[D^2u_1 + t(D^2u_2 - D^2u_1)]dt
= a_{ij}(x)\partial_i\partial_j(u_2 - u_1) = 0.
$$

By the assumption $\|u_i\|_{C^4} \leq C_0$, the operator $L = a_{ij}(x)\partial_i\partial_j$ is linear, uniformly elliptic, with $C^2$ coefficients. By the Schauder estimates of linear elliptic equations, we obtain (2.3). $\square$

We also need the following regularity for the Monge-Ampère equation [GT, P].

**Lemma 2.3.** Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$. Let $u$ be a convex solution of $\det D^2u = 1$ in $\Omega$, vanishing on $\partial \Omega$. If $B_r(0) \subset \Omega \subset B_R(0)$, then for any $\Omega' \subset \subset \Omega$, there is a constant $C > 0$, depending only on $n, r, R$, and $\text{dist}(\Omega', \partial \Omega)$, such that

$$
\|u\|_{C^4(\Omega')} \leq C.
$$

**Remark 2.1.** Lemma 2.3 also implies that if $u$ is a convex solution of $\det D^2u = 1$ in $\Omega$ which vanishes on $\partial \Omega$, and if $D^2u(0)$ is the unit matrix (or uniformly bounded), then the domain $\Omega$ must have a good shape. For if not, one makes a unimodular linear transform $y = Tx$ to normalize $\Omega$. Then the ratio of the largest and least eigenvalues of $T$ will be large. By Lemma 2.3, $D^2_yu(0)$ uniformly bounded. Hence $D^2_xu(0) = T'D^2_y(0)T$ cannot be the unit matrix (or uniformly bounded).

**Proof of Theorem 1.** By subtracting a linear function we suppose

$$
u(0) = 0, \quad Du(0) = 0.$$
We consider the solution \( u \) in the level set \( S^0_h \), where \( h > 0 \) is chosen small such that \( S^0_h \) is compactly supported in \( B_1(0) \). By Lemma 2.1 there is a unique unimodular linear transform \( T_h \) such \( T_h(S^0_h) \) is normalized. Hence by making the change \( x \to T_h x / \sqrt{h} \) and \( u \to u / h \), we may suppose \( h = 1 \), \( S^0_1 \) is normalized, and

\[
\int_0^1 \frac{\omega(r)}{r} \leq \varepsilon,
\]

where \( \omega(r) = \omega_f(r) \), \( \varepsilon \) can be as small as we want, provided \( h \) is sufficiently small. Note that the Monge-Ampère equation is invariant under the change \( x \to Tx \) and \( u \to (\det T)^{2/n} u \) for any non-degenerate linear transform \( T \).

Let \( u_k, k = 0, 1, \ldots \), be the solution of

\[
\det D^2 u_k = f(0) \quad \text{in} \quad S^0_{4^{-k},u}, \quad u_k = u (= 4^{-k}) \quad \text{on} \quad \partial S^0_{4^{-k},u}.
\]

Denote

\[
\nu(t) = \sup_{z \in B_1} \{|f(x) - f(y)| : x, y \in S^0_{t^2,u}(z)\}, \quad \nu_k = \nu(2^{-k}),
\]

which is invariant under unimodular linear transformation of \( x \). If \( S^0_{1,u} \) has good shape, then we have \( \nu(t) \leq \omega(Ct) \).

Since \( S^0_{1,u} \) has a good shape, by Lemma 2.3, \( \|u_0\|_{C^4(S^0_{3/4,u})} \leq C \). By the comparison principle, \( |u - u_0| \leq C \nu_0 \) and \( |u - u_1| \leq C \nu_1 \). We obtain \( |u_1 - u_0| \leq C \nu_0 \). Since \( S^0_{1,u} \) has a good shape, so does \( S^0_{4^{-1},u} \). It follows \( \|u_1\|_{C^4(S^0_{3/16,u})} \leq C \). Replace the balls \( B_1 \) and \( B_{1/2} \) in Lemma 2.2 by the level sets \( S^0_{1/8,u} \) and \( S^0_{1/16,u} \), we obtain

\[
|D^k u_0(x) - D^k u_1(x)| \leq C \nu_0
\]

for \( x \in S^0_{4^{-k},u}, \) where \( 1 \leq k \leq 3 \). By Remark 2.1, the estimate also implies that \( S^0_{4^{-2},u} \) has a good shape.

By induction we assume that \( S^0_{4^{-k-1},u} \) has good shape, with the constant \( c^* \) (see (2.1)) independent of \( k \). Hence \( \nu_k \leq \omega(C2^{-k}) \) for some \( C > 0 \) independent of \( k \) (depending on \( c^* \)). Applying the same argument to \( \tilde{u}_0 := 4^k u_k(\frac{x}{2}) \) and \( \tilde{u}_1 := 4^k u_{k+1}(\frac{x}{2}) \), we obtain, for \( x \in S^0_{4^{-k-2},u_{k+1}}, \)

\[
\begin{align*}
|D u_k(x) - D u_{k+1}(x)| &\leq C 2^{-k} \nu_k, \\
|D^2 u_k(x) - D^2 u_{k+1}(x)| &\leq C \nu_k, \\
|D^3 u_k(x) - D^3 u_{k+1}(x)| &\leq C 2^k \nu_k,
\end{align*}
\]
where $2^k$ in (2.9) is the scaling constant. Hence

\begin{equation}
|D^2u_0(x) - D^2u_{k+1}(x)| \leq C \sum_{i=0}^{k} \nu_i \leq C \int_{2^{-k}}^{1} \frac{\omega(r)}{r} dr
\end{equation}

for $x \in S_{4^{-k-2},u_{k+1}}^0$, where $C > 0$ is independent of $k$.

Estimate (2.10), together with (2.5) and Remark 2.1, implies that $S_{4^{-k-2},u_{k+1}}^0$ has a good shape, with the constant $c^*$ independent of $k$. Denote

\begin{equation}
\hat{u} = 4^{k+1}u\left(\frac{x}{2^{k+1}}\right), \quad \hat{u}_{k+1} = 4^{k+1}u_{k+1}\left(\frac{x}{2^{k+1}}\right).
\end{equation}

Then $\hat{u}$ and $\hat{u}_{k+1}$ satisfy the equations $\det D^2\hat{u} = f(2^{-k-1}x)$ and $\det D^2\hat{u} = f(0)$ respectively. By the comparison principle,

\begin{equation}
|\hat{u} - \hat{u}_{k+1}| \leq C \nu_{k+1}.
\end{equation}

Hence $S_{4^{-1},\hat{u}}^0$ has a good shape, and so also $S_{4^{-k-2},u}^0$ has a good shape.

For any given point $z$ near the origin,

\begin{equation}
|D^2u(z) - D^2u(0)| \leq I_1 + I_2 + I_3 =: |D^2u_k(z) - D^2u_k(0)| + |D^2u_k(0) - D^2u(0)| + |D^2u(z) - D^2u_k(z)|.
\end{equation}

Let $k \geq 1$ such that $4^{-k-4} \leq u(z) \leq 4^{-k-3}$. Then by (2.9b),

\begin{equation}
I_2 \leq C \sum_{j=k}^{\infty} \nu_j \leq C \int_0^{\infty} \frac{|z|}{r} \omega(r) rdr.
\end{equation}

Next we estimate $I_3$. Let $u_{z,j}$ be the solution of

\begin{equation}
\det D^2u_{z,j} = f(z) \quad \text{in} \quad S_{4^{-j},u}^0(z),
\end{equation}

\begin{equation}
u_{z,j} = u \quad \text{on} \quad \partial S_{4^{-j},u}^0(z).
\end{equation}

Let $j_k = \inf\{ j : S_{4^{-j},u}^0(z) \subset S_{4^{-k-2},u}^0(0) \}$. Obviously $j_k \geq k$. We claim that $j_k \leq k + l_0$ for some fixed $l_0$ independent of $k$. Indeed, by making the dilation $x \rightarrow 2^k x$ and $u \rightarrow 4^k u$, we may assume that $k = 0$ and $u(z) \leq 4^{-3}$. From Caffarelli’s strict convexity [C1], we have $S_{4^{-l_0},u}^0(z) \subset S_{1,u}^0(0)$, which implies that $j_k \leq k + l_0$. Note that $|u_k - u_{z,k+l_0}| \leq C \nu_k$.

Applying Lemma 2.2 to $u_k$ and $u_{z,k+l_0}$ in $S_{4^{-k-l_0},u}^0(z)$ we have

\begin{equation}
|D^2u_k(z) - D^2u_{z,k+l_0}(z)| \leq C \nu_k.
\end{equation}

Similarly to (2.14) we have

\begin{equation}
|D^2u(z) - D^2u_{z,k+l_0}(z)| \leq C \sum_{j=k+l_0}^{\infty} \nu_j \leq C \int_0^{\infty} \frac{|z|}{r} \omega(r) rdr.
\end{equation}
Combining (2.16) and (2.17) we obtain an estimate for \( I_3 \).

To estimate \( I_1 \), denote \( h_j = u_j - u_{j-1} \). By (2.9c),
\[
|D^2 h_j(z) - D^2 h_j(0)| \leq C 2^j \nu_j |z|.
\]

Hence
\[
I_1 \leq |D^2 u_{k-1}(z) - D^2 u_{k-1}(0)| + |D^2 h_k(z) - D^2 h_k(0)|
\leq |D^2 u_0(z) - D^2 u_0(0)| + \sum_{j=1}^k |D^2 h_j(z) - D^2 h_j(0)|
\leq C |z| (1 + \sum_{j=1}^k 2^j \nu_j)
\leq C |z| (1 + \int_{|z|}^1 \frac{\omega(r)}{r^2} dr).
\]

Hence we obtain (1.5). Note that (1.6) and (1.7) follows readily from (1.5). This completes the proof of Theorem 1. □

§3. Proof of Theorem 2

The proof of Theorem 2 is divided into two parts. The first part is the proof of (1.11) with a large constant \( \theta \), and the second one is an estimate for \( \theta \).

The first part is a modification of the proof of Theorem 1. A difference is that in the proof of Theorem 1, the level sets \( S_{4-k}^0 \) have a good shape for all \( k > 0 \), and we don’t need to make linear transforms to normalize them. But in the proof of Theorem 2, we have to make linear transforms for every \( k \) to keep these level sets in good shape.

§3.1. By subtracting a linear function we suppose \( u(0) = 0 \), \( Du(0) = 0 \). By making a dilation of the axes, we may assume that \( f(0) = 1 \). Consider \( u \) in the level set \( S_h^0 \) for some small \( h > 0 \), such that \( S_h^0 \subset B_1(0) \). By making a linear transform as in Section 2, we may assume that \( h = 1 \) and \( S_1^0 \) is normalized. Let \( \nu_k = \nu(2^{-k}) \) be as in (2.7). Let \( u_k, k = 0, 1, \cdots \), be the solution of
\[
\det D^2 u_k = f_k \quad \text{in} \quad S_{4-k}^0,
\quad u_k = u \quad \text{on} \quad \partial S_{4-k}^0,
\]
where \( f_k \) is a constant,
\[
f_k = \frac{1}{2} \left[ \inf \{f(x) : x \in S_{4-k}^0\} + \sup \{f(x) : x \in S_{4-k}^0\} \right].
\]

We choose such a special constant \( f_k \) to get a better (smaller) upper bound for the constant \( \theta \) in (1.11). In this subsection we will assume \( f_k = 1 \) by making a dilation of the axes.
First we make a unimodular linear transform \( x^{(0)} = T_0 x \), such that \( D^2_{x^{(0)}} u_0(0) = I \), where \( I \) denotes the unit matrix. Here and below we use \( D \) to denote derivatives in \( x \) and \( D_{x^{(i)}} \) to denote derivatives in the new coordinates \( x^{(i)} \). By Remark 2.1, the level set \( S_{4^{-k},u}^0 \) has a good shape.

We then make a unimodular linear transform \( x^{(1)} = T_1 x^{(0)} \), such that \( D^2_{x^{(1)}} u_1(0) = I \). Then the level sets \( S_{4^{-k+1},u}^0 \) has a good shape, and estimates \( (2.9) \) (for \( k = 1 \)) hold in the new coordinates \( x^{(1)} \).

By induction we assume that \( D^2_{x^{(k-1)}} u_{k-1}(0) = I \). Then by Remark 2.1, the level set \( S_{4^{-k+1},u}^0 \) has a good shape. Hence from the proof of Theorem 1 (see (2.11) and (2.12)), \( S_{4^{-k+1},u}^0 \) has a good shape. We then make a unimodular linear transform \( x^{(k)} = T_k x^{(k-1)} \), such that \( D^2_{x^{(k)}} u_k(0) = I \). More precisely, assume that in the coordinates \( x^{(k-1)} \), \( u_k \) has the expansion (after a rotation of axes such that \( D^2_{x^{(k)}} u_k(0) \) is diagonal)

\[
u_k(x) = u_k(0) + a_i x_i + \frac{1}{2} b_i x_i^2 + O(|x|^3).
\]

Then the transform \( x^{(k)} = T_k x^{(k-1)} \) is given by

\[Tx = (b_1^k x_1, \ldots, b_n^k x_n),\]

so that in \( x^{(k)} \),

\[u_k(x) = u_k(0) + \frac{a_i}{b_i^k} x_i + \frac{1}{2} b_i^k x_i^2 + O(|x|^3),\]

and the largest eigenvalue of \( T_k \) is

\[\lambda_{max}(T_k) = \max b_i^k 1/2.\]

Remark 3.1. Here we assume the constant \( f_k \) in (3.1) is equal to 1, so that \( \prod_{i=1}^n b_i = 1 \). If \( f_k \neq 1 \), then \( T_k \) should be given by

\[T_k x = \frac{1}{f_k^{1/2n}} (b_1^k x_1, \ldots, b_n^k x_n),\]

so that \( T_k \) is unimodular, namely \( \det T_k = 1 \).

After the transform \( T_k \), the level set \( S_{4^{-k-1},u}^0 \) has a good shape, and estimates (2.9) hold in the new coordinates \( x^{(k)} \). That is

\[ |D_{x^{(k)}} u_k(x) - D_{x^{(k)}} u_{k+1}(x)| \leq C 2^{-k} \nu_k,\]

\[ |D^2_{x^{(k)}} u_k(x) - D^2_{x^{(k)}} u_{k+1}(x)| \leq C \nu_k,\]

for \( x \in S_{4^{-k-1},u}^0 \).
For any given point \(z\) near the origin,
\[
|Du(z) - Du(0)| \leq I_1 + I_2 + I_3 =:\n|Du_k(z) - Du_k(0)| + |Du_k(0) - Du(0)| + |Du(z) - Du_k(z)|,
\]
where we choose \(k = k_z \geq 1\) such that \(4^{-k-4} \leq u(z) \leq 4^{-k-3}\). For the estimate of \(I_2\), we have
\[
I_2 = |Du_k(0) - Du(0)| \leq \sum_{i=k}^{\infty} |Du_i(0) - Du_{i+1}(0)|.
\]
Denote \(T^{(i)} = T_i \cdot T_{i-1} \cdots T_1 \cdot T_0\). Let \(\lambda_i\) be the largest eigenvalue of \(T^{(i)}\). Then
\[
|Du_i(0) - Du_{i+1}(0)| \leq \lambda_i |D_{x(i)} u_i(0) - D_{x(i)} u_{i+1}(0)|.
\]
By (3.3a),
\[
|D_{x(i)} u_i(0) - D_{x(i)} u_{i+1}(0)| \leq C 2^{-i} \nu_i.
\]
It follows that
\[
|Du_i(0) - Du_{i+1}(0)| \leq C \lambda_i 2^{-i} \nu_i,
\]
where \(C\) is independent of \(i\). Hence we obtain
\[(3.4)\]
\[
I_2 \leq C \sum_{i=k}^{\infty} \lambda_i 2^{-i} \nu_i.
\]
Similarly we have \(I_3 \leq C \sum_{i=k}^{\infty} \lambda_i 2^{-i} \nu_i\). To estimate \(I_1\), by (3.3b) we have
\[
|D^2_{x(i)} u_i(x^{(i)}) - D^2_{x(i)} u_{i+1}(x^{(i)})| \leq C \nu_i
\]
for any \(i = 0, 1, \ldots, k\) and \(x^{(i)} \in T^{(i)}(S^0_{4-i-2, u})\). Hence
\[
|D^2 u_i(x) - D^2 u_{i+1}(x)| \leq C \lambda_i^2 \nu_i
\]
for any \(x \in S^0_{4-i-2, u}\).

Denote \(h_i = u_i - u_{i-1}\). We have
\[
|Dh_i(z) - Dh_i(0)| \leq |D^2 h_i| |z| \leq C \lambda_i^2 \nu_i |z|.
\]
Hence
\[(3.5)\]
\[
I_1 \leq |Du_k-1(z) - Du_k-1(0)| + |Dh_k(z) - Dh_k(0)|
\leq |Du_0(z) - Du_0(0)| + \sum_{i=1}^{k} |Dh_i(z) - Dh_i(0)|
\leq |Du_0(z) - Du_0(0)| + C |z| \sum_{i=0}^{k} \lambda_i^2 \nu_i
\leq C |z| [1 + \sum_{i=0}^{k} \lambda_i^2 \nu_i].
\]
We obtain

\[ |Du(z) - Du(0)| \leq C \sum_{i=k}^{\infty} \lambda_i 2^{-i} \nu_i + C|z|[1 + \sum_{i=0}^{k} \lambda_i^2 \nu_i]. \tag{3.6} \]

Next we estimate \( \lambda_i \). For a fixed \( i \), denote

\[ \hat{u} = 4^i u(\frac{x^{(i)}}{2^i}), \quad \hat{u}_i = 4^i u_i(\frac{x^{(i)}}{2^i}), \quad u^*_i = 4^i u_{i+1}(\frac{x^{(i)}}{2^i}). \tag{3.7} \]

Then \( \hat{u}, \hat{u}_i, \) and \( u^*_i \) satisfy respectively the equation \( \det D^2 \hat{u} = f(2^{-i}x^{(i)}), \) \( f_i \), and \( f_{i+1} \). By definition, \( \nu_i \geq \sup\{|f(2^{-i}x^{(i)}) - f(0)| : x^{(i)} \in S^0_{i,\hat{u}}\} \). Hence by the comparison principle,

\[ |\hat{u} - \hat{u}_i| \leq C\nu_i, \quad |\hat{u} - u^*_i| \leq C\nu_{i+1}. \]

It follows that

\[ |\hat{u}_i - u^*_i| \leq C\nu_i. \]

Note that \( D^2_{x^{(i)}} u_i(0) = I \). Hence by Lemma 2.2, \( |D^2_{x^{(i)}} u_{i+1}(0) - I| \leq C\nu_i \). We obtain

\[ \lambda_{\max}(T_i) \leq 1 + \theta\nu_i \tag{3.8} \]

for some constant \( \theta \) independent of \( i \) (but later we will give a more precise upper bound of \( \theta \) for large \( i \)). Hence

\[ \lambda_i \leq \prod_{j=0}^{i} \lambda_{\max}(T_j) \]

\[ \leq \prod_{j=0}^{i} (1 + \theta\nu_j) \]

\[ = e^{\sum_{j=0}^{i} \log(1 + \theta\nu_j)} \]

\[ \leq e^{\theta \sum_{j=0}^{i} \nu_j}. \]

We have therefore established (recall that \( Du(0) = 0 \))

\[ |Du(z)| \leq C \sum_{i=k}^{\infty} 2^{-i} \nu_i e^{\theta \sum_{j=0}^{i} \nu_j} + C|z|[1 + \sum_{i=0}^{k} \nu_i e^{2\theta \sum_{j=0}^{i} \nu_j}] \]

\[ \leq C \int_0^{2^{-k}} \nu(t) e^{\theta \int_1^{t} \frac{\nu(s)}{s} ds} dt + C|z|[1 + \int_{2^{-k}}^{1} \frac{\nu(t)}{t} e^{2\theta \int_1^{t} \frac{\nu(s)}{s} ds} dt]. \tag{3.9} \]

**Remark 3.2.** From (3.9) we see that if \( \lim_{t \to 0} \nu(t) < \varepsilon \) for some small \( \varepsilon > 0 \), then \( u \in C^{1,\alpha} \) for some \( \alpha \in (1 - 2\theta \varepsilon, 1) \), which also follows from Caffarelli’s \( W^{2, p} \) estimate.
If furthermore \( \int_0^1 \frac{\nu(t)}{t} < \infty \), the above estimate implies that \( Du \) is Lipschitz continuous, namely \( D^2 u \) is uniformly bounded. In the following we assume \( \lim_{t \to 0} \nu(t) = 0 \).

The right hand side of (3.9) can be simplified as follows. Denote

\[
\varphi(t) = -\int_t^1 \frac{\nu(s)}{s} ds.
\]

Assume that \( \nu(t) \to 0 \) at \( t \to 0 \), so that \( \varphi(t) = o(|\log t|) \) as \( t \to 0 \). The first integral on the right hand side of (3.9) is equal to

\[
\int_0^r \nu(t)e^\theta \int_0^t \frac{\nu(s)}{s} ds dt = \int_0^r t \varphi'(t)e^{-\theta \varphi(t)} dt
= \frac{-r}{\theta} e^{-\theta \varphi(r)} + \frac{1}{\theta} \int_0^r e^{-\theta \varphi(t)},
\]

where \( r = 2^{-k} \). The second integral on the right hand side of (3.9) is equal to

\[
\int_r^1 \varphi'(t)e^{-2\theta \varphi(t)} dt = \frac{1}{2\theta} [e^{-2\theta \varphi(r)} - e^{-2\theta \varphi(1)}].
\]

We claim that if \( \varphi(0) = -\infty \),

\[
\int_0^r e^{-\theta \varphi(t)} = O(r e^{-\theta \varphi(r)}) \quad as \quad r \to 0.
\]

Indeed, noting that \( r \varphi'(r) = \nu(r) = o(1) \), we have

\[
\lim_{r \to 0} \int_0^r e^{-\theta \varphi(t)} = \lim_{r \to 0} \frac{\int_0^r e^{-\theta \varphi(t)}}{r e^{-\theta \varphi(r)}} \frac{(\int_0^r e^{-\theta \varphi(t)})'}{(r e^{-\theta \varphi(r)})'}
= \lim_{r \to 0} \frac{e^{-\theta \varphi(r)}}{e^{-\theta \varphi(r)}(1 - \theta r \varphi'(r))}
= 1.
\]

Therefore from (3.9),

\[
(3.10) \quad |Du(z)| \leq C2^{-k}[1 + e^{-\theta \varphi(2^{-k})}] + C|z|[1 + e^{-2\theta \varphi(2^{-k})}].
\]

Claim:

\[
(3.11) \quad 2^{-k}[1 + e^{-\theta \varphi(2^{-k})}] \leq C|z|[1 + e^{-2\theta \varphi(2^{-k})}].
\]

Indeed, (3.11) is obvious if \( 2^{-k} \leq |z| \). If \( |z| \leq 2^{-k} \), denote \( h(t) = u(t \frac{z}{|z|}) \), \( \alpha = 2^{-k}/|z| \), \( \beta = e^{-\theta \varphi(2^{-k})} \). Since \( h(0) = h'(0) = 0 \), by convexity and since \( u(z) \geq 4^{-k-4} \), we have

\[
h'(t) \geq \frac{1}{t}[h(t) - h(0)] \geq \frac{4^{-k-4}}{|z|} = 2^{-k-8}\alpha \quad at \quad t = |z|.
\]
From (3.10),
\[ h'(t) \leq C2^{-k}(1 + \beta) + C|z|(1 + \beta^2). \]
Combining the above two inequalities, we obtain
\[ \alpha \leq C(1 + \beta) + \frac{C}{\alpha}(1 + \beta^2). \]
Hence \( \alpha \leq C(1 + \beta) \) and so
\[ 2^{-k}(1 + \beta) = |z|\alpha(1 + \beta) \leq C|z|(1 + \beta^2). \]
We also obtain (3.11).

We have therefore proved that
\[ (3.12) \quad |Du(z) - Du(0)| \leq C|z|[1 + e^{-2\theta\varphi(2^{-k})}]. \]
Note that estimate (3.12) still holds and \( Du \) is Lipschitz continuous if \( \varphi(0) > -\infty \). This is the case treated in §2.

Estimate (1.11) now follows from (3.12). Indeed, by Remark 3.2, \( u \in C^{1,\alpha} \) for any \( \alpha \) close to 1. Hence for any \( \varepsilon > 0 \), the level set \( S_{t,\varepsilon}^0(y) \) is contained in the ball \( B_{t,\varepsilon}(y) \) provided \( t > 0 \) is sufficiently small. In particular we have \( 2^{-k} \geq |z|^{1+\varepsilon} \) and \( \nu(t) \leq \omega(t^{1-\varepsilon}) \). With \( d = |z| \) we then obtain
\[
|\varphi(2^{-k})| = \int_{d^{1+\varepsilon}}^{1} \frac{\nu(t)}{t} dt \leq \int_{d^{1+\varepsilon}}^{1} \frac{\omega(t^{1-\varepsilon})}{t} dt = \frac{1}{1 - \varepsilon} \int_{d^{1+\varepsilon}}^{1} \frac{\omega(t^{1-\varepsilon})}{t^{1-\varepsilon}} dt^{1-\varepsilon} \leq \frac{1}{1 - \varepsilon} \int_{d^{1+\varepsilon}}^{1} \frac{\omega(t)}{t^{1-\varepsilon}} dt \leq \frac{1}{1 - \varepsilon} |\psi(d^{1-\varepsilon^2})| \leq \frac{1}{1 - \varepsilon} |\psi(d)|.
\]
We also indicate that the modulus of continuity of \( Du \) in (1.11) is determined by \( \omega_{\log f} \). This is because for a general positive function \( f \), letting \( w = \frac{u}{f^{1/n}(0)} \) such that \( \det D^2 w = g =: \frac{f}{f(0)} \), we have
\[
|g(x) - g(0)| = \left| \frac{f(x)}{f(0)} - 1 \right| \approx \log \frac{f(x)}{f(0)} = \log f(x) - \log f(0)
\]
for \( x \) near 0.

From (3.12) we see that if
\[ (3.13) \quad \nu(t) \leq \frac{1}{2\theta |\log t|} \]
for \( t > 0 \) small, then \( Du \) is log-Lipschitz continuous.

§3.2. To finish the proof of Theorem 2, it remains to prove that \( \theta < \frac{1}{2} \).
Lemma 3.1. Assume \( \lim_{x \to 0} f(x) = 1 \). For any \( \varepsilon > 0 \), there exists \( h_\varepsilon > 0 \) such that when \( 0 < h < h_\varepsilon \), \( S_{h,u}(0) \) is in the \( \varepsilon h^{2} \)-neighborhood of a sphere of radius \( (2h)^{1/2} \) after normalization.

This was proved in [C2], Lemma 7. Moreover, the condition \( \lim_{x \to 0} f(x) = 1 \) can be relaxed to
\[
\lim_{x \to 0} |f(x) - 1| \leq \delta
\]
for some \( \delta > 0 \) depending on \( \varepsilon \). We note that Lemma 3.1 can also be proved by a blow-up argument, as a convex solution of \( \det D^2 u = 1 \) must be a quadratic function if its graph is complete.

Lemma 3.2. Let \( f(0) = 1 \) and \( \nu = \text{osc}_{S_{0}^{1},u} f \). Let \( \overline{u} \in C_{\text{loc}}^{4}(S_{0}^{1},u) \) be the solution of
\[
(3.14) \quad \det D^2 v = \overline{f} \quad \text{in} \quad S_{1,u}^{0}, \quad v = 1 \quad \text{on} \quad \partial S_{1,u}^{0},
\]
where \( \overline{f} \) is a constant, \( \overline{f} = \frac{1}{2}(f_{\min} + f_{\max}) \), \( f_{\min} = \inf\{f(x) : x \in S_{1,u}^{0}\} \) and \( f_{\max} = \sup\{f(x) : x \in S_{1,u}^{0}\} \). Then
\[
(3.15) \quad |u - \overline{u}| \leq \frac{3\nu}{8n} + C\nu^{2} \quad \text{on} \quad \partial S_{\frac{1}{4},u}^{0}.
\]

Proof. Let \( u_{\min} \) (\( u_{\max} \), resp.) be the solution of \( \det D^2 v = f_{\min} \) (\( f_{\max} \), resp.) in \( S_{1,u}^{0} \) such that \( v = 1 \) on \( \partial S_{1,u}^{0} \). Then
\[
u_{\max} - 1 = \left( \frac{f_{\max}}{f_{\min}} \right)^{\frac{1}{n}} (u_{\min} - 1).
\]

Observe that \( \frac{f_{\max}}{f_{\min}} = 1 + \nu + O(\nu^{2}) \). We have \( \left( \frac{f_{\max}}{f_{\min}} \right)^{\frac{1}{n}} = 1 + \frac{\nu}{n} + O(\nu^{2}) \). By the comparison principle,
\[
u_{\max} \leq u \leq u_{\min}.
\]
Hence on \( S_{\frac{1}{4},u}^{0} \),
\[
(3.16) \quad |u_{\max}| \leq |u_{\min}| + \frac{3\nu}{4n} + C\nu^{2}.
\]
By our choice, \( \overline{f} = \frac{1}{2}(f_{\max} + f_{\min}) \). Hence
\[
\overline{u} = \frac{1}{2}(u_{\min} + u_{\max}) + O(\nu^{2}).
\]
We obtain (3.15). Alternatively we have
\[
\overline{u} - 1 = \left( \frac{\overline{f}}{f_{\min}} \right)^{\frac{1}{n}} (u_{\min} - 1)
\]
and similarly to (3.16), $|\bar{u}| \leq |u_{\min}| + \frac{3\nu}{8n} + C\nu^2$. We also obtain (3.15). □

Denote

\begin{equation}
\beta_n = \sup |D^2u(0)|,
\end{equation}

where $|D^2u| = \max_{|\xi| = 1} u_{\xi\xi}$, and the sup is taken among all harmonic functions in the unit ball $B_1(0) \subset \mathbb{R}^n$ satisfying $|u| \leq 1$ on $\partial B_1$.

Let $\hat{u}_i, \hat{u}_i$, and $u_{i+1}$ be the functions given in (3.7). Then $\hat{u}_i (u_{i+1}^*, \text{resp.})$ satisfies

$$\det D^2u = f_i \ (f_i+1, \text{resp.}) \text{ in } S^0_{1,\hat{u}} \ (\text{in } S^0_{1/4,\hat{u}}, \text{resp}),$$

where by Lemma 3.1, the set $S^0_{1/4,\hat{u}}$ is a small perturbation of a ball of radius $r$. As $f_i+1$ may differ from $f_i$, we introduce a new function $\hat{v}_{i+1}$, which is the solution of

$$\det D^2v = f_i \ (f_i+1, \text{resp.}) \text{ in } S^0_{1,\hat{u}}, \ v = \hat{u} \text{ on } S^0_{1/4,\hat{u}}.$$  \hspace{1cm} (3.19)

Then

\begin{equation}
\hat{v}_{i+1} - \frac{1}{4} = \left( \frac{f_i}{f_{i+1}} \right)^{\frac{1}{2}} (u_{i+1}^* - \frac{1}{4}).
\end{equation}

Let

$$v = \frac{\hat{u}_i - \hat{v}_{i+1}}{\|\hat{u}_i - \hat{v}_{i+1}\|_{L^\infty(S^0_{1/4,\hat{u}})}}.$$

Then $v$ satisfies a linearized Monge-Ampère equation, that is

\begin{equation}
\det D^2\hat{v}_{i+1} - \det D^2\hat{u}_i = \int_0^1 \frac{d}{dt} \det [D^2\hat{v}_i + t(D^2\hat{v}_{i+1} - D^2\hat{u}_i)] \ dt = a_{ij}(x) \partial_i \partial_j (\hat{v}_{i+1} - \hat{u}_i) = 0. \hspace{1cm} (3.19)
\end{equation}

Notice that by Lemma 3.1, both $\hat{u}_i$ and $\hat{v}_{i+1}$ converges to $\frac{1}{2}|x|^2$ as $i \to \infty$. By the regularity of the Monge-Ampère equation (Lemma 2.3), the matrix $\{a_{ij}\}$ converges to the unit matrix. Hence from (3.17),

\begin{equation}
|D^2v(0)| \leq 2(\beta_n + \varepsilon), \hspace{1cm} (3.20)
\end{equation}

where $\varepsilon > 0$ can be as small as we want, provided $i$ is large enough. The coefficient 2 is due to that $v_{i+1}$ is defined in $S^0_{1/4,\hat{u}}$, which is a small perturbation of $B_{1/\sqrt{2}}$.

By the homogeneous equation (3.19), sup$\{|\hat{u}_i - \hat{v}_{i+1}|(x) : \ x \in S^0_{1/4,\hat{u}}\}$ is attained on the boundary $S^0_{1/4,\hat{u}}$. Hence by Lemma 3.2,

$$\sup\{|\hat{u}_i - \hat{v}_{i+1}|(x) : \ x \in S^0_{1/4,\hat{u}}\} \leq \sup\{|\hat{u}_i - \hat{u}|(x) : \ x \in S^0_{1/4,\hat{u}}\} \leq \frac{3\nu_i}{8n} + O(\nu_i^2).$$
Recall that $D^2 \hat{u}_i(0) = I$. Hence by (3.20),

$$|D^2 \hat{v}_{i+1}(0) - I| \leq (\beta_n + \varepsilon) \frac{3\nu_i}{4n} + O(\nu_i^2).$$

Note that by (3.18),

$$D^2 \hat{v}_{i+1} = \left(\frac{f_i}{f_{i+1}}\right)^{\frac{1}{n}} D^2 u_i^*.$$ 

Hence by a dilation $x \rightarrow \left(\frac{f_i}{f_{i+1}}\right)^{-1/2n} x$, we may cancel the coefficient $\left(\frac{f_i}{f_{i+1}}\right)^{\frac{1}{n}}$. It is obvious that the dilation does not affect the eigenvalues of the mapping $T_i$ in (3.8) (because $T_i$ is unimodular). Hence by (3.2) and (3.21),

$$\lambda_{\text{max}}(T_i) \leq \left(1 + (\beta_n + \varepsilon) \frac{3\nu_i}{4n}\right)^{\frac{1}{2}}.$$ 

Therefore we obtain an upper bound for the constant $\theta$ in (3.8) (for large $i$)

$$\theta \leq \frac{3\beta_n}{8n} + \varepsilon.$$

Next we give an upper bound for $\beta_n$.

**Lemma 3.3.** Let $\beta_n$ be given in (3.17). Then we have the estimate

$$\beta_n = \frac{4(n+2)\omega_n^{-1}}{\omega_n \sqrt{n}} \left(\frac{n-1}{n}\right)^{\frac{n-1}{2}},$$

where $\omega_n$ is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

**Proof.** For any small $\varepsilon > 0$, let $u$ be a harmonic function satisfying

$$u_{nn}(0) \geq \sup |D^2 v(0)| - \varepsilon,$$

where the sup is taken among all harmonic functions $v$ in the unit ball with $|v| \leq 1$, and $|D^2 v|$ denotes the largest eigenvalue of the matrix $D^2 v$. By a rotation of axes we assume that $D^2 u(0)$ is diagonal and $|D^2 u(0)| = u_{nn}(0)$. By Green’s representation,

$$u(x) = \frac{1-|x|^2}{\omega_n} \int_{\partial B_1} g(y)|x-y|^n dy,$$

where $g$ is the boundary value of $u$ on $\partial B_1$. Hence

$$u_{nn}(0) = \frac{n+2}{\omega_n} \left[ \int_{\partial B_1} n y_n^2 g - \int_{\partial B_1} g \right].$$

To compute the above integrals, we make a rearrangement of the function $g$, which keep the integral $\int_{\partial B_1} g$ invariant, such that $g$ is rotationally symmetric in $x' = (x_1, \cdots, x_{n-1})$, where $x_n = 0$. To accomplish this, we use a rotation of axes $x_n \rightarrow y_n$ to obtain

$$g(x) = \frac{\sqrt{n} - x_n^2}{\omega_n} \int_{\partial B_1} g(y)|x-y|^n dy.$$
even in \( x_n \), and is monotone increasing in \( x_n \) for \( x_n \in (0,1) \). It is easy to see that the rearrangement will increases the value \( u_{nn}(0) \).

After the arrangement, \( g \) is a function of \( x_n \). There exists a constant \( t \in (0,1) \) such that \( g > 0 \) when \( x_n > t \) and \( g \leq 0 \) when \( x_n \leq t \). If \( g \) is strictly positive or negative, we take \( t = 0 \) or 1. Let \( h \) be a function on \( \partial B_1 \) which is rotationally symmetric in \( x' \), even in \( x_n \), and monotone increasing in \( x_n \) for \( x_n \in (0,1) \). It is easy to see that the rearrangement will increases the value \( u_{nn}(0) \).

Hence to compute \( \sup |D^2v(0)| \), we may assume furthermore that
\[
(3.25) \quad g = g_t = \begin{cases} 
1 & \text{when } x_n \in (t,1], \\
-1 & \text{when } x_n \in [0,t) 
\end{cases}
\]
for a different \( t \in (0,1) \). We have now the family of functions \( \{g_t\} \). From the integrand in (3.24), one easily verifies that among all the functions \( g_t \), the sup is attained when \( t = \frac{1}{\sqrt{n}} \). Therefore
\[
(3.26) \quad u_{nn}(0) \leq \frac{n + 2}{\omega_n} \left\{ \int_{S^{n-1} \cap \{|x_n| > \frac{1}{\sqrt{n}}\}} (ny_n^2 - 1) - \int_{S^{n-1} \cap \{|x_n| < \frac{1}{\sqrt{n}}\}} (ny_n^2 - 1) \right\}.
\]
Notice that \( u_{nn}(0) \) is invariant if we add a constant to \( g \). Hence
\[
\int_{S^{n-1}} (ny_n^2 - 1) = 0.
\]
We obtain
\[
u_{nn}(0) \leq \frac{2(n + 2)}{\omega_n} \int_{S^{n-1} \cap \{|x_n| > \frac{1}{\sqrt{n}}\}} (ny_n^2 - 1).
\]
Denote \( r = |x'|, a = \sqrt{1 - \frac{1}{n}} \). Then \( y_n^2 = 1 - r^2 \) and
\[
\int_{S^{n-1} \cap \{|x_n| > \frac{1}{\sqrt{n}}\}} (ny_n^2 - 1) = 2\omega_n - 1 \int_0^a \frac{n(1 - r^2) - 1}{\sqrt{1 - r^2}} r^{n-2} dr.
\]
We have
\[
\int_0^a \frac{r^n}{\sqrt{1 - r^2}} = \frac{n - 1}{n} \int_0^a \frac{r^{n-2}}{\sqrt{1 - r^2}} - \frac{1}{n} a^{n-1} \sqrt{1 - a^2}.
\]
Hence
\[
\int_0^a \frac{n(1 - r^2) - 1}{\sqrt{1 - r^2}} r^{n-2} dr = a^{n-1} \sqrt{1 - a^2}.
\]
We obtain
\[
u_{nn}(0) \leq \frac{4(n + 2)\omega_n - 1}{\omega_n} a^{n-1} \sqrt{1 - a^2} = \frac{4(n + 2)\omega_n - 1}{\omega_n \sqrt{n}} \left( \frac{n - 1}{n} \right)^{n-1}.
\]
This completes the proof. \( \square \)

The upper bound in (3.23) can be simplified.
Lemma 3.4. Let $\beta_n$ be given in (3.17). Then $\beta_2 = \frac{4}{\pi}$, $\beta_3 = \frac{20\sqrt{3}}{9}$, $\beta_4 = \frac{9\sqrt{3}}{\pi}$, $\beta_5 = \frac{168}{25}\sqrt{\frac{4}{5}}$, and for $n \geq 6$,

(3.27) \quad \beta_n < n + 2.

Proof. Since $\omega_2 = 2\pi$, $\omega_3 = 4\pi$, we have obviously $\beta_2 = \frac{4}{\pi}$, $\beta_3 = \frac{20\sqrt{3}}{9}$.

For $n \geq 4$, denote

(3.28) \quad \beta_n^* := \frac{\beta_n}{n+2} = \frac{4}{\sqrt{n-1}} \frac{\omega_{n-1} (n-1)^n}{\omega_n}.

We have

$$
\omega_n = 2 \int_{|x'|<1} \frac{dx'}{\sqrt{1 - |x'|^2}} = 2\omega_{n-1} \int_0^1 \frac{r^{n-2}}{\sqrt{1-r^2}} dr,
$$

where $x' = (x_1, \cdots, x_{n-1})$. Integration by parts gives

$$
\int_0^1 \frac{r^{n-2}}{\sqrt{1-r^2}} dr = \frac{n-3}{n-2} \int_0^1 \frac{r^{n-4}}{\sqrt{1-r^2}} dr.
$$

Hence we have

(3.29) \quad \int_0^1 \frac{r^{n-2}}{\sqrt{1-r^2}} dr = \begin{cases}
\frac{n-3}{n-2} \cdots \frac{2}{3} \pi & \text{if } n \text{ is odd},
\frac{n-3}{n-2} \cdots \frac{1}{2} \pi & \text{if } n \text{ is even}.
\end{cases}

By direct computation, we obtain $\beta_4 = \frac{9\sqrt{3}}{\pi}$, $\beta_5 = \frac{168}{25}\sqrt{\frac{4}{5}}$, and (3.27) for $n \leq 10$. When $n > 10$, by (3.28) we have

$$
\beta_n^* < \frac{4}{\sqrt{e}\sqrt{n-1}} \frac{\omega_{n-1}}{\omega_n}.
$$

If $n = 2k > 10$ is even, then

$$
\frac{\omega_{n-1}}{\omega_n} = \frac{1}{\pi} \frac{(n-2)(n-4)\cdots2}{(n-3)(n-5)\cdots1} := \frac{1}{\pi} I_n.
$$

Since $\frac{k+1}{k} \leq \frac{k}{k-1}$ for all $k$,

$$
I_n < \left(\frac{n-2}{n-3} \cdots \frac{10}{9} \frac{9}{8}\right)^{1/2} \frac{8642}{7531} = \frac{\sqrt{n-2} 8642}{\sqrt{8} 7531}.
$$

We obtain

$$
\beta_n^* \leq \sqrt{\frac{286421}{e7531}} \pi < 1.
$$
If \( n = 2k + 1 \) is odd, then similarly
\[
\beta_n^* \leq \frac{2}{3\sqrt{e}} \sqrt[9]{8 6 4 2} < 1.
\]
This proves (3.27). □

Let \( \varepsilon > 0 \) be sufficiently small, from (3.22) and Lemma 3.4 we have
\[
\theta < \frac{1}{2}
\]
(3.30)
in all dimensions. From (3.30) and (3.13), we see that \( Du \) is log-Lipschitz continuous when \( \nu(t) \leq \frac{1}{|\log \mathcal{P}|} \). This completes the proof of Theorem 2.

§4. Remarks

The log-Lipschitz continuity in Theorem 2 can be used to prove the local existence of continuous solutions to the Cauchy problem of the semi-geostrophic equation, which is the transport equation
\[
\partial_t \rho + \nabla(v \rho) = 0, \quad v = \nabla^\perp u
\]
coupled with the Monge-Ampère equation
\[
\det(D^2 u + I) = \rho,
\]
where \( \nabla^\perp u = (-u_{x_2}, u_{x_1}) \) in \( \mathbb{R}^2 \) or \( \nabla^\perp u = (-u_{x_2}, u_{x_1}, 0) \) in \( \mathbb{R}^3 \) [BB, C5, CRD, CuF, E, L]. We assume the initial condition \( \rho(\cdot, 0) = \rho_0 \in C^0(\mathbb{R}^n) \) and the boundary condition \( T_u(\mathbb{R}^n) = \Omega^* \), where \( T_u(x) = Du(x) + x \) and \( \Omega^* \) is a given bounded, convex domain. As in [L] we consider the periodic case. That is for any \( \vec{p} \in \mathbb{Z}^n \), \( \rho_0(x + \vec{p}) = \rho_0(x) \).

As the quasi-geostrophic equation, which is the transport equation (4.1) coupled with the equation \( u = (-\Delta)^{-\frac{1}{2}} \rho \) [CF], the semi-geostrophic equation has also been used as an approximation to the Euler equation [BB, C5, CRD, CuF, E, L]. Recall that for the 2-d Euler equation, which can be written as a system of the transport equation (4.1) and the Posvion equation \( u = (-\Delta)^{-1} \rho \), the global existence and uniqueness of smooth solutions were first derived by using the log-Lipschitz continuity for the Possion equation [Y]. For the semi-geostrophic equation, since the log-Lipschitz estimate (1.11) depends of the modulus of continuity of \( \log f \), our Theorem 2 implies the local existence of continuous solutions when the initial data \( \rho_0 \) satisfies
\[
\omega_{\log \rho_0}(r) < \frac{1}{|\log r|}
\]
for small \( r \). The proof is similar to that in [L], where it is assumed that \( f \) is Dini continuous. The local existence of continuous solutions was first obtained in [L] when \( \rho_0 \) is Dini continuous. We also refer the reader to [Br, C4] for the existence and regularity of solutions to the Monge-Ampère equation (4.2) subject to the boundary condition \( T_u(\Omega) = \Omega^* \), where \( \Omega \) is a convex domain in \( \mathbb{R}^n \).

Next we remark that the \( L^\infty \)-oscillation \( \omega_{\log f} \) in the estimate (3.9) can be replaced by the \( L^1 \)-oscillation, using the following \( L^\infty \)-estimate for the Monge-Ampère equation.
Lemma 4.1. Let \( u \) (resp \( \overline{u} \)) be a solution to the Monge-Ampère equation \( \det D^2 u = f \geq 0 \) (resp \( \overline{f} \)) in \( \Omega \). Suppose \( u = \overline{u} \) on \( \partial \Omega \). Then we have the estimate

\[
(4.4) \quad \sup_{\Omega} |u - \overline{u}| \leq C|f - \overline{f}|^{1/n}_{L^1(\Omega)},
\]

where \( C \) depends only on \( n \) and \( \text{diam}(\Omega) \).

Proof. Let \( \varphi \) be the convex solution to

\[
\det D^2 \varphi = |f^{1/n} - \overline{f}^{1/n}|^n \quad \text{in} \quad \Omega, \\
\varphi = 0 \quad \text{on} \quad \partial \Omega.
\]

Then we have the estimate

\[
(4.5) \quad |\inf \varphi|^n \leq C \int_{\Omega} |f - \overline{f}|.
\]

Estimate (4.5) can be established easily as follows. Suppose \( \inf \varphi \) is attained at \( x_0 \in \Omega \). Consider a convex function \( \psi \) whose graph is the convex cone with vertex at \( (x_0, \varphi(x_0)) \) and base \( \partial \Omega \times \{x_{n+1} = 0\} \). Then \( \psi = \varphi \) on \( \partial \Omega \) and \( \psi \geq \varphi \) in \( \Omega \). Denote by \( N_{\varphi} \) the normal mapping of \( \varphi \) [P]. Then \( N_{\psi}(\Omega) \subseteq N_{\varphi}(\Omega) \). It follows that

\[
|\inf \varphi|^n \leq C|N_{\psi}(\Omega)| \leq C|N_{\varphi}(\Omega)| = C \int_{\Omega} |f^{1/n} - \overline{f}^{1/n}|^n.
\]

Noting that \( |f^{1/n} - \overline{f}^{1/n}|^n \leq |f - \overline{f}| \), we obtain

\[
|\inf \varphi|^n \leq C \int_{\Omega} |f - \overline{f}|,
\]

where \( C \) depends only on \( n \) and \( \text{diam}(\Omega) \).

Next observe that the Monge-Ampère operator \( M(u) = \det^{1/n} D^2 u \) is concave, we have \( M(\frac{u+v}{2}) \geq \frac{1}{2}(M(u) + M(v)) \) for any convex function \( u, v \). Hence

\[
\det^{1/n} D^2 (\overline{u} + \varphi) \geq \det^{1/n} D^2 \overline{u} + \det^{1/n} D^2 \varphi \geq f^{1/n}.
\]

Similarly, \( \det^{1/n} D^2 (u + \varphi) \geq \overline{f}^{1/n} \). By the comparison principle, \( |u - \overline{u}| \leq |\varphi| \). We obtain the estimate. \( \Box \)

Finally we would like to point out that estimate (1.11) should be optimal, in the sense that the gradient \( Du \) may not be log-Lipschitz continuous if \( \omega_{\log f} \geq \frac{C}{|\log r|} \) for some large \( C \). Indeed, consider the case when \( u \) and \( f \) are even functions in dimension two such that \( u \) and \( u_i \) (solutions of (3.1)) attain their minimum at \( 0 \). Then for an appropriate \( f \), there is a positive constant \( c_0 > 0 \) independent of \( i \) such that \( |\tilde{u}_i - u^*_{i+1}| \geq c_0 \nu_i \) and \( \lambda_{\max}(T_i) \geq 1 + c_0 \nu_i \) (see (3.8) and the formula before it).
References


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