ON THE SECOND BOUNDARY VALUE
PROBLEM FOR MONGE-AMPÈRE TYPE
EQUATIONS AND OPTIMAL TRANSPORTATION

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Abstract. This paper is concerned with the existence of globally smooth solutions for the second boundary value problem for Monge-Ampère equations and the application to regularity of potentials in optimal transportation. The cost functions satisfy a weak form of our condition A3, under which we proved interior regularity in a recent paper with Xi-nan Ma. Consequently they include the quadratic cost function case of Caffarelli and Urbas as well as the various examples in the earlier work. The approach is through the derivation of global estimates for second derivatives of solutions.

1. Introduction

This paper is concerned with the global regularity of solutions of the second boundary value problem for equations of Monge-Ampère type and its applications to the regularity of potentials in optimal transportation problems with non-quadratic cost functions.

The Monge-Ampère equations under consideration have the general form

\begin{equation}
\det \{ D^2u - A(\cdot, u, Du) \} = B(\cdot, u, Du),
\end{equation}

where \( A \) and \( B \) are given \( n \times n \) matrix and scalar valued function defined on \( \Omega \times \mathbb{R} \times \mathbb{R}^n \), where \( \Omega \) is a domain in Euclidean \( n \)-space, \( \mathbb{R}^n \). We use \((x, z, p)\) to denote points in \( \Omega \times \mathbb{R} \times \mathbb{R}^n \) so that \( A(x, z, p) \in S^{n \times n}, B(x, z, p) \in \mathbb{R} \) and \((x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \). The equation (1.1) will be elliptic, (degenerate elliptic), with respect to a solution \( u \in C^2(\Omega) \) whenever

\begin{equation}
D^2u - A(\cdot, u, Du) > 0 \quad (\geq 0),
\end{equation}

whence also \( B > 0 \) (\( \geq 0 \)).
A special case of (1.1) arises from the prescription of the Jacobian determinant of a mapping $Tu$ defined by

$$\text{(1.3)} \quad Tu = Y(\cdot, u, Du),$$

where $Y$ is a given vector valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, namely

$$\text{(1.4)} \quad \det DY(\cdot, u, Du) = \psi(\cdot, u, Du).$$

Assuming that the matrix

$$\text{(1.5)} \quad Y_p = [D_p Y^i]$$

is non-singular, we may write (1.4) in the form (1.1), that is,

$$\text{(1.6)} \quad \det\{D^2 u + Y_p^{-1}(Y_x + Y_z \otimes Du)\} = \frac{\psi}{|\det Y_p|},$$

for degenerate elliptic solutions $u$.

The second boundary value problem for equation (1.4) is to prescribe the image

$$\text{(1.7)} \quad Tu(\Omega) = \Omega^\ast,$$

where $\Omega^\ast$ is a given domain in $\mathbb{R}^n$. When $Y$ and $\psi$ are independent of $z$ and $\psi$ is separable in the sense that

$$\text{(1.8)} \quad \psi(x, p) = f(x)/g \circ Y(x, p)$$

for positive $f, g \in L^1(\Omega), L^1(\Omega^\ast)$ respectively, then a necessary condition for the existence of an elliptic solution to the second boundary value problem (1.4) (1.7) is the mass balance condition

$$\text{(1.9)} \quad \int_{\Omega} f = \int_{\Omega^\ast} g.$$

The second boundary value problem (1.4) (1.7) arises naturally in optimal transportation. Here we are given a cost function $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and the vector field $Y$ is generated by the equation

$$\text{(1.10)} \quad c_x(x, Y(x, p)) = p,$$

which we assume to be uniquely solvable for $p \in \mathbb{R}^n$, with non-vanishing determinant, that is

$$\text{(1.11)} \quad \det c_{x,y}(x, y) \neq 0$$
for all \( x, y \in \Omega \times \Omega^* \). Using the notation

\[
\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \ldots \frac{\partial}{\partial y_k} \frac{\partial}{\partial y_l} \ldots c
\]

we have

\[
Y_p(x, p) = [c^{i,j}(x, Y(x, p))],
\]

where \([c^{i,j}]\) is the inverse of \([c_{i,j}]\). The corresponding Monge-Ampère equation can now be written as

\[
\det \{D^2u - c_{xx}(\cdot, Y(\cdot, Du))\} = |\det c_{x,y}|\psi,
\]

that is in the form (1.1) with

\[
A(x, z, p) = c_{xx}(x, Y(x, p)),
\]

\[
B(x, z, p) = |\det c_{x,y}(x, Y(x, p))|\psi(x, z, p).
\]

In the case of the (quadratic) cost function

\[
c(x, y) = x \cdot y,
\]

we have

\[
Y(x, p) = p, \quad Tu = Du,
\]

and equation (1.14) reduces to the standard Monge-Ampère equation

\[
\det D^2u = \psi.
\]

For this case global regularity of solutions was proved by Delanoë [D], Caffarelli [C2] and Urbas [U1], with (conditional) interior regularity shown earlier by Caffarelli [C1]. Interior regularity for a class of non-quadratic cost functions was proved in [MTW]. In this paper we will prove global estimates and regularity under corresponding conditions. In particular, we will assume that the cost function \( c \in C^4(\mathbb{R}^n \times \mathbb{R}^n) \) satisfies the following conditions:

1. **(A1)** For each \( p, q \in \mathbb{R}^n \), there exists unique \( y = Y(x, p), x = X(q, y) \) such that

   \[
   c_x(x, y) = p \quad \forall x \in \Omega,
   \]

   \[
   c_y(x, y) = q \quad \forall y \in \Omega^*.
   \]

2. **(A2)** \( \det c_{x,y}(x, y) \neq 0, \quad \forall x \in \overline{\Omega}, y \in \overline{\Omega}^* \).
Conditions A1 and A2 are precisely the same conditions in [MTW] but condition A3w is the degenerate form of condition A3 in [MTW].

In our paper [MTW], we also introduced a notion of convexity of domains with respect to cost functions, namely $\Omega$ is $c$-convex, with respect to $\Omega^*$, if the image $c_y(\cdot, y)(\Omega)$ is convex in $\mathbb{R}^n$ for each $y \in \Omega^*$, while analogously $\Omega^*$ is $c^*$-convex, with respect to $\Omega$, if the image $c_x(x, \cdot)(\Omega^*)$ is convex for each $x$ in $\Omega$. For global regularity we need to strengthen these conditions, in the same way that convexity is strengthened to uniform convexity. Namely we define $\Omega$ to be uniformly $c$-convex, with respect to $\Omega^*$, if $\Omega$ is $c$-convex, with respect to $\Omega^*$, $\partial \Omega \in C^2$ and there exists a positive constant $\delta_0$ such that

\[
[D_i \gamma_j(x) - c^j_{i,k} c_{ij,l}(x,y) \gamma_k(x)] \tau_i \tau_j(x) \geq \delta_0
\]

for all $x \in \partial \Omega$, $y \in \Omega^*$, unit tangent vector $\tau$ and outer unit normal $\gamma$. Similarly we call $\Omega^*$ uniformly $c^*$-convex, with respect to $\Omega$, when $c^*(x,y) = c(y,x)$.

We can now formulate our main theorem.

**Theorem 1.1.** Let $c$ be a cost function satisfying hypotheses A1, A2, A3w, with respect to bounded $C^4$ domains $\Omega, \Omega^* \in \mathbb{R}^n$ which are respectively uniformly $c$-convex, $c^*$-convex with respect to each other. Let $\psi$ be a positive function in $C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. Then any elliptic solution $u \in C^3(\Omega)$ of the second boundary value problem (1.14), (1.7) satisfies the a priori estimate

\[
|D^2 u| \leq C,
\]

where $C$ depends on $c, \psi, \Omega$ and $\Omega^*$.

As we will indicate later, the smoothness assumption on the solution and the data may be reduced. Further regularity also follows from the theory of linear elliptic equations for example if $c, \Omega, \Omega^*, \psi$ are $C^\infty$ then the solution $u \in C^\infty(\overline{\Omega})$.

As a consequence of Theorem 1.1, we may conclude existence theorems for classical solutions.

**Theorem 1.2.** Suppose in addition to the above hypotheses that the function $\psi$ satisfies (1.8) (1.9). Then there exists a unique (up to additive constants) elliptic solution $u \in C^3(\overline{\Omega})$ of the second boundary value problem (1.14), (1.7).

From Theorem 1.2, we also obtain an existence result for classical solutions of the Monge-Kantorovich problem in optimal transportation. As above we let $c \in C^4(\mathbb{R}^n \times \mathbb{R}^n)$ be a cost function and $\Omega, \Omega^*$ be two bounded domains in $\mathbb{R}^n$ satisfying the hypotheses

\[
D_{p,q} A_{kl}(x,p) \xi_i \xi_j \eta_k \eta_l \geq 0 \quad \forall \ x \in \Omega, p \in \mathbb{R}^n, \xi \perp \eta \in \mathbb{R}^n.
\]
of Theorem 1.1. Let $f > 0 \in C^2(\Omega)$, $g > 0 \in C^2(\Omega^*)$ be positive densities satisfying the mass balance condition (1.9). Then the corresponding optimal transportation problem is to find a measure preserving mapping $T_0 : \Omega \to \Omega^*$ which maximizes the cost functional

$$C(T) = \int_{\Omega} f(x)c(x, T(x))dx$$

among all measure preserving mappings $T$ from $\Omega$ to $\Omega^*$. A mapping $T : \Omega \to \Omega^*$ is called measure preserving if it is Borel measurable and for any Borel set $E \subset \Omega^*$,

$$\int_{T^{-1}(E)} f = \int_E g.$$

**Theorem 1.3.** Under the above hypotheses, there exists a unique maximizer $T \in C^2(\overline{\Omega})$ to the functional (1.21), given by

$$T(x) = Y(x, Du(x)),$$

where $u$ is an elliptic solution of the boundary problem (1.7), (1.14).

The solution $u$ of (1.7), (1.14) is called a potential. Note that in [MTW] the cost function and potentials are the negatives of those here and the optimal transportation problem is written, (in its usual form), as a minimization problem.

The plan of this paper is as follows. In Section 2, we prove that boundary conditions of the form (1.7) are oblique with respect to functions for which the Jacobian $DT$ is non-singular and we estimate the obliqueness for solutions of the boundary value problem (1.14), (1.7) under hypotheses A1 and A2, (Theorem 2.1). Here the twin assumptions of $\Omega$ and $\Omega^*$ being $c$ and $c^*$-convex with respect to each other are critical. In Section 3, we prove that second derivatives of solutions of equation (1.14) can be estimated in terms of their boundary values under hypothesis A3w, (Theorem 3.1). This estimation is already immediate from [MTW] when the non-degenerate condition A3 is satisfied. The argument is carried out for equations of the general form (1.1) (with symmetric $A$), in the presence of a global barrier which is not necessary in the optimal transportation case (Theorem 3.2). This estimation is also used in our treatment of the classical Dirichlet problem in [TW]. The proof of the global second derivative estimates for solutions of the boundary value problem (1.14), (1.7) is completed in Section 4. Here the procedure is similar to that in [LTU] and [U1]. In Section 5, we complete the proof of the existence result, Theorem 1.2, by adapting the method of continuity [GT]. Section 6 is devoted to the applications to optimal transportation and the proof of Theorem 1.3, which implies the global regularity of the potential functions in [MTW], with condition A3 also relaxed to A3w. Finally in Section 7, we discuss our results in the light of examples already given in [MTW].
2. Obliqueness

In this section, we prove that the boundary condition (1.7) implies an oblique boundary condition and estimate the obliqueness. First we recall that a boundary condition of the form

\[(2.1)\quad G(x, u, Du) = 0 \text{ on } \partial \Omega\]

for a second order partial differential equation in a domain \(\Omega\) is called oblique if

\[(2.2)\quad G_p \cdot \gamma > 0 \quad \text{for all } (x, z, p) \in \partial \Omega \times \mathbb{R} \times \mathbb{R}^n,\]

where \(\gamma\) denotes the unit outer normal to \(\partial \Omega\). Let us now assume that \(\varphi\) and \(\varphi^*\) are \(C^2\) defining functions for \(\Omega\) and \(\Omega^*\) respectively, with \(\varphi, \varphi^* < 0\) near \(\partial \Omega, \partial \Omega^*,\) \(\varphi = 0\) on \(\partial \Omega,\) \(\varphi^* = 0\) on \(\partial \Omega^*,\) \(\nabla \varphi, \nabla \varphi^* \neq 0\) near \(\partial \Omega, \partial \Omega^*\). Then if \(u \in C^2(\Omega)\) is an elliptic solution of the second boundary value problem (1.4), (1.7), we must have

\[(2.2)\quad \varphi^*(Tu) = 0 \text{ on } \partial \Omega, \quad \varphi^*(Tu) < 0 \text{ near } \partial \Omega.\]

By tangential differentiation, we obtain

\[(2.3)\quad \varphi^*_i (D_j T^i u) \tau_j = 0\]

for all unit tangent vectors \(\tau\), whence

\[(2.4)\quad \varphi^*_i (D_j T^i) = \chi \gamma_j\]

for some \(\chi \geq 0\). Consequently

\[(2.5)\quad \varphi^*_i c^{i,k} (u_{jk} - c_{jk}) = \chi \gamma_j,\]

that is

\[(2.6)\quad \varphi^*_i c^{i,k} w_{jk} = \chi \gamma_j,\]

where

\[(2.7)\quad w_{ij} = u_{ij} - c_{ij}.\]

At this point we observe that \(\chi > 0\) on \(\partial \Omega\) since \(|\nabla \varphi^*| \neq 0\) on \(\partial \Omega\) and \(\det DT \neq 0\). Using the ellipticity of (1.4) and letting \([w^{ij}]\) denote the inverse matrix of \([w_{ij}]\), we then have

\[(2.8)\quad \varphi^*_i c^{i,k} \gamma_j = \chi w^{jk} \gamma_j.\]
Now writing

\[(2.9)\]
\[G(x, p) = \varphi^*(Y(x, p)),\]
we have

\[(2.10)\]
\[\beta_k := G_{p_k}(\cdot, Du) = \chi w^{jk}\gamma_j,\]
whence

\[(2.11)\]
\[\beta \cdot \gamma = \chi w^{ij}\gamma_i\gamma_j > 0\]
on \(\partial \Omega\). We obtain a further formula for \(\beta \cdot \gamma\), from (2.6), namely

\[(2.12)\]
\[\varphi^*_i c^{i,k}w_{jk}\varphi^*_l c^{l,j} = \chi\varphi^*_i c^{i,j}\gamma_j = \chi(\beta \cdot \gamma).\]

Eliminating \(\chi\) from (2.11) and (2.12), we have

\[(2.13)\]
\[(\beta \cdot \gamma)^2 = (w^{ij}\gamma_i\gamma_j)(w_{kl}c^{i,k}c^{j,l}\varphi^*_i \varphi^*_j).\]

We call (2.13) a formula of Urbas type, as it was proved by Urbas [U1] for the special case, \(c(x, y) = x \cdot y, Y(\xi, p) = p\), of the Monge-Ampère equation. Note that to prove (2.13), we only used conditions A1 and A2 and moreover (2.13) continues to hold in the generality of (1.4).

Our main task now is to estimate \(\beta \cdot \gamma\) from below for solutions of (1.14), (1.7). For this in addition to conditions A1, A2, we also need the uniform \(c\) and \(c^*\) convexity of \(\Omega\) and \(\Omega^*\) respectively. Our approach is similar to [U1] for the special case of the Monge-Ampère equation and begins by invoking the key idea from [T] for estimating double normal derivatives of solution of the Dirichlet problem. Namely we fix a point \(x_0\) on \(\partial \Omega\) where \(\beta \cdot \gamma\) is minimized, for an elliptic solution \(u \in C^3(\Omega)\), and use a comparison argument to estimate \(\gamma \cdot D(\beta \cdot \gamma)\) from above. Without some concavity condition in \(p\) the quantity \(\beta \cdot \gamma\) does not satisfy a nice differential inequality so we will get around this by considering instead the function

\[(2.14)\]
\[v = \beta \cdot \gamma - \kappa \varphi^*(Tu)\]
for sufficiently large \(\kappa\), where now the defining function \(\varphi^*\) is chosen so that

\[(2.15)\]
\[(D_{ij}\varphi^*(Tu) - c^{k,l}c_{l,ij}(\cdot, Tu)D_k\varphi^*(Tu)\xi_i\xi_j \geq \delta_0|\xi|^2)\]
near \( \partial \Omega \), for all \( \xi \in \mathbb{R}^n \) and some positive constant \( \delta^*_0 \). Inequality (2.15) is possible by virtue of the uniform \( c^* \)-convexity of \( \Omega^* \). By differentiation of equation (1.14), in the form (1.1), we obtain, for \( r = 1, \ldots, n \),

\[
(2.16) \quad w^{ij} \{ D_{ij} u_r - D_{pk} A_{ij} (x, Du) D_k u_r - D_{x_r} A_{ij} (x, Du) \} = D_r \log B.
\]

Introducing the linearized operator \( L \),

\[
(2.17) \quad Lv = w^{ij} (D_{ij} v - D_{pk} A_{ij} D_k v),
\]

we need to compute \( Lv \) for \( v \) given by (2.14). Setting

\[
(2.18) \quad F(x, p) = G_p (x, p) \cdot \gamma (x) - \kappa G(x, p),
\]

where \( G \) is defined by (2.9), we see that

\[
(2.19) \quad v(x) = F(x, Du(x)).
\]

Writing \( b^{ij}_k = - D_{pk} A_{ij} \), we then have

\[
(2.20) \quad Lv = w^{ij} \{ F_{pk} D_{ij} u_r + F_{p_r p_s} D_{ir} u D_{js} u \\
+ F_{x_i x_j} + 2 F_{x_i p_r} D_{jr} u + b^{ij}_k (F_{x_k} + F_{p_r} D_{k u_r}) \}
\]

In the ensuing calculations, we will often employ the following formulae,

\[
(2.21) \quad c^{i,j}_k (x, y) = D_{x_k} c^{i,j} (x, y) \\
= - c^{i,l} c^{r,j} c_{kl,r} (x, y),
\]

\[
= c^{i,l} c^{r,j} c_{l,kr} (x, y),
\]

as well as (1.13). Indeed, using (1.13) and (2.21), we have

\[
(2.22) \quad G_{p_r p_j} = D_{pj} (\varphi^* k^{i,k}) \\
= \varphi^* k^{i,k} c^{r,j} - \varphi^* c^{s,j} c^{k,i} \\
= c^{k,i} c^{l,j} \{ \varphi_{kl} - \varphi^* c^{r,s} c_{s,kl} \}
\]

so that

\[
(2.23) \quad G_{p_r p_j} (x, Du) \xi_i \xi_j \geq \delta^*_0 \sum |c^{i,j} \xi_j|^2 \\
\geq \kappa^*_0 |\xi|^2
\]
for a further positive constant $\kappa_0^*$. By choosing $\kappa$ sufficiently large, we can then ensure that

\begin{equation}
F_{p,p_j}(x, Du)\xi_i\xi_j \leq -\frac{1}{2}\kappa|\xi|^2
\end{equation}

near $\partial\Omega$. Substituting into (2.20) and using (2.16), it follows that

\begin{equation}
Lv \leq -\frac{1}{4}\kappa w_{ii} + C(w^{ii} + 1) + D_{pk}\log BD_kv,
\end{equation}

where $C$ is a constant depending on $c, \psi, \Omega$ and $\Omega^*$, as well as $\kappa$.

A suitable barrier is now provided by the uniform $c$-convexity of $\Omega$ which implies, analogously to the case of $\Omega^*$ above, that there exists a defining function $\varphi$ for $\Omega$ satisfying

\begin{equation}
[D_{ij}\varphi - c_l^k c_{ij,l}(x, Tu)D_k\varphi]\xi_i\xi_j \geq \delta_0|\xi|^2,
\end{equation}

near $\partial\Omega$, (for a constant $\delta_0 > 0$), whence, by virtue of (2.21), we have

\begin{equation}
L\varphi \geq \delta_0 w_{ii}.
\end{equation}

Combining (2.25) and (2.27), and using the positivity of $B$, we then infer by the usual barrier argument [GT]

\begin{equation}
\gamma \cdot Dv(x_0) \leq C,
\end{equation}

where again $C$ is a constant depending on $c, \Omega, \Omega^*$ and $\psi$. From (2.28) and since $x_0$ is a minimum point of $v$ on $\partial\Omega$, we can write

\begin{equation}
Dv(x_0) = \tau \gamma(x_0)
\end{equation}

where $\tau \leq C$. To use the information embodied in (2.29), we need to calculate

\begin{equation}
D_i(\beta \cdot \gamma) = D_i\{\varphi_k^* c^{k,j} \gamma_j\}
\end{equation}

Multiplying by $\varphi_i^* c^{i,j}$ and summing over $i$, we obtain

\begin{equation}
\varphi_i^* c^{i,j} D_i(\beta \cdot \gamma) = \varphi_k^* \varphi_i^* c^{k,j} c^{i,j}(D_{ij}\gamma_j - c^{s,r} c_{ij,s} \gamma_r) + \varphi_i^* c^{i,j} c^{k,j} \gamma_j D_{ij} T_k u
\end{equation}

\begin{equation}
\geq \delta_0 \sum |\varphi_i^* c^{i,j}|^2
\end{equation}
by virtue of the uniform \( c \)-convexity of \( \Omega \), the \( c^* \)-convexity of \( \Omega^* \) and (2.6). Consequently, from (2.18) and (2.29), we obtain at \( x_0 \),

\[
(2.32) \quad -\kappa w_{kl} c^{i,k} c^{j,l} \phi_i \phi_j^* \leq C(\beta \cdot \gamma) - \tau_0
\]

for positive constants, \( C \) and \( \tau_0 \). Hence if \( \beta \cdot \gamma \leq \tau_0 / 2C \), we have the lower bound

\[
(2.33) \quad w_{kl} c^{i,k} c^{j,l} \phi_i \phi_j^* \geq \frac{\tau_0}{2\kappa}.
\]

To complete the estimation of \( \beta \cdot \gamma \) we invoke the dual problem to estimate \( w_{ij} \gamma_i \gamma_j \) at \( x_0 \). Assuming for the moment that \( Au \) is one to one, we let \( u^* \) denote the \( c \)-transform of \( u \), defined for \( y = Tu(x) \in \Omega^* \) by

\[
(2.34) \quad u^*(y) = c(x, y) - u(x).
\]

It follows that

\[
(2.35) \quad Du^*(y) = c_y(x, y) = c_y(T^* u^*(y), y),
\]

where

\[
(2.36) \quad T^* u^*(y) = X(Du^*, y) = (Tu)^{-1}(y),
\]

and the second boundary value problem (1.14), (1.7) is equivalent to

\[
(2.37) \quad |\det D_y(T^* u^*)| = g(y)/f(T^* u^*) \quad \text{in} \quad \Omega^*,
\]
\[
(2.38) \quad T^* \Omega^* = \Omega.
\]

Noting that the defining functions \( \varphi \) and \( \varphi^* \) may be chosen so that \( \nabla \varphi = \gamma, \nabla \varphi^* = \gamma^* \) on \( \partial \Omega, \partial \Omega^* \) respectively, we clearly have for \( x \in \partial \Omega, y \in Tu(x) \in \partial \Omega^* \),

\[
(2.39) \quad \beta \cdot \gamma(x) = c^{k,i}(x, y) \varphi_i \varphi^*_k(y) = \beta^* \cdot \gamma^*(y),
\]

where

\[
(2.40) \quad \beta^*(y) = D_q \varphi(Y^*(D_y u^*, y)).
\]
Hence the quantity $\beta^* \cdot \gamma^*$ is minimized on $\partial \Omega^*$ at the point $y_0 = Tu(x_0)$. Furthermore, for $y = Tu(x), x \in \partial \Omega$,

$$(2.41) \quad w^{ij} \gamma_i \varphi_j(x) = w^*_{kl}(y)c^{k,i}c^{l,j}(x,y)\varphi_i\varphi_j(x),$$

where

$$(2.42) \quad w^*_{kl}(y) = u^*_{yk}y_l(y) - c_{kl}(x,y).$$

Applying now the estimate (2.33) to $u^*$ at the point $y_0 \in \partial \Omega^*$, we finally conclude from (2.13) the desired obliqueness estimate

$$(2.43) \quad \beta \cdot \gamma \geq \delta$$

on $\partial \Omega$ for some positive constant $\delta$ depending only on $\Omega, \Omega^*, c, \text{and } \psi$.

The above argument clearly extends to arbitrary positive terms $B$ in (1.15). Accordingly we have the following theorem.

**Theorem 2.1.** Let $c \in C^3(\mathbb{R}^n \times \mathbb{R}^n)$ be a cost function satisfying hypotheses A1, A2, with respect to bounded $C^3$ domains $\Omega, \Omega^* \subset \mathbb{R}^n$, which are respectively uniformly $c$-convex, $c^*$-convex with respect to each other. Let $\psi$ be a positive function in $C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Then any elliptic solution $u \in C^3(\Omega)$, for which $Tu$ is one-to-one, of the second boundary value problem (1.14), (1.7) satisfies the obliqueness estimate (2.43).

Later on, we will show that $Tu$ is automatically one-to-one under the hypotheses of Theorem 2.1. In an ensuing work [TW] we will consider the extension of Theorem 2.1 to the more general prescribed Jacobian equation (1.4). The main difference is that we cannot directly use the $c$-transform to get the complementary estimate to (2.33) Instead the quantities there are transformed using the diffeomorphism $Tu$.

**3. Global second derivative bounds**

In this section we show that the second derivatives of elliptic solutions of equation (1.14) may be estimated in terms of their boundary values. For this estimation and the boundary estimates in the next section, it suffices to consider the general form (1.1) under the assumption that the matrix valued function $A \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfies condition A3w, that is

$$(3.1) \quad D_{p_k p_l} A_{ij}(x,z,p) \xi_i \xi_j \eta_k \eta_l \geq 0$$

for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \xi, \eta \in \mathbb{R}^n, \xi \perp \eta$. We also assume $A$ is symmetric, which is the case for the optimal transportation equation (1.14). For non-symmetric $A$, we need
to modify condition A3w. When (3.1) is strengthened to the condition A3 in [MTW], that is
\[
D_{pk}A_{ij}(x, z, p)\xi_i\xi_j \eta_k \eta_l \geq \delta |\xi|^2 |\eta|^2
\]
for some constant \(\delta > 0\), for all \((x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \xi, \eta \in \mathbb{R}^n, \xi \perp \eta\), then the global second derivative estimate follows immediately from our derivation of interior estimates in [MTW]. In the general case the proof is much more complicated and we need to also assume some kind of barrier condition, namely that there exists a function \(\bar{\varphi} \in C^2(\overline{\Omega})\) satisfying
\[
[D_{ij}\bar{\varphi}(x) - D_{pk}A_{ij}(x, z, p)D_k\bar{\varphi}(x)]\xi_i\xi_j \geq \tilde{\delta}|\xi|^2
\]
for some positive \(\tilde{\delta} > 0\) and for all \(\xi \in \mathbb{R}^n, x \in \overline{\Omega}, z, p \in \text{some set } P \subset \mathbb{R} \times \mathbb{R}^n\). In general condition (3.1) implies some restriction on the domain \(\Omega\), but for the case of equations arising in optimal transportation it can be avoided by a duality argument.

Our reduction to the boundary estimation follows the approach in [GT], originating with Pogorelov, with some modification analogous to that in [LTU]. Let \(u \in C^4(\Omega)\) be an elliptic solution of equation (1.1), with \(u, Du(\Omega) \subset P\) and \(\xi\) a unit vector in \(\mathbb{R}^n\). Let \(v\) be the auxiliary function given by
\[
v = v(\cdot, \xi) = \log(w_{ij}\xi_i\xi_j) + \tau|Du|^2 + \kappa\bar{\varphi},
\]
where \(w_{ij} = D_{ij}u - A_{ij}\). By differentiation of equation (1.1), we have
\[
w^{ij}[D_{ij}u_{\xi\xi} - D_{\xi\xi}A_{ij} - (D_{zz}A_{ij})u_{\xi} - (D_{pk}A_{ij})D_ku_{\xi}] = D_{\xi\xi}\bar{B} + (D_{zz}\bar{B})u_{\xi} + (D_{pk}\bar{B})D_ku_{\xi},
\]
where \(\bar{B} = \log B\). A further differentiation yields
\[
w^{ij}[D_{ij}u_{\xi\xi} - D_{\xi\xi}A_{ij} - (D_{zz}A_{ij})u_{\xi}^2 - (D_{pk}A_{ij})D_ku_{\xi}D_{l}u_{\xi} - (D_{zz}A_{ij})u_{\xi} - (D_{pk}A_{ij})D_ku_{\xi}\xi_k - 2(D_{\xi\xi\xi}A_{ij})u_{\xi} - 2(D_{\xi\xi\xi}A_{ij})u_{\xi} - 2(D_{\xi\xi\xi}A_{ij})u_{\xi} - 2(D_{\xi\xi\xi}A_{ij})u_{\xi}] - w^{lk}w^{jl}D_{\xi}w_{ij}D_{\xi}w_{kl} = D_{\xi\xi}\bar{B} + (D_{zz}\bar{B})u_{\xi}^2 + (D_{pk}\bar{B})D_ku_{\xi}D_{l}u_{\xi} + 2(D_{\xi\xi}\bar{B})D_ku_{\xi} + 2(D_{\xi\xi}\bar{B})D_ku_{\xi} + 2(D_{\xi\xi}\bar{B})D_ku_{\xi}.
\]
Furthermore differentiating (3.4) we have
\[
D_{ij}v = \frac{D_{ij}w_{\xi\xi}}{w_{\xi\xi}} + 2\tau D_{k}u D_{l}u + \kappa D_{ij}\bar{\varphi},
\]
and
\[
D_{ij}v = \frac{D_{ij}w_{\xi\xi}}{w_{\xi\xi}} - \frac{D_{ij}w_{\xi\xi}D_{j}w_{\xi\xi}}{w_{\xi\xi}^2} + 2\tau(D_{ik}u D_{jk}u + D_{k}u D_{ijk}u) + \kappa\bar{\varphi}.
\]
where we have written \(w_{\xi\xi} = D_{ij}w_{\xi\xi_j}\). Using condition A3w in (3.6) and retaining all terms involving third derivatives, we estimate

\[
Lu_{\xi\xi} = w^{ij}(D_{ij}u_{\xi\xi} + b_k^{ij}D_ku_{\xi\xi}) - (D_{pk}\bar{B})D_ku_{\xi\xi} \\
\geq w^{ik}w^{jl}D_{\xi}w_{ij}D_{\xi}w_{kl} - C\{(1 + w_{ii})w^{ii} + (w_{ii})^2\}
\]

where, as in the previous section, \(b_k^{ij} = -D_{pk}A_{ij}\) and \(C\) is a constant depending on the first and second derivatives of \(A\) and \(\log B\) and \(\sup_\Omega(|u| + |Du|)\). To apply A3w, we fix a point \(x \in \Omega\) and choose coordinate vectors as the eigenfunctions of the matrix \([w_{ij}]\) corresponding to eigenvalues \(0 < \lambda_1 \leq \cdots \leq \lambda_n\). Writing \(A_{ij,kl} = D_{p,n}A_{ij}\), we then estimate

\[
w^{ij}A_{ij,kl}w_{kl}w_{\xi\xi} \geq w^{ij}A_{ij,kl}w_{k\xi}w_{l\xi} - Cw^{ii}(1 + w_{ii}) \\
\geq \sum_{k \text{or } l = r} \frac{1}{\lambda_r} A_{rr,kl} (\lambda_k \xi_k)(\lambda_l \xi_l) - Cw^{ii}(1 + w_{ii}) \\
\geq -C\{w^{ii}(1 + w_{ii}) + w_{ii}\}
\]

From (3.9), we obtain also

\[
Lu_{\xi\xi} \geq w^{ik}w^{jl}D_{\xi}w_{ij}D_{\xi}w_{kl} - C\{(1 + w_{ii})w^{ii} + (w_{ii})^2\}
\]

for a further constant \(C\). Here we use equation (3.5) to control the third derivative term arising from differentiating \(A_{kl}\xi_k\xi_l\). From (3.8) and (3.10), we obtain, after some reduction,

\[
Lv \geq \frac{1}{w_{\xi\xi}} w^{ik}w^{jl}D_{\xi}w_{ij}D_{\xi}w_{kl} - \frac{1}{w_{\xi\xi}} w^{ij}D_{ij}w_{\xi\xi}D_jw_{\xi\xi} \\
+ 2\tau w_{ii} + \kappa w^{ii} - C\{\frac{1}{w_{\xi\xi}} [(1 + w_{ii})w^{ii} + (w_{ii})^2] + \tau + \kappa\}
\]

Now suppose \(v\) takes its maximum at a point \(x_0 \in \Omega\) and a vector \(\xi\), which we take to be \(e_1\). We need to control the first two terms on the right hand side of (3.11). To do this we choose remaining coordinates so that \([w_{ij}]\) is diagonal at \(x_0\). Then we estimate

\[
\Delta := \frac{1}{w_{\xi\xi}} w^{ik}w^{jl}D_{\xi}w_{ij}D_{\xi}w_{kl} - \frac{1}{w_{\xi\xi}} w^{ij}D_{ij}w_{\xi\xi}D_jw_{\xi\xi} \\
= \frac{1}{w_{11}} w^{ii}w^{ij}(D_1w_{ij})^2 - \frac{1}{w_{11}} w^{ij}(D_iw_{11})^2 \\
\geq \frac{1}{w_{11}^2} \sum_{i > 1} [2w^{ii}(D_1w_{11}^2) - w^{ii}(D_iw_{11})^2] \\
= \frac{1}{w_{11}^2} \sum_{i > 1} w^{ii}(D_iw_{11})^2 + \frac{2}{w_{11}^2} \sum_{i > 1} w^{ii}[D_iw_{11} - D_iw_{11}] [D_1w_{11} + D_iw_{11}] \\
\geq \frac{1}{w_{11}^2} \sum_{i > 1} w^{ii}(D_iw_{11})^2 + \frac{2}{w_{11}^2} \sum_{i > 1} w^{ii}[D_iA_{11} - D_1A_{11}] [2D_iw_{11} + D_iA_{11} - D_1A_{11}] \\
\geq -Cw^{ii}
\]
Combining (3.11) with (3.12) and taking \( \tau, \kappa \) sufficiently large, we finally obtain an estimate from above for \( w_{ii}(x_0) \). Accordingly, we have the following estimate.

**Theorem 3.1.** Let \( u \in C^4(\Omega) \) be an elliptic solution of equation (1.1) in \( \Omega \), with \( u, Du \in P \) Suppose the conditions \( A3w \) and (3.3) hold and \( B \) is a positive function in \( C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \). Then we have the estimate

\[
(3.13) \quad \sup_{\Omega} |D^2 u| \leq C(1 + \sup_{\partial \Omega} |D^2 u|),
\]

where the constant \( C \) depends on \( c, B, \Omega \) and \( \Omega^* \).

From the proof of Theorem 3.1 we obtain the corresponding estimate for equation (1.14), without the barrier condition (3.3).

**Theorem 3.2.** Let \( u \in C^4(\Omega) \) be an elliptic solution of equation (1.14) in \( \Omega \) with \( Tu \) one to one and \( Tu(\Omega) \subset \Omega^* \). Suppose the cost function \( c \) satisfies hypotheses \( A1, A2, A3w \) and \( B \) is a positive function in \( C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \). Then we have the estimate (3.13).

To prove Theorem 3.2, we take \( \kappa = 0 \) in the proof of Theorem 3.1, to obtain an estimate for \( w_{ii} \) in terms of \( w^{ii} \), that is

\[
(3.14) \quad w_{ii} \leq \varepsilon \sup_{\Omega} w^{ii} + C_\varepsilon,
\]

for arbitrary \( \varepsilon > 0 \), with constant \( C_\varepsilon \) also depending on \( \varepsilon \). We then conclude (3.13), in the optimal transportation case, by using the dual problem (2.37), (2.38).

The estimate (3.13) arose from our treatment of the Dirichlet problem in [TW].

### 4. Boundary estimates for the second derivatives

This part of our argument is similar to the treatment of the oblique boundary value problems for Monge-Ampère equations in [LTU, U3]. The paper [LTU] concerned the Neumann problem, utilizing a delicate argument which did not extend to other linear oblique boundary conditions. For nonlinear oblique conditions of the form (2.1) where the function \( G \) is uniformly convex in the gradient, the twice tangential differentiation of (2.1) yields quadratic terms in second derivatives which compensate for the deviation of \( \beta = G_p \) from the geometric normal and permit some technical simplification for general inhomogeneous terms \( \psi \) [U3].

First we deal with the non-tangential second derivatives. Letting \( F \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \) and \( v = F(\cdot, u, Du) \), where \( u \in C^3(\overline{\Omega}) \) is an elliptic solution of equation (1.1), we have from our calculation in Section 2,

\[
(4.1) \quad |Lv| \leq C(w^{ii} + w_{ii} + 1),
\]
where \( L \) is given by (2.17) and \( C \) is a constant depending on \( A, B, G, \Omega \) and \( |u|_{1;\Omega} \). Now using the equation (1.1) itself, we may estimate

\[
(4.2) \quad w_{ii}^{\frac{1}{n-1}} \leq C w^{ii},
\]

so that, writing \( M = \sup_{\Omega} w_{ii} \), we have from (4.1)

\[
(4.3) \quad |Lv| \leq C (1 + M) \frac{n-2}{n-1} w^{ii}.
\]

Hence, if there exists a \( C^2 \) defining function \( \varphi \) satisfying (3.3) near \( \partial \Omega \), together with \( \varphi = 0 \) on \( \partial \Omega \), we obtain by the usual barrier argument, taking \( F = G \),

\[
(4.4) \quad |D(\beta \cdot Du)| \leq C (1 + M) \frac{n-2}{n-1}
\]

on \( \partial \Omega \), so that in particular

\[
(4.5) \quad w_{\beta\beta} \leq C (1 + M) \frac{n-2}{n-1}
\]

on \( \partial \Omega \). Now for any vector \( \xi \in \mathbb{R}^n \), we have

\[
(4.6) \quad w_{\xi\xi} = w_{\tau\tau} + b(w_{\tau\beta} + w_{\beta\tau}) + b^2 w_{\beta\beta},
\]

where

\[
(4.7) \quad b = \frac{\xi \cdot \gamma}{\beta \cdot \gamma}, \quad \tau = \xi - b\beta.
\]

Suppose \( w_{\xi\xi} \) takes its maximum over \( \partial \Omega \) and tangential \( \xi \), \( |\xi| = 1 \) at \( x_0 \in \partial \Omega \) and \( \xi = e_1 \). Then from (4.5) and (4.6) and tangential differentiation of the boundary condition (2.1) we have on \( \partial \Omega \),

\[
(4.8) \quad w_{11} \leq |e_1 - b\beta|^2 w_{11}(0) + bF(\cdot, u, Du) + Ch^2 (1 + M) \frac{n-2}{n-1},
\]

for a given function \( F \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n) \). Combining (2.43), (3.3), (3.10), (4.1), (4.2), we thus obtain the third derivative estimate

\[
(4.9) \quad -D_\beta w_{11}(x_0) \leq C (1 + M) \frac{2n-3}{n-1},
\]

Differentiating (2.1) twice in a tangential direction \( \tau \), with \( \tau(x_0) = e_1 \), we obtain at \( x_0 \),

\[
(4.10) \quad (D_{p_k p_l} G) u_{1k} u_{1l} + (D_{p_k} G) u_{11k} \leq C (1 + M),
\]

whence we conclude from (4.9)

\[
(4.11) \quad \max_{\partial \Omega} |D^2 u| \leq C(1 + \sup_{\Omega} |D^2 u|) \frac{2n-3}{n-1},
\]

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by virtue of the uniform convexity of \( G \) with respect to \( p \). Taking account of the global estimate (3.13), we complete the proof of Theorem 1.1, when \( Tu \) is one-to-one. \( \square \)

Once the second derivatives are bounded, the equation (1.1) is effectively uniformly elliptic so that from the obliqueness estimate (2.43), we obtain global \( C^{2,\alpha} \) estimates from the theory of oblique boundary value problems for uniformly elliptic equations in [LT]. By the theory of linear elliptic equations with oblique boundary conditions [GT], we then infer estimates in \( C^{3,\alpha}(\Omega) \) for any \( \alpha < 1 \) from the assumed smoothness of our data. We may also have assumed that our solution \( u \in C^2(\Omega) \).

As in the previous section, the technicalities are simpler when condition A3w is strengthened to condition A3 and we also obtain local boundary estimates for the second derivatives. To see this we estimate the tangential second derivatives first by differentiating the equation (1.1) and boundary condition (2.1) twice with respect to a tangential vector field \( \tau \) near a point \( y \in \partial \Omega \). We then obtain an estimate for \( \eta D_{\tau\tau} u \), for an appropriately chosen cut-off function \( \eta \). The mixed tangential-normal second derivatives \( D_{\tau n} u \) are estimated as above by a single tangential differentiation of (2.1) so that the double normal derivative may be obtained either from (4.5) or from the equation (1.1) itself and the estimates in Section 2 for \( w^{ij}\gamma_i\gamma_j \) from below, similarly to the Dirichlet problem, see [T1].

5. Proof of Theorem 1.2

To complete the proof of Theorem 1.2, we adapt the method of continuity for nonlinear oblique boundary value problem, presented in [GT] and already used in the special case (1.16) (1.17) [U1]. The situation here is much more complicated as we need to vary the domain \( \Omega \), unless we know in advance that there exists a smooth function \( u_0 \), satisfying the ellipticity condition (1.2), whose image \( Tu_0(\Omega) \) is uniformly \( c^* \)-convex. In order to proceed, we fix a point \( x_0 \in \Omega \). Then for sufficiently small radius \( r > 0 \), the ball \( \Omega_0 = B_r(x_0) \subset \Omega \) will be uniformly \( c \)-convex with respect to \( \Omega^* \) and the function \( u_0 \), given by

\[
(5.1) \quad u_0(x) = \frac{\kappa}{2}|x - x_0|^2 + p_0 \cdot (x - x_0),
\]

will satisfy the ellipticity condition (1.2). Moreover the image \( \Omega_0^* = Tu_0(\Omega_0) \) will be uniformly \( c^* \)-convex with respect to \( \Omega \). Moreover, \( Tu \) is a diffeomorphism from \( \Omega_0 \) to \( \Omega_0 \). To see this we observe that

\[
(5.2) \quad c_\tau(x_0, \Omega_0^*) = B_{\kappa r}(p_0),
\]

so that by taking \( \kappa r \) small enough, we can fulfill condition (1.19) on \( \partial \Omega_0^* \), with respect to \( x_0 \in \Omega \), for constant \( \delta_0 = \frac{1}{\kappa r} \) as large as we wish. We now choose increasing families
of domains \( \{\Omega_t\}, \{\Omega^*_t\}, 0 \leq t \leq 1 \), such that

(i) \( \Omega_t \subset \Omega, \Omega^*_t \subset \Omega^* \),

(ii) \( \Omega_1 = \Omega, \Omega^*_1 = \Omega^* \),

(iii) \( \partial \Omega_t, \partial \Omega^*_t \in C^4 \), uniformly with respect to \( t \),

(iv) \( \Omega_t, \Omega^*_t \) are uniformly \( c \)-convex, \( c^* \)-convex with respect to \( \Omega^*, \Omega \), respectively.

The construction of such a family is discussed at the end of this section.

Having defined our families of domains \( \Omega_t, \Omega^*_t \), \( 0 < t \leq 1 \), we need to define corresponding equations. Let \( B \) be a positive function in \( C^2(\overline{\Omega} \times \mathbb{R}^n) \) and \( f \) a positive function in \( C^2(\Omega) \) such that

\[
(5.3) \quad f = -\sigma u_0 + \log[\det\{D^2u_0 - c_{xx}(\cdot, Y(\cdot, Du_0))\}/B(\cdot, Du_0)]
\]

in \( \Omega_0 \), for some fixed constant \( \sigma > 0 \). We then consider the family of boundary value problems:

\[
(5.4) \quad F[u] := \det\{D^2u - c_{xx}(\cdot, Y(\cdot, Du))\} = e^{\sigma u + (1-t) f} B(\cdot, Du),
\]

\[
Tu(\Omega_t) = Y(\cdot, Du)(\Omega_t) = \Omega^*_t,
\]

From our construction and the obliqueness, we see that \( u_0 \) is the unique elliptic solution of (5.4) at \( t = 0 \).

From Section 2, we also see that the boundary condition in (5.4) is equivalent to the oblique condition

\[
(5.5) \quad G_t(\cdot, Du) := \varphi^*_t(Y(\cdot, Du)) = 0 \quad \text{on} \quad \partial \Omega_t.
\]

To adapt the method of continuity from [GT], we fix \( \alpha \in (0, 1) \) and let \( \Sigma \) denote the subset of \( [0, 1] \) for which the problem (5.4) is solvable for an elliptic solution \( u = u_t \in C^{2,\alpha}(\overline{\Omega_t}) \), with \( Tu \) invertible. We then need to show that \( \Sigma \) is both closed and open in \( [0, 1] \). First we note that the boundary condition (5.4) implies a uniform bound for \( Du_t \). Integrating the equation (5.4), we then obtain uniform bounds for the quantities

\[
\int_{\Omega_t} e^{\sigma u_t},
\]

so that the solutions \( u_t \) will be uniformly bounded for \( \sigma > 0 \). Uniform estimates in \( C^{2,1}(\overline{\Omega}) \) now follow from our a priori estimates in Section 4, which are also clearly independent of \( t \in [0, 1] \). By compactness, we then infer that \( \Sigma \) is closed. To show \( \Sigma \) is open, we use the implicit function theorem and the linear theory of oblique boundary value problems, as in [GT]. The varying domains \( \{\Omega_t\} \) may be handled by means of diffeomorphisms approximating the identity, which transfer the problem (5.4) for \( t \) close to some \( t_0 \in \Sigma \) to a problem in \( \Omega_{t_0} \). We then conclude the solvability of (5.4) for all
$t \in [0, 1]$, which implies there exists a unique elliptic solution $u = u_\sigma \in C^3(\Omega)$ of the boundary value problem

\begin{align*}
F[u] &= e^{\sigma u} B(\cdot, Du), \\
Tu(\Omega) &= \Omega^*
\end{align*}

for arbitrary $\sigma > 0$, with $Tu$ one-to-one. To complete the proof of Theorem 1.2, we assume that $B$ satisfies (1.5), (1.8) and (1.9). As above we see that the integrals

$$
\int_\Omega e^{\sigma u_\sigma}
$$

are uniformly bounded, with $D(\sigma u_\sigma) \to 0$ as $\sigma \to 0$. Consequently $\sigma u_\sigma \to \text{constant} = 0$ by (1.19) and modulo additive constants, $u_\sigma \to u$ as $\sigma \to 0$, where $u$ is the solution of (1.14), (1.7), as required. □

To construct the family of domains $\{\Omega_t\}$ used above, we may take any increasing family $\{\Omega'_t\}$ satisfying conditions (i) to (iii). We then define a family $\{\Omega''_t\}$ where for each $t$, $\Omega''_t$ is the intersection of all uniformly $c$-convex domains with constant $\delta_0$, containing $\Omega'_t$. By suitable approximation, we obtain the desired family $\{\Omega_t\}$. Alternatively if we are given a defining function $C^4$ defining function $\varphi$, satisfying

\begin{align*}
[D_{ij}\varphi(x) - c^{l,k}c_{ij,l}(x,y)D_k\varphi(x)]\xi_i\xi_j &\geq \delta_0|\xi|^2 \\
\end{align*}

for all $x \in \Omega, y \in \Omega^*, \xi \in \mathbb{R}^n$, which takes its minimum at $x_0$, we may choose

$$
\Omega_t = \{x \in \Omega \mid \varphi_t(x) < 0\}
$$

where $\varphi_t$ is defined by

\begin{align*}
\varphi_t &= Kt\varphi + (1-t)\varphi_0,
\end{align*}

for $\varphi_0(x) = |x - x_0|^2 - r^2$ and $K$ sufficiently large. The domains $\Omega^*_t$ may be similarly constructed.

Note that our proof of Theorem 1.2 shows that the mapping $Tu$ is a diffeomorphism between $\Omega$ and $\Omega^*$. Through a uniqueness argument, we may then infer the full strength of Theorem 1.1.
6. Optimal Transportation

The interior regularity of solutions to the optimal transportation problem is considered in [MTW], where we prove they are locally smooth under conditions A1, A2, A3 and the $c^\ast-$convexity of the target domain $\Omega^\ast$. Our approach is to first show that the Kantorovich potentials are generalized solutions of the boundary value problem (1.14), (1.7) in the sense of Aleksandrov and Bakelman. The $c^\ast-$convexity of $\Omega^\ast$ is used to show the image of the generalized normal mapping lies in $\overline{\Omega}^\ast$ and condition A3 is employed to obtain a priori second derivative estimates from which the desired regularity follows.

The potential functions $u$ and $v$ solve the dual problem of minimizing the functional

$$I(u, v) = \int_{\Omega} fu + \int_{\Omega^\ast} gv$$

over the set $K$ given by

$$(6.2) K = \{(u, v) \mid u, v \in C^0(\Omega), C^0(\Omega^\ast) \text{ resp. } u(x) + v(y) \geq c(x, y) \text{ for all } x \in \Omega, y \in \Omega^\ast\}.$$ 

The potential functions $(u, v)$ satisfy the relations

$$(6.3) u(x) = \sup_{y \in \Omega} \{c(x, y) - v(y)\},$$

$$(6.3) v(y) = \sup_{x \in \Omega} \{c(x, y) - u(y)\},$$

that is they are the $c^\ast$ and $c$ transforms of each other. Since $c \in C^{1.1}$, they will be semi-convex. The optimal mapping $T$ is then given almost everywhere by (1.23) and the equation (1.14) will be satisfied with elliptic solution $u$ almost everywhere in $\Omega$. The function $u$ and $v$ are respectively $c$ and $c^\ast$ convex. A function $u \in C^0(\Omega)$ is called $c -$convex in $\Omega$ if for each $x_0 \in \Omega$, there exists $y_0 \in \mathbb{R}^n$ such that

$$u(x) \geq u(x_0) + c(x, y_0) - c(x_0, y_0)$$

for all $x \in \Omega$. If $u$ is a $c-$convex function, for which the mapping $T$ given by (1.23) is measure preserving, then it follows that $u$ is a potential and again $T$ is the unique optimal mapping. These results hold under the hypotheses A1 and A2; the reader is referred to [GM], [MTW], [U2], [V] for further details.

From the above discussion we see that the solution of the boundary value problem (1.14), (1.7) will automatically furnish a potential for the optimal transportation problem if it is $c-$convex. Note that ellipticity only implies that the solution is locally $c-$convex and we need a further argument to conclude the global property, unlike the case of quadratic cost functions and convex solutions. First we recall the concept of generalized
solution introduced in [MTW]. Let \( u \) be a \( c \)-convex function on the domain \( \Omega \). The generalized normal mapping \( \chi_u \) is defined by

\[
(6.5) \quad \chi_u(x_0) = \{ y_0 \in \mathbb{R}^n \mid u(x) \geq u(x_0) + c(x, y) - c(x_0, y_0), \text{ for all } x \in \Omega \}.
\]

Clearly, \( \chi_u(x_0) \subset Y(x_0, \partial u(x_0)) \) where \( \partial \) denotes the subgradient of \( u \). For \( g \geq 0, \in L^1_{\text{loc}}(\mathbb{R}^n) \), the generalized Monge-Ampère measure \( \mu[u, g] \) is then defined by

\[
(6.6) \quad \mu[u, g](e) = \int_{\chi_u(e)} g
\]

for any Borel set \( e \in \Omega \), so that \( u \) satisfies equation (1.14) in the generalized sense if

\[
(6.7) \quad \mu[u, g] = f \, dx.
\]

The boundary condition (1.7) is satisfied in the generalized sense if

\[
(6.8) \quad \Omega^* \subset \chi_u(\overline{\Omega}), \quad \left| \left\{ x \in \Omega \mid f(x) > 0 \text{ and } \chi_u(x) - \overline{\Omega}^* \neq \emptyset \right\} \right| = 0
\]

The theory of generalized solutions replicates that for the convex case, \( c(x, y) = x \cdot y \), [MTW]. If \( f \) and \( g \) are positive, bounded measurable functions on \( \Omega, \Omega^* \) respectively satisfying the mass balance condition (1.9), and \( c \) satisfies A1, A2, then there exists a unique (up to constants) generalized solution of (6.7), (6.8), (with \( (6.1), (6.2)[\text{MTW}] \)). Furthermore from Theorem 2.1, [MTW] implies that \( u \in C^3(\Omega) \) if \( f, g \) satisfy the hypotheses of Theorem 1.3, \( c \) satisfies condition A3 and \( \Omega^* \) is \( c^* \)-convex.

Now let \( u \in C^2(\overline{\Omega}) \) be an elliptic solution of the boundary value problem (1.7), (1.14) and \( v \) a \( c \)-convex solution of the corresponding generalized problem. By adding constants, we may assume \( \inf_{\Omega}(u - v) = 0 \). We need to prove \( u = v \) in \( \Omega \), that is the strong comparison principle holds. Let \( \Omega' \) denote the subset of \( \Omega \) where \( u > v \) and first suppose that \( \partial \Omega' \cap \Omega \neq \emptyset \). Note that if \( c \) satisfies A3, then \( v \in C^2(\Omega) \) and this situation is immediately ruled out by the classical strong maximum principle [GT]. Otherwise we may follow the proof of the strong maximum principle as there will exist a point \( x_0 \in \partial \Omega' \cap \Omega \), where \( \Omega' \) satisfies an interior sphere condition, that is there exists a ball \( B \subset \Omega - \Omega' \) such that \( x_0 \in \partial \Omega' \cap \partial B \), \( u(x_0) = v(x_0) \) and \( u > v \) in \( B \). Since \( v \) is semi-convex, \( v \) will be twice differentiable at \( x_0 \), with \( Dv(x_0) = Du(x_0) \). Moreover by passing to a smaller ball if necessary we may assume both \( u \) and \( v \) are \( c \)-convex in \( B \). Since \( u \) is a smooth elliptic solution of (1.14), there will exist a strict supersolution \( w \in C^2(\overline{B} - B_\rho) \), for some concentric ball \( B_\rho \) of radius \( \rho < R \), satisfying \( w(x_0) = u(x_0) \), \( w \geq v \) on \( \partial B \cup \partial B_\rho \), \( Dw(x_0) \neq Du(x_0) \). By the comparison principle, [MTW], Lemma 5.2, we have \( w \geq v \) in \( B - B_\rho \), and hence \( Dw(x_0) = Du(x_0) \), which is a contradiction.
Thus we may assume $\partial \Omega' \cap \Omega = \emptyset$, that is $u > v$ in $\Omega$ with $u(x_0) = v(x_0)$ for some point $x_0 \in \partial \Omega$. From our argument above, we obtain a function $w \in C^2(\overline{B} - B_\rho)$ satisfying $w(x_0) = u(x_0) = v(x_0)$, $v \leq w \leq u$ in $\overline{B} - B_\rho$, together with
\begin{equation}
(6.9) \quad u(x) - w(x) \geq \epsilon |x - x_0|
\end{equation}
for all $x \in B_R - B_\rho$. Since $v \leq w$ in $B_R - B_\rho$, this contradicts the obliqueness condition (2.43) if $\Omega^*$ is $c^*$-convex.

Alternatively we may proceed directly as follows to show that the solution $u$ is $c$-convex, when $Tu$ is one-to-one. Let $x_0 \in \Omega$ and $y_0 = Tu(x_0)$. Suppose there exists a point $x_1 \in \Omega$, where
\begin{equation}
(6.10) \quad u(x_1) < c(x_1, y_0) - c(x_0, y_0).
\end{equation}
By downwards vertical translation, there exists a point $x_2 \in \partial \Omega$, satisfying
\begin{equation}
(6.11) \quad u(x) > u(x_2) + c(x, y_0) - c(x_2, y_0).
\end{equation}
for all $x \in \Omega$. Putting $y_2 = Tu(x_2)$, we must also have
\begin{equation}
(6.12) \quad c_x(x_2, y_2).\gamma(x_2) < c_x(x_2, y_0).\gamma(x_2),
\end{equation}
which again contradicts the $c^*$-convexity of $\Omega^*$.

**Remarks.**

In the first proof above, we employed a comparison result that if $u$ is a classical elliptic supersolution of (1.14) dominating a generalized subsolution $v$ on the boundary of a subdomain $\Omega'$, then $u \geq v$ in $\Omega'$. In our local uniqueness argument in [MTW], we also used implicitly the complementing result that if $u$ is an elliptic subsolution dominated by a generalized supersolution $v$ on $\partial \Omega'$, then $u \leq v$ in $\Omega'$. However, in this case, we cannot apply Lemma 5.2 in [MTW] directly as local $c$-convexity of $v$ may not imply global $c$-convexity in $\Omega$, unless $v \in C^1(\Omega)$. For this we also need to use our assumption A3, which implies that the contact set with a $c$-support function is $c$-convex with respect to $\Omega^*$, and hence connected, if $\Omega^*$ is $c^*$-convex with respect to $\Omega$. This property is further examined in the recent preprint of G.Loeper [L], which uses geometric characterizations of condition A3 to infer $C^{1,\alpha}(\Omega)$ regularity for non-smooth densities.

**7. Examples**

We repeat and expand somewhat our examples in [MTW], taking account that our cost functions are the negatives of those there.
Example 1.

\begin{equation}
(7.1) \quad c(x, y) = -\sqrt{1 + |x - y|^2}
\end{equation}

Here the vector field $Y$ and matrix $A$ are given by

\begin{equation}
(7.2) \quad Y(x, p) = x + \frac{p}{\sqrt{1 - |p|^2}},
A(x, p) = A(p) = -\left(1 - |p|^2\right)^{1/2} (I - p \otimes p).
\end{equation}

The cost function satisfies condition A3. We remark that condition A1 is only satisfied for $|p| < 1$ but this does not prohibit application of our results as the boundedness of target domain $\Omega^*$ ensures that $|Du| < 1$ for solutions of (1.7), (1.14).

Example 2.

\begin{equation}
(7.3) \quad c(x, y) = -\sqrt{1 - |x - y|^2}
\end{equation}

Here $c$ is only defined for $|x - y| \leq 1$. The vector field $Y$ and matrix $A$ are given by

\begin{equation}
(7.4) \quad Y(x, p) = -x + \frac{p}{\sqrt{1 + |p|^2}},
A(x, p) = A(p) = \left(1 + |p|^2\right)^{1/2} (I + p \otimes p).
\end{equation}

The cost function satisfies condition A3. In order apply our results we need to assume $\Omega$ and $\Omega^*$ are strictly contained in a ball of radius 1.

Example 3. Let $f, g \in C^2(\Omega), C^2(\Omega^*)$ respectively and

\begin{equation}
(7.5) \quad c(x, y) = x \cdot y + f(x)g(y).
\end{equation}

If $|\nabla f|, |\nabla g| < 1$, then $c$ satisfies A1, A2. If $f, g$ are convex, then $c$ satisfies A3w, while if $f, g$ are uniformly convex, then $c$ satisfies A3. As indicated in [MTW], the function (7.5) is equivalent to the square of the distance between points on the graphs of $f$ and $g$.

Example 4. Power costs.

\begin{equation}
(7.6) \quad c(x, y) = \pm \frac{1}{m} |x - y|^m, \quad m \neq 0, \quad \log|x - y|, \quad m = 0).
\end{equation}

For $m \neq 1$ and $x \neq y$, when $m < 1$, the vector fields $Y$ and matrices $A$ are given by

\begin{equation}
(7.7) \quad Y(x, p) = x \pm |p|^\frac{2-m}{m-1} p,
A(x, p) = A(p) = \pm \left\{ |p|^\frac{m-2}{m-1} I + (m - 2)|p|^{-\frac{m}{m-1}} p \otimes p \right\}.
\end{equation}
The only cases for which condition A3w is satisfied are $m = 2(\pm)$ and $-\frac{1}{2} \leq m < 1$ (+ only). For the latter, condition A3 holds for $-\frac{1}{2} < m < 1$. To apply our results directly in the latter cases, we need to assume $\Omega$ and $\Omega^*$ are disjoint.

In [MTW] we also considered the cost function

\begin{equation}
(7.8) \quad c(x, y) = -(1 + |x - y|^2)^{p/2}
\end{equation}

for $1 \leq p \leq 2$, extending Example 1 to $p > 1$. We point out here that these functions only satisfy A3 under the restriction $|x - y|^2 < \frac{1}{p-1}$. This condition was omitted in [MTW].

**Example 5. Reflector antenna problem**

Corresponding results and examples may be obtained on other manifolds such as the spheres $S^n$. Indeed the interior regularity results in [MTW] stemmed from the solution of the reflector antenna problem by Wang in [W1], which may be represented as an optimal transportation problem on the sphere $S^n$ with cost function

\begin{equation}
(7.9) \quad c(x, y) = -\log(1 - x \cdot y),
\end{equation}

which is simply the spherical analogue of the case $m = 0$ in Example 4 above. The corresponding vector field $Y$ is now given by

\begin{equation}
(7.10) \quad Y(x, p) = x - \frac{2}{1 + |p|^2}(x + p),
\end{equation}

where now $p$ belongs to the tangent space of $S^n$ at $x$, while the matrix $A$ is given by

\begin{equation}
(7.11) \quad A = \frac{1}{2}(|p|^2 - 1)g_0 - p \times p,
\end{equation}

where $g_0$ denotes the metric on $S^n$. See [W1, W2, MTW] for more details. When the domains $\Omega$ and $\Omega^*$ have disjoint closures, and spherically uniformly convex boundaries, we obtain the global regularity of potentials. We will defer further examination and extensions to intersecting domains and other cost functions in a future work. We also point out here that Example 4 provides regularity for quadratic cost functions on spheres when the points $x$ and $y$ are not orthogonal.
References


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