ON STRICT CONVEXITY AND $C^1$ REGULARITY
OF POTENTIAL FUNCTIONS IN OPTIMAL TRANSPORTATION

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ABSTRACT. This note concerns the relationship between conditions on cost functions and
domains and the convexity properties of potentials in optimal transportation and the contin-
uity of the associated optimal mappings. In particular, we prove that if the cost function
satisfies the condition (A3), introduced in our previous work with Xinan Ma, the densities
and their reciprocals are bounded and the target domain is convex with respect to the
cost function, then the potential is continuously differentiable and its dual potential strictly
concave with respect to the cost function. Our result extends, by different and more direct
proof, similar results of Loeper proved by approximation from our earlier work on global
regularity.

§1. Introduction

We continue our investigation on the regularity of potential functions in the optimal
transportation problem [MTW, TW]. In this paper we prove strict convexity and $C^1$
regularity of potential functions for non-smooth densities. The strict convexity and $C^1$
regularity for solutions to the Monge-Ampere equation were established by Caffarelli
[C1]. For the reflector design problem, which is a special optimal transportation problem
[W2], these results were obtained in [CGH]. For more general optimal transportation
problems, the $C^{1,\alpha}$ regularity has been obtained by Loeper [L]. In this paper we prove
the strict convexity for potential functions, and obtain the $C^1$ regularity under weaker
conditions on the domains. We also use our results to plug a gap in [MTW] pertaining
to the use of a comparison argument in the proof of interior regularity.

Let $\Omega, \Omega^*$ be two bounded domains in $\mathbb{R}^n$, and $f, g$ be two nonnegative integrable
functions on $\Omega, \Omega^*$ satisfying the mass balance condition

\[(1.1) \quad \int_{\Omega} f = \int_{\Omega^*} g.\]

Let $(u, v)$ be potential functions to the optimal transportation problem, namely $(u, v)$ is
a maximizer of

\[(1.2) \quad \sup\{I(\varphi, \psi) : (\varphi, \psi) \in K\},\]

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where

\begin{equation}
I(\varphi, \psi) = \int_{\Omega} f(x) \varphi(x) + \int_{\Omega^*} g(y) \psi(y),
\end{equation}

\[ K = \{(\varphi, \psi) \in C^0(\Omega) \times C^0(\Omega^*) : \varphi(x) + \psi(y) \leq c(x, y)\}. \]

We assume that the cost function is smooth, \( c \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \), and satisfies (A1)-(A3) below. It is known [C4, GM] that there is a maximizer \((u, v)\) to (1.2), which is also unique up to a constant if \( f, g \) are positive. The potentials \((u, v)\) are semi-concave and satisfy

\begin{equation}
\begin{aligned}
&u(x) = \inf_{y \in \Omega^*} \{c(x, y) - v(y)\}, \\
v(y) = \inf_{x \in \Omega} \{c(x, y) - u(x)\}.
\end{aligned}
\end{equation}

The optimal mapping \( T : x \in \Omega \to y \in \Omega^* \) can be determined, a.e. in \( \Omega \), by

\begin{equation}
Du(x) = D_x c(x, y),
\end{equation}

where \( y \) also attains the infimum in (1.4). If \( u \in C^2(\Omega) \), it satisfies the equation

\begin{equation}
\det(D^2_x c - D^2 u) = |\det c_{i,j}| \frac{f}{g} \cdot T \quad \text{in} \quad \Omega,
\end{equation}

where \( c_i = \partial x_i c, c_{i,j} = \partial x_i \partial y_j c \). The main result of this note is

**Theorem 1.** Suppose that \( \Omega^* \) is \( c \)-convex with respect to \( \Omega \), that the cost function \( c \) satisfies (A1)-(A3), and that \( f, g \) satisfy

\begin{equation}
C_1 \leq f, g \leq C_2
\end{equation}

for some positive constants \( C_1, C_2 \). Then \( v \) is strictly \( c \)-concave and \( u \) is \( C^1 \) smooth.

We refer the reader to §2.1 for definitions of various convexity notions relative to cost functions. By approximation and the uniqueness of potential functions (when \( f, g > 0 \)), condition (1.7) can be weakened to \( 0 \leq \frac{f}{g} \cdot T < C \) and \( f, g > 0 \). From the \( C^1 \) smoothness of \( u \) it follows that the optimal mapping \( T \) is continuous.

For the reflector design problem (in the far field case), Theorem 1 was obtained in [CGH]. For the optimal transportation problem, the \( C^1 \) smoothness of \( u \) essentially belongs to Loeper [L]. His proof uses approximation by globally smooth solutions to the optimal transportation problem, established in [TW], and accordingly assumes additional conditions such as the \( c \)-convexity of both domains \( \Omega \) and \( \Omega^* \). See also remarks after Corollary 1 in §2.8. Here we give a more direct proof. Our proof is also completely different from that in [C1], in which Caffarelli proved the strict convexity and \( C^1 \) regularity for solutions to the Monge-Ampere equation with constant boundary condition. For higher regularity, the interior and global \( C^{2,\alpha} \) estimates for solutions of (1.6), in the case of the quadratic cost function, were established in [C2, C3, U1], and earlier in [D] for \( n = 2 \).
For cost functions satisfying (A1)-(A3) below, the regularity of potential functions was obtained in [MTW, TW].

By approximation and the $C^1$ regularity in Theorem 1, it follows that the potential function $u$ is fully $c$-concave if $f, g > 0$ and $c$ satisfies (A1)-(A3). In particular, every local $c$-support is a global one. See §2.8 for more discussion.

The assumption (A1)-(A3) are as follows.

(A1) For any $x, z \in \mathbb{R}^n$, there exists a unique $y = y(x, z)$ such that $D_x c(x, y) = z$.

(A2) For any $x, y \in \mathbb{R}^n$, $\det \{c_{ij}(x, y)\} \neq 0$.

(A3) For any $x \in \Omega, y \in \Omega^*$, and $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$,

\[ \sum_{i,j,k,l,p,q,r,s} (c^{p,q}c_{ij,p}c_{q,rs} - c_{ij,rs})c^{r,k}c^{s,l}\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2, \]

where $c_0$ is a positive constant, and $(c^{i,j})$ is the inverse matrix of $(c_{i,j})$.

We also assume the above conditions hold after exchanging $x$ and $y$. Formula (1.8) is equivalent to

\[ \partial_{z_k} z_c(x, y)\xi_i\xi_j\eta_k\eta_l \leq -c_0|\xi|^2|\eta|^2 \]

for all $x \in \Omega, y \in \Omega^*$ and $\xi \perp \eta \in \mathbb{R}^n$, where $y = y(x, z)$ is given by (A1), which is smooth in $x$ and $z$ by (A2). Note that for global regularity in [TW] and the subsequent application in [L], condition (A3) can be relaxed to its degenerate form, $c_0 = 0$, called A3w in [TW].

We divide the proof of Theorem 1 into several short sections. We first introduce in §2.1 various notions of convexities and concavities relative to the cost function $c$. We then indicate in §2.2 a geometric property of (A3) (see also [L]). In §2.3 we give an analytic formulation of the $c$-convexity of domains. A geometric characterization of $c$-convex domains (under condition A3) is given in §2.4. In §2.5 we prove that a local $c$-concave function is fully $c$-concave if the domain is $c$-convex and the cost function $c$ satisfies (A3). This is a key ingredient in the proof of Theorem 1. In §2.6 we show that $u$ is $C^1$ if and only if $v$ is strictly $c$-concave. We then prove Theorem 1 in §2.7. Various remarks are given in §2.8.

2. Proof of Theorem 1

2.1. Convexities relative to cost functions [MTW]. Let $u$ be a semi-concave function in $\Omega$, namely $u - C|x|^2$ is concave for a large positive constant $C$. The supergradient $\partial^+ u$ [GM] and $c$-supergradient $\partial^+_c u$ are defined by

\[ \partial^+ u(x_0) = \{ p \in \mathbb{R}^n : u(x) \leq u(x_0) + p \cdot (x - x_0) + o(|x - x_0|) \}, \]

\[ \partial^+_c u(x_0) = \{ y \in \mathbb{R}^n : u(x) \leq c(x, y) - c(x_0, y) + u(x_0) + o(|x - x_0|) \} \]
for $x$ near $x_0$, where $x_0 \in \Omega$. For a set $E \subset \Omega$, we denote $\partial^+ u(E) = \bigcup_{x \in E} \partial^+ u(x)$ and $\partial^+_c u(E) = \bigcup_{x \in E} \partial^+_c u(x)$. By (1.5) we have

$$(2.3) \quad c_x(x_0, \partial^+_c u(x_0)) = \partial^+ u(x_0).$$

Note that $\partial^+ u(x_0)$ is a closed, convex set. Hence $\partial^+_c u(x_0)$ is closed and $c$-convex with respect to $x_0$.

We may extend the above mappings to boundary points. Let $x_0 \in \partial \Omega$ be a boundary point, we denote $\partial^+ u(x_0) = \{ p \in \mathbb{R}^n : p = \lim_{k \to \infty} p_k \}$, where $p_k \in \partial^+ u(x_k)$ and $\{x_k\}$ is a sequence of interior points of $\Omega$ such that $x_k \to x_0$, and let $\partial^+_c u(x_0)$ be given by (2.3).

The $c$-normal mapping $T_u$ is defined by

$$(2.4) \quad T_u(x_0) = \{ y \in \mathbb{R}^n : u(x) \leq c(x, y) - c(x_0, y) + u(x_0) \text{ for all } x \in \Omega \}.$$ 

Note that $T_u(x_0) \subset \partial^+_c u(x_0)$.

c-support: Let $u$ be a semi-concave function in $\Omega$. A local $c$-support of $u$ at $x_0 \in \overline{\Omega}$ is a function of the form

$$h = c(\cdot, y_0) + a_0,$$

where $a_0$ is a constant and $y_0 \in \mathbb{R}^n$, such that $u(x_0) = h(x_0)$ and $u(x) \leq h(x)$ near $x_0$. If $u(x) \leq h(x)$ for all $x \in \Omega$, then $h$ is a global $c$-support (or $c$-support for short) of $u$ at $x_0$. If $h$ is a local $c$-support of $u$ at $x_0$, then $y_0 \in \partial^+_c u(x_0)$ and $a_0 = u(x_0) - c(x_0, y_0)$.

c-concavity of functions: We say a semi-concave function $u$ is locally $c$-concave if for any point $x_0 \in \overline{\Omega}$ and any $y \in \partial^+_c u(x_0)$, $h = c(\cdot, y) - c(x_0, y) + u(x_0)$ is a local $c$-support of $u$ at $x_0$. We say $u$ is $c$-concave if for any point $x_0 \in \overline{\Omega}$, there exists a global $c$-support at $x_0$ in $\Omega$. We say $u$ is strongly $c$-concave if it is both locally $c$-concave and $c$-concave. We say $u$ is fully $c$-concave if it is locally $c$-concave and every local $c$-support of $u$ is a global $c$-support. We say $u$ is strictly $c$-concave if it is fully $c$-concave and every $c$-support of $u$ contacts its graph at one point only.

c-segment: A set of points $\ell \subset \mathbb{R}^n$ is a $c$-segment with respect to a point $y_0 \in \mathbb{R}^n$ if $D_y c(\ell, y_0)$ is a line segment in $\mathbb{R}^n$.

c-convexity of domains: We say a set $U$ is $c$-convex with respect to another set $V$ if the image $c_y(U, y)$ is convex for each $y \in V$. Equivalently, $U$ is $c$-convex with respect to $V$ if for any two points $x_0, x_1 \in U$ and any $y \in V$, the $c$-segment relative to $y$ connecting $x_0$ and $x_1$ lies in $\Omega$. By (2.3), a $c$-convex domain is topologically a ball.

By definition, a $c$-concave function $u$ can be represented as [GM]

$$u(x) = \inf \{ c(x, y) - a(y) : y \in T_u(\Omega) \},$$

where $a$ is a function of $y$ only. By (1.4), a potential function $u$ is $c$-concave [GM]. Our Theorem 1 implies that $u$ is furthermore fully $c$-concave under assumption (A3). If $u$ is $C^1$, then local $c$-support is unique and $c$-concavity is equivalent to full $c$-concavity. We also remark that a potential function may not be locally $c$-concave in general.
Similarly we can define $c^*$-segment, $c^*$-support, $c^*$-convexity and $c^*$-concavity by exchanging variables $x$ and $y$ [MTW]. In this paper, we will generally omit the superscript * when the meaning is clear.

2.2. A geometric property of (A3). Let $y_0, y_1$ be two points in $\Omega^*$. Let $\overline{y_0y_1}$ be the $c$-segment relative to a point $x_0 \in \Omega$, connecting $y_0$ and $y_1$. By definition,

\[
\overline{y_0y_1} = \{y_t : c_x(x_0, y_t) = p_t, \ t \in [0, 1]\},
\]

where $p_t = tp_1 + (1 - t)p_0$, and $p_0 = c_x(x_0, y_0)$, $p_1 = c_x(x_0, y_1)$. Let

\[
h_i(x) = c(x, y_i) - a_i, \quad i = 0, 1,
\]

\[
h_t(x) = c(x, y_t) - a_t,
\]

where $t \in (0, 1)$, $a_0$, $a_1$ and $a_t$ are constants such that $h_0(x_0) = h_1(x_0) = h_t(x_0)$. Suppose (A3) holds. Then for $x \neq x_0$, near $x_0$, we have the inequality

\[
h_t(x) > \min\{h_0(x), h_1(x)\},
\]

which is crucial for the remaining analysis of this paper.

Inequality (2.7) follows from (1.9). Indeed, by a rotation of axes, we assume that $p_1 = p_0 + \delta e_n$, where $\delta$ is a positive constant and $e_n = (0, \cdots, 0, 1)$ is the unit vector in the $x_n$-axis. By (1.9),

\[
\frac{d^2}{dt^2} c_{ij}(x, y(x, p_t))\xi_i\xi_j \leq -c_0\delta^2
\]

for any unit vector $\xi$ orthogonal to $e_n$, where $y = y(x_0, p_t)$ is given in (A1). Now (2.7) follows from (2.8); for details see [L].

2.3. An analytic formulation of the $c$-convexity of domains. If $\Omega$ is $c$-convex with respect to $\Omega^*$, by definition, $c_y(\Omega, y)$ is convex for any $y \in \Omega^*$. Suppose $0 \in \partial \Omega$ and locally $\partial \Omega$ is given by

\[
x_n = \rho(x')
\]

with $D\rho(0) = 0$ such that $e_n$ is the inner normal at 0, where $x' = (x_1, \cdots, x_{n-1})$. Then at $(0, y)$ (for a fixed point $y$),

\[
\partial_{x_i} c_y = c_{i,y} + c_{n,y} \rho_{x_i},
\]

\[
\partial_{x_i x_j} \langle \gamma, \gamma \rangle = \langle c_{ij,y}, \gamma \rangle + \langle c_{n,y}, \gamma \rangle \rho_{x_i x_j} \geq 0,
\]

where $\gamma$ is the inner normal of $c_y(\Omega, y)$ at $c_y(0, y)$. We may write (2.10) explicitly,

\[
c_{ij,y} \gamma_t + c_{n,y} \gamma_t \rho_{ij} \geq 0.
\]
Make the linear transformation 
\[ \hat{y}_k = a_{kl} y_l. \]

Then we have
\[ (2.11) \]
\[ c_{ij,\hat{y}_k} a_{kl} a_{ml}^{-1} \hat{y}_m + c_{n,\hat{y}_k} a_{kl} a_{ml}^{-1} \hat{y}_m \rho_{ij} \geq 0. \]

Let \( a_{ij} = c_{i,j}(0, y) \). Then \( c_{x_i,\hat{y}_j} = \delta_{ij} \) and \( \hat{\gamma} = e_n \). We obtain
\[ (2.12) \]
\[ c_{ij,\hat{y}_n} + \rho_{x_i x_j} \geq 0, \]
which is equivalent to
\[ (2.13) \]
\[ c_{ij,\hat{y}_n} c^{k,n} + \rho_{x_i x_j} \geq 0. \]

Let \( \varphi \in C^2(\Omega) \) be a defining function of \( \Omega \). That is \( \varphi = 0, |\nabla \varphi| \neq 0 \) on \( \partial \Omega \) and \( \varphi < 0 \) in \( \Omega \). From (2.13), we obtain an analytic formulation of the \( c \)-convexity of \( \Omega \) relative to \( \Omega^* \) [TW],
\[ (2.14) \]
\[ \varphi_{ij}(x) - c_{i,l}(x, y) c_{j,k}(x, y) \varphi_l(x) \geq 0 \quad \forall \ x \in \partial \Omega, y \in \Omega^*. \]

Conversely, if \( \Omega \) is simply connected and (2.14) holds, then \( \Omega \) is \( c \)-convex. Following [TW], we call \( \Omega \) uniformly \( c \)-convex with respect to \( \Omega^* \) if the matrix in (2.14) is uniformly positive.

### 2.4. Geometric properties of the \( c \)-convexity of domains

Let \( y_0, y_1 \) be any two given points in \( \Omega^* \). Denote
\[ (2.15) \]
\[ \mathcal{N} = \mathcal{N}_{y_0, y_1, a} = \{ x \in \mathbb{R}^n : c(x, y_0) = c(x, y_1) + a \}, \]
where \( a \) is a constant. Assume that the origin \( 0 \in \mathcal{N} \) and locally \( \mathcal{N} \) is represented as
\[ (2.16) \]
\[ x_n = \eta(x') \]
such that \( \eta(0) = 0 \) and \( D \eta(0) = 0 \) (obviously \( \eta \) also depends on \( y_0, y_1 \) and \( a \)). Then
\[ (2.17) \]
\[ c(x', \eta(x'), y_0) = c(x', \eta(x'), y_1) + a. \]

Differentiating (2.17) gives
\[ (2.18) \]
\[ c_i(0, y_0) + c_n \eta_i = c_i(0, y_1) + c_n \eta_i, \]
\[ c_{ij}(0, y_0) + c_n \eta_{ij} = c_{ij}(0, y_1) + c_n \eta_{ij}. \]

We obtain
\[ (2.19) \]
\[ [c_n(0, y_1) - c_n(0, y_0)] \eta_{ij} + [c_{ij}(0, y_1) - c_{ij}(0, y_0)] = 0. \]
Now let $y_0$ be fixed but let $y_1$ and $a$ vary in such a way that $y_1 \to y_0$ and the set $\mathcal{N} = \mathcal{N}_{y_0, y_1, a}$, given by (2.15), is tangential to $\{x_n = 0\}$, namely $\eta(0) = 0$ and $D\eta(0) = 0$. By a linear transform as in §2.3 we assume that $c_{x_i, y_j} = \delta_{ij}$ at $x = 0$ and $y = y_0$. Then

$$\frac{y_1 - y_0}{|y_1 - y_0|} \to e_n.$$ 

We obtain

(2.20) \hspace{1cm} c_{n,n}(0, y_0)\eta^0_{ij} + c_{ij,n}(0, y_0) = 0,$$

where $\eta^0$ is the limit of $\eta_{y_0, y_1, a}$ as $y_1$ and $a$ vary as above.

Now let $\Omega$ be $c$-convex, (uniformly $c$-convex), with respect to $\Omega^*$. Suppose that $\partial\Omega$ is given by (2.9) with $D\rho(0) = 0$ so that $\partial\Omega$ is tangential to $\mathcal{N}$ at the origin. Then by (2.12) and (2.20) we obtain

(2.21) \hspace{1cm} D^2(\rho - \eta^0) \geq 0, \hspace{0.5cm} (> 0), \hspace{0.5cm} \text{at } 0.$$

From (2.21) we obtain some useful geometric properties of $c$-convex domains, assuming that $c$ satisfies (A3).

First, if $\Omega$ is $c$-convex with respect to $\Omega^*$, then for any compact subset $G \subset \Omega^*$, $\Omega$ is uniformly $c$-convex with respect to $G$. Indeed, let $y_0, y_1$ be two points in $\Omega^*$. Let $p_0 = c_x(0, y_0)$, $p_1 = c_x(0, y_1)$. For $t \in (0, 1)$, let $p_t = tp_1 + (1 - t)p_0$ and $y_t$ satisfy $c_x(0, y_t) = p_t$. Then the set $\{p_t : 0 \leq t \leq 1\}$ is a line segment and the set $\{y_t : 0 \leq t \leq 1\}$ is a $c$-segment. Suppose $p_1 = p_0 + \delta e_n$ for some $\delta > 0$. Let $h_t(x) = c(x, y_t) + a_t$, where $a_t = c(0, y_0) - c(0, y_t)$ such that $h_t(0) = h_0(0)$ for all $t \in [0, 1]$. Let $\mathcal{N}_t = \{x \in \mathbb{R}^n : h_t(x) = h_0(x)\}$ and suppose that near $0$, $\mathcal{N}_t$ is given by $x_n = \eta_t(x')$. Since $c_x(0, y_t) = p_t$, we see that $\mathcal{N}_t$ is tangential to $\{x_n = 0\}$, namely $D(\eta_t - \eta_t')(0) = 0$. By (2.8) we have furthermore the monotonicity formula

(2.22) \hspace{1cm} D^2(\eta_t - \eta_t')(0) > 0$$

for any $t > t'$ and $t, t' \in [0, 1]$. Geometrically it implies that $\mathcal{N}_t$ lies above $\mathcal{N}_{t'}$ if $t > t'$, namely $\eta_t(x') \geq \eta_{t'}(x')$ for $x'$ near $0$, and equality holds only at $x' = 0$. Consequently we obtain from (2.21) the strict inequality

(2.23) \hspace{1cm} D^2(\rho - \eta) > 0 \hspace{0.5cm} \text{at } 0.$$

From (2.23) we obtain the above mentioned property.

Next, for any $y_0, y_1 \in \Omega^*$, if $\mathcal{N}$ (given in (2.15)) is tangent to $\partial\Omega$ at some point $x_0$ and if $\Omega$ is $c$-convex with respect to $\Omega^*$, then the whole domain $\Omega$ lies on one side of $\mathcal{N}$. Indeed, we may assume $x_0 = 0$, locally $\partial\Omega$ and $\mathcal{N}$ are given respectively by (2.9) and (2.16), such that $D\rho(0) = D\eta(0) = 0$. By (2.23), $\rho(x) > \eta(x)$ for $x$ near $0$, $x \neq 0$. Denote

$$U = U_{y_0, y_1, a} = \{x \in \mathbb{R}^n : c(x, y_0) < c(x, y_1) + a\}$$
so that $\mathcal{N} = \partial U$. If $\Omega$ does not lie on one side of $\mathcal{N}$, namely $\Omega$ is not contained in $U$, then $\overline{\Omega} - U$ contains two disconnected components (one is the origin). Since $\partial \Omega$ is a closed, compact hypersurface, we decrease the constant $a$ (shrinking the set $U$) until a moment when two components of $\overline{\Omega} - U$ meet each other at some point $x^* \in \partial \Omega$. But since $\mathcal{N}$ is tangent to $\partial \Omega$ at $x^*$, we reach a contradiction by (2.23).

It follows that if $\Omega$ is $c$-convex with respect to $\Omega^*$, then

$$
\Omega = \bigcap U_{y_0, y_1, a},
$$

where the intersection is for all $y_0, y_1 \in \Omega^*$ and constant $a$ such that $U_{y_0, y_1, a} \supset \Omega$. However, we don’t know if the converse is true, namely whether $\Omega$ is $c$-convex with respect to $\Omega^*$ if it is given by (2.24).

The above properties also extend to cost functions satisfying A3w and uniformly $c$-convex domains.

### 2.5. Local $c$-support is global.

Let $u$ be a locally $c$-concave function in $\Omega$ with $\partial^+ u(\Omega) \subset \Omega^*$. Suppose $\Omega$ is $c$-convex with respect to $\Omega^*$. Let $h(x) = h_a(x) = c(x, y_0) + a$ be a local $c$-support of $u$ at $x_0$. Then $h$ is a global $c$-support of $u$, namely

$$
u(x) \leq h(x) \quad \forall x \in \Omega. \quad (2.25)$$

Indeed, if this is not true, then for $\varepsilon > 0$ small, the set $\{x \in \Omega : u(x) > h_{a-\varepsilon}(x)\}$ contains at least two disconnected components. We increase $\varepsilon$ (moving the graph of $h$ vertically downwards) until at a moment $\varepsilon = \varepsilon_0 > 0$, two components first time touch each other at some point $x_0$. If $x_0$ is an interior point of $\Omega$, by definition $h_{a-\varepsilon_0}$ cannot be a local $c$-support at $x_0$, which implies that $y_0 \notin \partial^+ u(x_0)$. We claim that for a sufficiently small $r > 0$, $y_0$ does not lie in the set $\partial^+_c u(B_r(x_0))$ either. Indeed, if $h_k = c(\cdot, y_0) + a_k$ for $k = 1, 2, \ldots$ is a sequence of local $c$-support of $u$ at $x_k$ and if $x_k, a_k \to x_0, a_0$, we have $y_0 \notin \partial^+_c u(x_0)$ as $u$ is semi-concave. Hence $h_0 = c(\cdot, y_0) + a_0$ is a local $c$-support of $u$ at $x_0$. This is a contradiction.

Therefore $h_{a-\varepsilon_0}$ and $u$ are transversal near $x_0$, and for $\varepsilon < \varepsilon_0$, close to $\varepsilon_0$, locally the set $\{x \in \Omega : u(x) > h_{a-\varepsilon}(x)\}$ cannot contain two disconnected components. Hence $x_0$ must be a boundary point of $\Omega$.

In case $x_0$ is a boundary point of $\Omega$, we will also reach a contradiction by (2.23). Similarly as above, $h_{a-\varepsilon_0}$ cannot be a local $c$-support at $x_0$, namely $y_0 \notin \partial^+_c u(x_0)$. Without loss of generality let us assume that $x_0 = 0$ and locally $\partial \Omega$ is tangent to $\{x_n = 0\}$ such that $e_n$ is an inner normal of $\Omega$ at $0$. As before we also assume that $c_{x_i, y_j} = \delta_{ij}$ at $x = 0$ and $y = y_0$. Let $p_0 = c_x(0, y_0)$. By subtracting a linear function of $x$ from both $c$ and $u$, we assume that $\{x_n = 0\}$ is a tangent plane of $h$ at $0$. Then we have $p_0 = 0$, $u(0) = 0$, and $u(x) \leq o(|x|)$ as $x \to 0$. Since $y_0 \notin \partial^+_c u(x_0)$, we have

$$
\beta = \lim_{t \to 0} \frac{1}{t} u(te_n) < 0. \quad (2.26)
$$

Let $p_1 = \beta e_n$, $p_t = tp_1 + (1 - t)p_0$ for $t \in [0, 1]$, and let $y_t \in \mathbb{R}^n$ be determined by $c_x(0, y_t) = p_t$. Then $p_1 \in \partial^+ u(0)$ and $y_1 \in \partial^+_c u(0)$. Since $u$ is locally $c$-concave,
\( c(x, y_1) + b \) is a local \( c \)-support of \( u \) at 0, where \( b \) is a constant such that \( c(0, y_1) + b = u(0) = 0 \). Denote

\[
U = \{ x \in \mathbb{R}^n : h_{a-\varepsilon_0}(x) > c(x, y_1) + b \}
\]

and \( \mathcal{N} = \partial U = \{ x \in \mathbb{R}^n : h_{a-\varepsilon_0}(x) = c(x, y_1) + b \} \), such that \( 0 \in \mathcal{N} \). Since \( \Omega \) is \( c \)-convex, by (2.23) we see that \( \Omega \subset U \). But since \( c(x, y_1) + b \) is a local \( c \)-support of \( u \) at 0, we have

(2.27)

\[
h_{a-\varepsilon_0}(x) > c(x, y_1) + b \geq u(x)
\]

for \( x \in \Omega \), near the origin. We reach a contradiction by our choice of \( \varepsilon_0 \). This completes the proof of (2.25).

From the above proof, we see that if \( h \) is a \( c \)-support of \( u \), then the contact set \( \{ x \in \Omega : h(x) = u(x) \} \) cannot contain two disconnected components (or points). In other words, the contact set is connected.

We also remark that if \( u \) is a potential function to the optimal transportation with positive mass distributions \( f \) and \( g \) so that \( u \) is uniquely determined up to a constant, the boundary point case \( x_0 \in \partial \Omega \) can be reduced to the case \( x_0 \in \Omega \) by extending \( u \) to larger domains, and it is not necessary to define the mapping \( \partial^+ u \) and \( \partial^+_c u \) on boundary points.

Taking account of our remark at the end of the previous section, we also see that (2.25) extends to A3w costs and uniformly \( c \)-convex domains. Consequently we obtain an alternate proof of the \( c \)-convexity of the solutions in Section 6 of [TW]. Furthermore by domain approximation, we may then extend (2.25) further to A3w costs and \( c \)-convex domains.

### 2.6. Potential functions.

Let \((u, v)\) be potential functions to the optimal transportation problem (1.2). Then for any point \( x_0 \in \Omega \), by (1.4), there exists a point \( y_0 \in \Omega^* \) such that \( u(x_0) + v(y_0) = c(x_0, y_0) \). Hence

(2.28)

\[
h(x) = c(x, y_0) - v(y_0)
\]

is a \( c \)-support of \( u \) at \( x_0 \), and

(2.29)

\[
h^*(y) = c(x_0, y) - u(x_0)
\]

is a \( c \)-support of \( v \) at \( y_0 \).

If \( u \) is \( C^1 \) at \( x_0 \), it has a unique \( c \)-support at \( x_0 \). As a potential function is semi-concave, it is twice differential a.e. in \( \Omega \). Hence \( u \) has a unique \( c \)-support almost everywhere.

Next we consider the case when \( u \) is not \( C^1 \) at \( x_0 \). Let us first introduce the terminology **extreme point**. Let \( E \) be a convex set in \( \mathbb{R}^k \). We say a point \( p \in \overline{E} \) is an extreme point of \( E \) if there exists a plane \( P \) such that \( P \cap \overline{E} \) contains only the point \( p \). It is easy to show, by induction on dimensions, that any interior point in a convex set can be expressed as a linear combination of extreme points.
If $u$ is not $C^1$ at an interior point $x_0$, since $u$ is semi-concave, $\partial^+ u(x_0)$ is a convex set of dimension $k$ for some integer $1 \leq k \leq n$. Let $N^c_u(x_0)$ denote the set of extreme points of $\partial^+ u(x_0)$. Let

$$T^c_u(x_0) = \{ y \in \mathbb{R}^n : c_x(x_0, y) \in N^c_u(x_0) \}. \quad (2.30)$$

Then for any $y_0 \in T^c_u(x_0)$, the function

$$h(x) = c(x, y_0) + a$$

is a global $c$-support of $u$ at $x_0$, where $a$ is a constant such that $h(x_0) = u(x_0)$.

This assertion follows from a similar one for concave functions, which can be proved by blowing up the graph of $u$ to a concave cone. That is for any $p \in T^c_u(x_0)$, there exists a sequence of $C^1$-smooth points $\{x_k\}$ of $u$, $x_k \to x_0$, such that the $c$-support of $u$ at $x_k$ converges to a $c$-support of $u$ at $x_0$. Recall that at $C^1$ points, $u$ has a unique global $c$-support.

From the above assertion and (2.7) it follows that a potential function is a local $c$-concave function if (A3) is satisfied. That is for any $y' \in \partial^+_c u(x_0)$, the function

$$h(x) = c(x, y') - c(x_0, y') + u(x_0)$$

is a local $c$-support of $u$ at $x_0$. By (1.4), $h$ is a global $c$-support if

$$u(x_0) + v(y') = c(x_0, y'). \quad (2.31)$$

We remark that in general a potential function may fail to be locally $c$-concave (with respect to the definition in §2.1), if (A3w) is violated.

From the above assertion it also follows that if $u$ is not $C^1$ at $x_0$, then the function $h^*$ in (2.29) is a $c$-support of $v$ at any point $y_0 \in T^c_u(x_0)$. In other words, $T^c_u(x_0)$ is contained in the contact set

$$C = \{ y \in \Omega^* : h^*(y) = v(y) \}. \quad (2.32)$$

By (2.31), a local $c$-support of $u$ at $x_0$ is a global one if and only if the contact set $C$ is $c$-convex (with respect to $x_0$).

Since $\Omega^*$ is $c$-convex with respect to $\Omega$, from the argument in §2.5, any local $c$-support of $v$ is a global one. Hence $v$ is fully $c$-concave in $\Omega^*$. Furthermore, $u$ is $C^1$ smooth if and only if its dual function $v$ is strictly $c$-concave.

**2.7 Proof of Theorem 1.** In the proof of Theorem 1 we will use the Perron lifting. Consider the Dirichlet problem

$$\det(D^2_y c - D^2 w) = \psi(x, Dw) \quad \text{in} \quad B_r(y_0), \quad w = \varphi \quad \text{on} \quad \partial B_r(y_0),$$

$$
(2.33)$$
where $\psi > 0$, $\varphi, \psi \in C^\infty$. Suppose $r > 0$ is sufficiently small and there is a supersolution to (2.33) (note that when $r$ is sufficiently small, there is always a supersolution). From [MTW], there is a solution $w \in C^\infty(B_r(y_0))$ to (2.33) such that the matrix $(D^2_y c - D^2 w) > 0$. By approximation and the interior a priori estimates [MTW], there is a solution $w \in C^\infty(B_r(y_0)) \cap C^0(B_r(y_0))$ if $\varphi \in C^0$ and $\psi \in C^{1,1}$. Obviously $w$ is locally $c$-concave. By §2.5, $w$ is $c$-concave in $B_r$.

Let $(u, v)$ be the potential functions to (1.2). To prove Theorem 1, it suffices to prove that $v$ is strictly $c$-concave. By approximation we assume that $f, g$ are positive and smooth. By §2.4 we may also assume that $\Omega^*$ is uniformly $c$-convex with respect to $\Omega$. From §2.6, $v$ is fully $c$-concave and every local $c$-support of $v$ is a global one. Suppose to the contrary that $v$ is not strictly $c$-concave. Then there is a $c$-support $h^*$ of $v$ at some point $y_0 \in \Omega^*$ such that the contact set

$$C^* = \{ y \in \Omega^* : h^*(y) = v(y) \}$$

contains more than one point.

From the argument in §2.5, $C^*$ cannot contain more than one disconnected component. In other words, $C^*$ is connected. Hence for any $r > 0$ small, the intersection $\partial B_{2r}(y_0) \cap C^*$ is not empty. Let $\{ B_j : j = 1, \cdots, k \}$ be finitely many balls with radius $r/2$, centered on $\partial B_r(y_0)$, such that $\bigcup_{j=1}^k B_j \supset \partial B_r(y_0)$.

Denote $v_0 = v$. For $j = 1, \cdots, k$, let $v_j$ be the solution of

(2.34) \[ \det(D^2_y c - D^2 w) = \frac{1}{2} \delta_0 \quad \text{in} \quad B_1, \]

\[ w = v_{j-1} \quad \text{on} \quad \partial B_r(0), \]

where

(2.35) \[ \delta_0 = \inf |\det c_{i,j}| \frac{f(x)}{g(T(x))}, \]

which is positive by assumption of Theorem 1. Extend $v_j$ to the whole $\Omega^*$ such that $v_j = v_{j-1}$ in $\Omega^* - B_j$. Then $\overline{v} =: v_k$ is locally $c$-concave in $\Omega^*$. From §2.5, it is fully $c$-concave.

By the a priori estimates in [MTW], $v_j$ is smooth in $B_j$, for $j = 1, \cdots, k$. By the comparison principle, we have $v_1 < v_0$ in $B_1$ and by induction, $v_j < v_0$ in $B_j$ for all $j = 2, \cdots, k$. It follows that $\overline{v} < v$ near $\partial B_r(y_0)$.

We have therefore obtained another fully $c$-concave function $\overline{v}$ which satisfies

$$\overline{v} = v \quad \text{in} \quad B_{r/2}(y_0) \cup \{ \Omega^* - B_{3r/2}(y_0) \},$$

$$\overline{v} < v \quad \text{near} \quad \partial B_r(y_0).$$

Hence the contact set $\{ y \in \Omega^* : h^*(y) = \overline{v}(y) \}$ cannot be connected. But this is impossible from the argument in §2.5 (as remarked at the end of §2.5). Hence $v$ is strictly $c$-concave. This completes the proof of Theorem 1.

2.8. Remarks. First we have the following result which follows from the $C^1$ regularity in Theorem 1 and approximation.
Corollary 1. Suppose that \( f, g > 0, f, g \in L^1 \), \( c \) satisfies (A1)-(A3), and \( \Omega^* \) is \( c \)-convex with respect to \( \Omega \). Then the potential function \( u \) is fully \( c \)-concave.

Corollary 1 is a complement to the paper [MTW]. In [MTW] we introduced a notion of generalized solution to the boundary value problem (1.6) and proved interior regularity under the conditions in Corollary 1 and assuming also the smoothness of \( f \) and \( g \). If the potential function \( u \) is not fully \( c \)-concave, a local \( c \)-support of \( u \) may not be a global one. In such case, the definition of generalized solution in [MTW] is not proper and the comparison principle may not hold in arbitrary sub-domains. Corollary 1 rules out the possibility provided the cost function satisfies (A1)-(A3) and \( \Omega^* \) is \( c \)-convex with respect to \( \Omega \), as assumed in [MTW]. However to get the full \( c \)-concavity of \( u \) we have to prove Theorem 1 first. Clearly a short and direct proof is desired. But in dimension 2, the full \( c \)-concavity is a direct consequence of (2.7).

Corollary 2. Suppose the cost function \( c \) satisfies (A1)-(A3). Then any potential functions \((u, v)\) defined in the whole \( \mathbb{R}^2 \) are fully \( c \)-concave.

Indeed, for any point \( x_0 \in \Omega \), if \( u \) is \( C^1 \) at \( x_0 \), there is a unique global \( c \)-support of \( u \) at \( x_0 \). Otherwise, let \( y_0, y_1 \) be any two points in \( \partial^+_- u(x_0) \). Let \( h_0(x) = c(x, y_0) - a_0 \) and \( h_1(x) = c(x, y_1) - a_1 \) be two \( c \)-supports of \( u \) at \( x_0 \). Denote \( \mathcal{N} = \{ x \in \mathbb{R}^n : h_0(x) = h_1(x) \} \). \( \mathcal{N} \) is a curve which divides \( \mathbb{R}^2 \) into two parts, and both are non-compact. It suffices to show that

\[
(2.36) \quad h_t \geq h_0 \quad (= h_1) \quad \text{on} \quad \mathcal{N},
\]

where \( t \in (0, 1) \) and \( h_t \) is as in (2.7). But if (2.36) is not true, by moving the graph of \( h_t \) downwards, we see that there is a constant \( b \) and a point \( x' \in \mathcal{N} \) such that

\[
(2.37) \quad h_t(x') - b = \min\{h_0(x'), h_1(x')\},
\]

\[
(2.38) \quad h_t(x) - b \leq \min\{h_0(x), h_1(x)\} \quad x \in \mathcal{N} \quad \text{near} \quad x'.
\]

But this is in contradiction with (2.7) at \( x' \). Hence Corollary 2 holds.

Note that for potential functions on bounded domains, by the uniqueness of potential functions when restricted to \( \{ f > 0 \} \), we see that \( u \) (and similarly \( v \)) is fully \( c \)-concave when restricted to \( \{ f > 0 \} \).

The proof of Corollary 2 does not extend to higher dimensions, as we don’t know if there is a point \( x' \in \mathcal{N} \) such that (2.37) and (2.38) hold. But Corollary 2 holds on compact manifolds of any dimension, as the set \( \mathcal{N} \) is compact. In particular it holds for the reflector design problem (in the far field case) [W1, W2]. But for the reflector design problem, a \( c \)-support is a paraboloid with focus at the origin and one can also verify (2.36) directly [CGH].

We remark that if the cost function \( c \) does not satisfy (A3), then (2.7) may not hold. Loeper [L] shows that if condition A3w is violated, there are potential functions which are not fully \( c \)-concave. Furthermore, the potential function \( u \) may not be \( C^1 \) smooth even if both \( f \) and \( g \) are positive and smooth. Note that Loeper’s potential functions are the negative of those here so that our \( c \)-concavity is equivalent to his \( c \)-convexity.
Remark. After this paper was finished, we learned that Kim and McCann found a direct proof of Corollary 1 above. They proved that under A3 (A3w, resp.), \( \frac{d^2}{dt^2} h_t > 0 \) (\( \geq 0 \), resp.), from which it follows that the contact set of the potential function \( u \) with its \( c \)-support is connected, where \( h_t \) was given in (2.6).

References


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