SCHAUDER ESTIMATES FOR ELLIPTIC AND PARABOLIC EQUATIONS

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Introduction

The Schauder estimate for the Laplace equation was traditionally built upon the Newton potential theory. Different proofs were found later by Campanato [Ca], in which he introduced the Campanato space; Peetre [P], who used the convolution of functions; Trudinger [T], who used the mollification of functions; and Simon [Si], who used a blow-up argument. Also a perturbation argument was found by Safonov [S1,S2] and Caffarelli [C1, CC] for fully nonlinear uniformly elliptic equations, which also applies to the Laplace equation.

In this note we give an elementary and simple proof for the Schauder estimates for elliptic and parabolic equations. Our proof allows the right hand side to be Dini continuous and also give a sharp estimate for the modulus of continuity of the second derivatives. It also yields the log-Lipschitz continuity of the gradient for equations with bounded right hand side. Moreover, it also applies to nonlinear equations.

1. The Laplace equation

Consider the Laplace equation

$$\Delta u = f \quad \text{in} \ \ B_1(0), \tag{1.1}$$

where $B_1(0)$ is the unit ball in the Euclidean space $\mathbb{R}^n$. Suppose f is Dini continuous, namely $\int_0^1 \frac{\omega(r)}{r} dr < \infty$, where $\omega(r) = \sup_{|x-y|<r} |f(x) - f(y)|$. Then we have the following estimate for the modulus of continuity of $D^2 u$.

Theorem 1. *Let $u \in C^2$ be a solution of (1.1). Then $\forall\ x, y \in B_{1/2}(0)$,*

$$|D^2 u(x) - D^2 u(y)| \le C_n \big[d \sup_{B_1} |u| + \int_0^d \frac{\omega(r)}{r} + d \int_d^1 \frac{\omega(r)}{r^2} \big], \tag{1.2}$$

where $d = |x - y|$, $C_n > 0$ depends only on n. It follows that if $f \in C^\alpha(B_1)$, then

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C_n \big[\sup_{B_1} |u| + \frac{\|f\|_{C^\alpha(B_1)}}{\alpha(1-\alpha)} \big] \quad \text{if} \ \ \alpha \in (0,1), \tag{1.3}$$

$$|D^2 u(x) - D^2 u(y)| \le C_n d \, (\sup_{B_1} |u| + \|f\|_{C^{0,1}} |\log d| \,) \quad \text{if} \ \ \alpha = 1. \tag{1.4}$$

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Proof. We will use the following elementary estimates for harmonic functions,

$$|D^k w(0)| \le C_{n,k} r^{-|k|} \sup_{B_r} |w|, \tag{1.5}$$

where $C_{n,k}$ depends only on n and k. Simple proofs of (1.5) can be found in [E2].

Denote $B_k = B_{\rho^k}(0)$ ($\rho = \frac{1}{2}$). For $k = 0, 1, \cdots$, let u_k be the solution of

$$\Delta u_k = f(0) \quad \text{in} \quad B_k, \qquad u_k = u \quad \text{on} \quad \partial B_k.$$

Then $\Delta(u_k - u) = f(0) - f$. By the maximum principle,

$$\|u_k - u\|_{L^\infty(B_k)} \le C\rho^{2k}\omega(\rho^k). \tag{1.6}$$

Hence

$$\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} \le C\rho^{2k}\omega(\rho^k). \tag{1.7}$$

Since $u_{k+1} - u_k$ is harmonic, by (1.5) we have

$$\begin{aligned} &\|D(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} \le C\rho^k\omega(\rho^k), \\ &\|D^2(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} \le C\omega(\rho^k). \end{aligned} \tag{1.8}$$

Since $u \in C^2$, by (1.6), u_k minus the quadratic part of u is harmonic and is equal to $o(\rho^{2k})$ in B_k. Hence by (1.5),

$$\begin{aligned} Du(0) &= \lim_{k\to\infty} Du_k(0), \\ D^2u(0) &= \lim_{k\to\infty} D^2u_k(0). \end{aligned} \tag{1.9}$$

For any given point z near the origin, we have

$$\begin{aligned} &|D^2u(z) - D^2u(0)| \le I_1 + I_2 + I_3 =: \qquad (1.10) \\ &|D^2u_k(z) - D^2u_k(0)| + |D^2u_k(0) - D^2u(0)| + |D^2u(z) - D^2u_k(z)|. \end{aligned}$$

Let $k \ge 1$ such that $\rho^{k+4} \le |z| \le \rho^{k+3}$. Then by (1.8), we have

$$I_2 \le C\sum_{j=k}^{\infty}\omega(\rho^k) \le C\int_0^{|z|}\frac{\omega(r)}{r}. \tag{1.11}$$

Similarly we can estimate I_3, through the solutions of $\Delta v = f(z)$ in $B_j(z)$ and $v = u$ on $\partial B_j(z)$ for $j = k, k+1, \cdots$. To estimate I_1, denote $h_j = u_j - u_{j-1}$. By (1.5) and (1.7) we have

$$|D^2h_j(z) - D^2h_j(0)| \le C\rho^{-j}\omega(\rho^j)|z|. \tag{1.12}$$

Hence

$$\begin{aligned} I_1 &\le |D^2u_{k-1}(z) - D^2u_{k-1}(0)| + |D^2h_k(z) - D^2h_k(0)| \\ &\le |D^2u_0(z) - D^2u_0(0)| + \sum_{j=1}^{k}|D^2h_j(z) - D^2h_j(0)| \\ &\le C|z|\big(\|u_0\|_{L^\infty} + C\sum\rho^{-j}\omega(\rho^j)\big) \\ &\le C|z|\big(\|u\|_{L^\infty} + C\int_{|z|}^1\frac{\omega(r)}{r^2}\big). \end{aligned} \tag{1.13}$$

Combining (1.10), (1.11), and (1.13) we obtain (1.2). This completes the proof. □

Similarly we have the estimate at the boundary.

Theorem 1′. *Let $u \in C^2(B_1 \cap \{x_n \geq 0\})$ be a solution of $\Delta u = f$ and $u = 0$ on $\{x_n = 0\}$. Suppose f is Dini continuous. Then $\forall\ x, y \in B_{1/2} \cap \{x_n \geq 0\}$, the estimate (1.2) holds.*

The proof is the same as that of Theorem 1, provided we replace B_k by $B_k \cap \{x_n \geq 0\}$ and note that if w is a harmonic function in $B_1 \cap \{x_n \geq 0\}$ and $w = 0$ on T, then w is harmonic in B_1 after odd extension in x_n.

Replacing the second derivatives in the proof of (1.4) by the first derivatives, and letting u_k be the solution of $\Delta u_k = 0$ in B_k, $u_k = u$ on ∂B_k, we also obtain the following log-Lipschitz continuity for Du, which was used in [Y] to establish the global existence of smooth solutions to the 2-d Euler equation.

Corollary 1. *Let $u \in C^1$ be a solution of (1.1). Then $\forall\ x, y \in B_{1/2}(0)$,*

$$|Du(x) - Du(y)| \leq C_n d\,(\sup_{B_1} |u| + \|f\|_{L^\infty} |\log d|\,). \tag{1.14}$$

2. Linear parabolic equations

The above proof also applies to equations with variable coefficients. Let us consider the linear parabolic equation

$$\sum a_{ij}(x,t) u_{x_i x_j} - u_t = f(x,t) \quad \text{in} \;\; Q_1. \tag{2.1}$$

We denote $Q_r = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}^1 : \; |x| < r, -r^2 < t \leq 0\}$.

Theorem 2. *Let $u \in C^{2,1}_{x,t}$ be a solution of (2.1). Suppose f and a_{ij} are Dini continuous. Then for any points $p_1 = (x_1, t_1), p_2 = (x_2, t_2) \in Q_{1/2}$,*

$$\begin{aligned}|\partial_x^2 u(p_1) - \partial_x^2 u(p_2)| \leq & C_n \left[d \sup_{Q_1} |u| + \int_0^d \frac{\omega_f(r)}{r} + d \int_d^1 \frac{\omega_f(r)}{r^2} \right] \\ & + C_n \sup_{Q_1} |\partial_x^2 u| \left[\int_0^d \frac{\omega_a(r)}{r} + d \int_d^1 \frac{\omega_a(r)}{r^2} \right],\end{aligned} \tag{2.2}$$

where $d = |p_1 - p_2|$ (parabolic distance), $\omega_f(r) = \sup_{|p_1 - p_2| < r} |f(p_1) - f(p_2)|$, and $\omega_a(r) = \sup_{i,j} \omega_{a_{ij}}(r)$.

Note that the modulus of continuity of $\partial_t u$ follows from (2.2) and equation (2.1). If a_{ij} and f are Hölder continuous, by the interpolation inequality [L], we obtain the Schauder estimate for parabolic equations. That is if $a_{ij}, f \in C^\alpha(Q_1)$ for some $\alpha \in (0,1)$, then

$$\|u\|_{C^{2+\alpha, 1+\alpha/2}_{x,t}(Q_{1/2})} \leq C\big[\sup_{Q_1} |u| + \|f\|_{C^\alpha(Q_1)}\big]. \tag{2.3}$$

If $\alpha = 1$, we have an estimate similar to (1.4).

Proof. Denote $Q_k = Q_{\rho^k}(0)$ ($\rho = \frac{1}{2}$). Let u_k be the solution of

$$\sum a_{ij}(0) u_{x_i x_j} - u_t = f(0) \quad \text{in} \quad Q_k, \quad u_k = u \quad \text{on} \quad \partial_p Q_k,$$

where ∂_p denotes the parabolic boundary. Then $v = u - u_k$ satisfies

$$\sum a_{ij}(0) v_{x_i x_j} - v_t = f - f(0) + \sum (a_{ij}(0) - a_{ij}(x)) u_{x_i x_j}. \tag{2.4}$$

By the maximum principle,

$$\|u_k - u\|_{L^\infty(Q_k)} \le C\rho^{2k}[\omega_f(\rho^k) + \omega_a(\rho^k)\eta],$$

where $\eta = \sup |\partial_x^2 u|$. Hence

$$\|u_k - u_{k+1}\|_{L^\infty(Q_{k+1})} \le C\rho^{2k}[\omega_f(\rho^k) + \omega_a(\rho^k)\eta]. \tag{2.5}$$

Therefore similarly as (1.8),

$$\sup_{Q_{k+2}}\{|\partial_x^2(u_k - u_{k+1})|, |\partial_t(u_k - u_{k+1})|\} \le C[\omega_f(\rho^k) + \omega_a(\rho^k)\eta]. \tag{2.6}$$

The rest of the proof is the same as that of Theorem 1 and is omitted here. □

3. Fully nonlinear equations

3.1. Fully nonlinear, uniformly elliptic equations.
The argument in §1 also applies to fully nonlinear uniformly elliptic equations. For simplicity we consider the equation

$$F(D^2 u) = f(x) \quad \text{in} \quad B_1(0), \tag{3.1}$$

where F is $C^{1,1}$. The estimates can be extended to operators of the form $F(D^2u, x)$ by the freezing coefficient method as in §2. We need an a priori estimate as (1.4).
Assumption (a). For any solution to

$$F(D^2 u + M) = c_0 \quad \text{in} \quad B_r,$$

where c_0 is a constant and M is a symmetric constant matrix such that $F(M) = c_0$, we have the estimate

$$\|u\|_{C^{2,\bar\alpha}(B_{r/2})} \le \bar C r^{-2-\bar\alpha} \|u\|_{L^\infty(B_1)}, \tag{3.2}$$

where $\bar\alpha \in (0, 1]$, $\bar C$ is independent of M, c_0 and r.

If F is concave or convex, the interior $C^{2,\bar\alpha}$ estimate for some $\bar\alpha \in (0, 1]$ was established independently by Evans [E1] and Krylov [Kr]. Similarly to Theorem 1 we then have

Theorem 3. *Let $u \in C^2$ be a solution of (3.1). Then $\forall\ x, y \in B_{1/2}(0)$,*

$$|D^2u(x) - D^2u(y)| \le C\big[d^{\bar\alpha} \sup_{B_1} |u| + \int_0^d \frac{\omega(r)}{r} + d^{\bar\alpha} \int_d^1 \frac{\omega(r)}{r^{1+\bar\alpha}}\big]. \tag{3.3}$$

If $f \in C^\alpha(B_1)$, we have

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C\big[\sup_{B_1} |u| + \|f\|_{C^\alpha(B_1)}\big] \quad \text{if } \ 0 < \alpha < \bar\alpha, \tag{3.4}$$

$$|D^2u(x) - D^2u(y)| \le Cd^{\bar\alpha}\big[\sup_{B_1} |u| + \|f\|_{C^\alpha} |\log d|\big] \quad \text{if } \ \alpha = \bar\alpha, \tag{3.5}$$

$$\|u\|_{C^{2,\bar\alpha}(B_{1/2})} \le C\big[\sup_{B_1} |u| + \|f\|_{C^\alpha(B_1)}\big] \quad \text{if } \ \bar\alpha < \alpha \le 1. \tag{3.6}$$

The constant C depends on n, $\bar\alpha, \bar C$ in (3.2), and the ellipticity constants (least and largest eigenvalues of $\{\frac{\partial}{\partial r_{ij}} F(r)\}$).

Proof. The proof is very similar to that of Theorem 1. Let u_k be the solution of

$$F(D^2u_k) = f(0) \quad \text{in } \ B_k, \qquad u_k = u \ \text{ on } \ \partial B_k. \tag{3.7}$$

By Assumption (a), $u_k - u_{k+1}$ satisfies a linearized equation of F with coefficients in $C^{\bar\alpha}$. Hence by the Schauder estimate for linear elliptic equations,

$$\|D^2(u_k - u_{k+1})\|_{B_{k+2}} \le C\rho^{-2k}\|u_k - u_{k+1}\|_{L^\infty} \le C\omega(\rho^k). \tag{3.8}$$

It follows that $D^2u_k(0)$ is convergent if f is Dini continuous. By Assumption (a) and since $u \in C^2$, we have $D^2u_k(0) \to D^2u(0)$. The only difference in the rest part of the proof is that (1.12) should be replaced by

$$|D^2h_k(z) - D^2h_k(0)| \le C\rho^{-k\bar\alpha}\omega(\rho^k)|z|^{\bar\alpha}, \tag{3.9}$$

where $h_k = u_k - u_{k+1}$. $\square$

3.2. The Monge-Ampère equations.
Estimate (1.2) (or (3.3) with $\bar\alpha = 1$) holds for strictly convex solutions to the Monge-Ampère equation

$$\det D^2u = f(x) \quad \text{in } \ B_1, \tag{3.10}$$

where $C_* \le f \le C^*$ for positive constants C_*, C^*, and $\omega(r) = \sup_{|x-y|<r} |f(x) - f(y)|$ (equivalent to $\sup_{|x-y|<r} |f^{1/n}(x) - f^{1/n}(y)|$). The constant C depends on n, C_*, C^*, and the modulus of convexity of u.

The proof is similar to that of Theorem 1, except that we first need to normalize the solution as follows. By subtracting a linear function, we assume $u(0) = 0$ and $Du(0) = 0$. Denote $S_h = \{x \in B_1,\ u(x) < h\}$. For a sufficiently small $h > 0$, first make unimodular linear transform T such that $B_R \subset T(S_h) \subset B_{nR}$. Then make a dilation $x \to x/R$ and $u \to u/R^2$ such that $B_1 \subset S_1 \subset B_n$ and $\int_0^1 \frac{\omega(r)}{r}$ is small. The Monge-Ampère operator is invariant under the changes.

Now define u_k as in (3.7) (for the Monge-Ampere equation it is more convenient to use level sets than balls in (3.7)). We need to verify Assumption (a) (with $\bar\alpha = 1$) for all k. It suffices to show that the set $E_k = \{x \in \mathbb{R}^n \mid u_k(x) < \inf u_k + \rho^{2(k+1)}\}$ has a good shape, namely $B_{R_k} \subset E_k \subset B_{2nR_k}$ for concentrated balls B_{R_k} and B_{2nR_k}. But this is guaranteed at $k = 0$ and also at $k > 0$ by induction, as long as ρ is chosen small and $\int_0^1 \frac{\omega(r)}{r}$ is sufficiently small. We wish to discuss the regularity of the Monge-Ampère equation with more details in a separate work.

3.3. Remarks.

(i) By Aleksandrov's maximum principle [GT], we can replace $\omega(r) = \mathrm{osc}_{B_r} f$ by $\omega(r) = r^{-n}\|f - f(0)\|_{L^n(B_r)}$ in the above proofs.

(ii) By the existence and uniqueness of weak or viscosity solutions to the Dirichlet problem, the above theorems also hold for weak or viscosity solutions.

(iii) The sharp estimate (1.2) for the Laplace equation was established in [B] by delicate singular integral estimates.

(iv) Theorem 3 with $f \in C^\alpha$ ($\alpha < \bar\alpha$) was proved by Safonov [S1, S2] and Caffarelli [C1, CC] by a perturbation argument, using approximation by quadratic polynomials. See also [K] for the case when f is Dini continuous. Our proof allow the case $\alpha \geq \bar\alpha$ in (3.5) and (3.6) above.

(v) The $C^{2,\alpha}$ estimate for strictly convex solutions to the Monge-Ampère equation (3.10) was proved in [C2]. When the Hölder continuity of f is relaxed to Dini continuity, the continuity of D^2u was proved in [W1].

(vi) Similar estimate to (1.14) also holds for the parabolic equation (2.1) and fully nonlinear equation (3.1), but it is not true for the Monge-Ampère equation (3.10), by an example in [W2].

References

[B] C. Burch, The Dini condition and regularity of weak solutions of elliptic equations, J. Diff. Equa. 30 (1978), 308–323.

[C1] L.A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, Ann. of Math. (2) 130 (1989), 189–213.

[C2] L.A. Caffarelli, Interior $W^{2,p}$ estimates for solutions of Monge-Ampère equations, Ann. Math., 131(1990), 135-150.

[CC] L.A. Caffarelli and X. Cabre, Fully nonlinear elliptic equations, Amer. Math. Soc., 1995.

[Ca] S. Campanato, Propriet di una famiglia di spazi funzionali, Ann. Scuola Norm. Sup. Pisa (3) 18(1964), 137–160.

[E1] L.C. Evans, Classical solutions of fully nonlinear, convex, second order elliptic equations, Comm. Pure Appl. Math., 25(1982), 333-363.

[E2] L.C. Evans, Partial Differential equations, Amer. Math. Soc., 1998.

[GT] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Springer, 1983.

[K] J. Kovats, Fully nonlinear elliptic equations and the Dini condition, Comm. PDE 22 (1997), no. 11-12, 1911–1927.

[Kr] N.V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations, Iz. Akad. Nauk. SSSR 46(1982), 487-523; English translation in Math. USSR (1983).

[L] G.M. Lieberman, Second order parabolic differential equations, World Scientific, 1996.

[P] J. Peetre, On convolution operators leaving $L^{p,\lambda}$ spaces invariant, Ann. Mat. Pura Appl. 72 (1966), 295–304.

[S1] M.V. Safonov, The classical solution of the elliptic Bellman equation, Dokl. Akad. Nauk SSSR 278(1984), 810–813; English translation in Soviet Math Doklady 30(1984), 482-485.

[S2] M.V. Safonov, Classical solution of second-order nonlinear elliptic equations, Izv. Akad. Nauk SSSR Ser. Mat. 52(1988), 1272–1287; English translation in Math. USSR-Izv. 33(1989), 597–612.

[Si] L. Simon, Schauder estimates by scaling, Calc. Var. PDE, 5(1997), 391–407.

[T] N.S. Trudinger, A new approach to the Schauder estimates for linear elliptic equations, Proc. Centre Math. Anal., Austral. Nat. Univ., 14(1986), 52–59.

[Y] V. Yudovich, Non-stationary flows of an ideal incompressible fluid (in Russian), Ž. Vyčisl. Mat. i Mat. Fiz. 3(1963), 1032-1066.

[W1] X.-J. Wang, Remarks on the Regularity of Monge-Ampère equations, Proc. Inter. Conf. Nonlinear PDE, eds: G.C. Dong and F.H. Lin, 1992, pp.257-263.

[W2] X.-J. Wang, Some counterexamples to the regularity of Monge-Ampère equations. Proc. Amer. Math. Soc. 123 (1995), 841-845.

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