THOM ISOMORPHISM AND PUSH-FORWARD MAP IN TWISTED K-THEORY

ALAN L. CAREY AND BAI-LING WANG

Abstract. We establish the Thom isomorphism in twisted K-theory for any real vector bundle and develop the push-forward map in twisted K-theory for any differentiable map $f : X \to Y$ (not necessarily K-oriented). The push-forward map generalizes the push-forward map in ordinary K-theory for any K-oriented map and the Atiyah-Singer index theorem of Dirac operators on Clifford modules. For D-branes satisfying Freed-Witten’s anomaly cancellation condition in a manifold with a non-trivial $B$-field, we associate a canonical element in the twisted K-group to get the so-called D-brane charges.

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1. Introduction

In complex K-theory, the push-forward map $f_1^{c_1} : K^*(X) \to K^*(Y)$ was established by Atiyah-Hirzebruch [1] for any differentiable $c_1$-map $f : X \to Y$ where $c_1 \in H^2(X, \mathbb{Z})$ satisfies

$$c_1 \equiv w_2(X) - f^*w_2(Y) \pmod{2}$$

with $w_2(X)$ and $w_2(Y)$ being the second Stiefel-Whitney classes of $X$ and $Y$ respectively. Such a map $f : X \to Y$ is called K-oriented.

The push-forward map $f_1^{c_1}$, also called the Gysin homomorphism in K-theory, gives rise to the following general Riemann-Roch theorem, for $a \in K^*(X)$,

$$Ch(f_1^{c_1}(a))\hat{A}(X) = f_*(Ch(a)e^{\frac{1}{2}w_1}\hat{A}(Y)),$$

where $f_*$ is the Gysin homomorphism in ordinary cohomology theory, $Ch$ is the Chern character in K-theory, $\hat{A}(X)$ and $\hat{A}(Y)$ are the A-hat genus (which can be expressed in terms of the Pontrjagin classes) of $X$ and $Y$ respectively. The set of K-orientations for $f : X \to Y$ is an affine space modeled on $2H^2(X, \mathbb{Z})$, hence $H^2(X, \mathbb{Z})$ acts (and transitively if $H^2(X, \mathbb{Z})$ has no 2-torsion elements) via

$$c_1 \mapsto c_1 + 2c,$$

for $c \in H^2(X, \mathbb{Z})$. The dependence of the push-forward map $f_1^{c_1}$ on the K-orientation $c_1$ can be described by

$$f_1^{c_1+2c}(a) = f_1^{c_1}(a \cdot [L_c])$$

where $[L_c] \in K^0(X)$ is the K-element defined by the equivalence class of the line bundle over $X$ with first Chern class $c \in H^2(X, \mathbb{Z})$.

In this paper, we will develop the push-forward map in twisted K-theory for any differentiable map $f : X \to Y$, which is not necessarily K-oriented. We aim to present the results in a fashion which demonstrates that for general manifolds and maps twisted K-theory provides the natural framework in which to study the push-forward map, Thom isomorphism and topological index.

Recall that twisted K-theory for a smooth manifold $X$ with a class $\sigma \in H^3(X, \mathbb{Z})$ is defined using a locally trivial projective Hilbert bundle over $X$ with infinite dimensional separable fibers, $P_\sigma$, whose Dixmier-Douady class is $\sigma \in H^3(X, \mathbb{Z})$.

$$K^*_P(X) = K^*_P(X) \oplus K^1_P(X).$$

As a $\mathbb{Z}_2$-graded cohomology theory it was fully developed by Atiyah and Segal in [2]. Earlier approaches may be found in [12][25][6]. Automorphisms of the $PU(H)$-principal bundle $P_\sigma$ result in a natural action of the Picard group $Pic(X)$ on the twisted K-group $K^*_P(X)$. Up to this natural action of $Pic(X)$, $K^*_P(X)$ depends only on the Dixmier-Douady class of $P_\sigma$. With this understood, we often denote the twisted K-group by $K^*_\sigma(X)$. Though in practice, we always fix a $PU(H)$-principal bundle $P_\sigma$ with Dixmier-Douady class $\sigma$.

Twisted K-theory for manifolds carrying a projective Hilbert bundle is a homotopy invariant and satisfies the Mayer-Vietoris property. Twisted K-theory has attracted considerable interest. It is intimately related to D-brane physics in superstring theory. Moreover the result of Freed-Hopkins-Teleman [13] which identifies the equivariant twisted K-theory of a compact Lie group...
G with the Verlinde ring of projective representations of the loop group \(LG\) has focussed the attention of many researchers.

We now recall the viewpoint of [6] which we will exploit later in the paper. Denote by \(W_3(f)\) the image of \(w_2(X) - f^*(w_2(Y)) \in H^2(X, \mathbb{Z}_2)\) under the Bockstein homomorphism
\[
\beta: H^2(X, \mathbb{Z}_2) \to H^3(X, \mathbb{Z}).
\]
For \(f : X \to pt\), \(W_3(f)\) is the third integral Stiefel-Whitney class \(W_3(X)\) of \(X\). Now [6] regards \(K^0_{W_3(X)}(X)\) as the Grothendieck group formed from bundle gerbe modules of a bundle gerbe over \(X\) with Dixmier-Douady class \(W_3(X)\).

The main theorem of this paper is the following.

**Theorem 1.1.** Let \(f : X \to Y\) be a differentiable map and \(\sigma \in H^3(Y, \mathbb{Z})\). There exists a push-forward map, compatible with the natural action of the Piccard group \(Pic(Y)\),
\[
f_! : K^*_f \sigma + W_3(f)(X) \to K^*_\sigma(Y)
\]
with grading shifted by \(\dim X - \dim Y (\text{mod 2})\). In particular,
1. if \(f : X \to Y\) is K-oriented, i.e., \(W_3(f) = 0\), with \(c_1 \in H^2(X, \mathbb{Z})\) satisfying
\[
c_1 \equiv w_2(X) - f^*w_2(Y) \pmod 2,
\]
then there exists a natural push-forward map
\[
f_!^{c_1} : K^*_{f, \sigma} (X) \to K^*_\sigma(Y)
\]
which agrees with the push-forward map for ordinary K-theory when \(\sigma\) is trivial.
2. if \(f : X \to pt\), then
\[
f_! : K^*_W(X)(X) \to K^*(pt) \cong \mathbb{Z}
\]
is the Atiyah-Singer index theorem for Dirac operators on Clifford modules (Theorem 4.3 in [4]), see also [21] and the introduction of [2].
3. if \(i : Q \to X\) is a closed submanifold \(Q\) of a spin manifold \(X\) such that
\[
i^* \sigma + W_3(Q) = 0,
\]
then the push-forward map defines the charge of a D-brane \((Q, \xi)\) supported on \(Q\) for \(\xi \in K(Q)\)
\[
i_! : K(Q) \to K^*_\xi(X).
\]
4. if \(f : X \to Y\) is a fibration with a closed oriented fiber, then the topological index in twisted K-theory is given by
\[
(1.1) \quad K^*_\pi \sigma_f(T(X/Y)) \xrightarrow{\cong} K^*_{f, \sigma + W_3(f)}(X) \xrightarrow{f_!} K^*_\sigma(Y),
\]
where \(\pi : T(X/Y) \to X\) is the vertical tangent bundle of \(f : X \to Y\) and \(W_3(\pi) = W_3(f)\).
For a differentiable map \( f : X \to Y \) with the \( K \)-orientation condition (i.e., \( W_3(f) = 0 \)), the above push-forward map in twisted K-theory is also established in \([18][3]\) by using a \( C^* \)-algebraic approach. The topological index (1.1) in twisted K-theory when \( W_3(f) = 0 \) is developed in \([18]\) using a different method.

One of the main technical issues is to establish the Thom isomorphism in twisted K-theory for general twistings \( \sigma \in H^3(M, \mathbb{Z}) \), which is done in section 3. This also generalizes the earlier result of Donovan-Karoubi in \([12]\) for a torsion twisting.

In order to give a uniform treatment in both the torsion and non-torsion cases we reprove the Thom isomorphism in twisted K-theory for torsion twisting using the language of bundle gerbes and bundle gerbe modules. For non-torsion twisting \( \sigma \), let \( P_\sigma \) be the corresponding principal \( PU(\mathcal{H}) \)-bundle. We associate a bundle

\[
\mathcal{A}_\sigma = P_\sigma \times_{PU(\mathcal{H})} \text{End}_{HS}(\mathcal{H})
\]

of Hilbert algebras, fiberwisely consisting of the Hilbert algebra \( \text{End}_{HS}(\mathcal{H}) \) of Hilbert-Schmidt operators with the conjugation action of \( PU(\mathcal{H}) \). This bundle of Hilbert algebras plays the role that the Azumaya bundle plays for the torsion case.

Let \( U_2 \) be the subgroup of \( U(\mathcal{H}) \) of unitaries of the form \( 1 + \text{Hilbert-Schmidt} \). The complex Hilbert bundles with structure group \( U_2 \) and admitting a fiberwise \( \mathcal{A}_\sigma \)-action form an additive category \( \mathcal{E}^{\mathcal{A}_\sigma}_{U_2}(M) \), whose Grothendieck group is isomorphic to the twisted K-group \( K_\sigma(M) \). From this we show the existence of the Thom isomorphism in twisted K-theory.

**Theorem 1.2.** Suppose that \( \pi : V \to M \) is an oriented real vector bundle over \( M \) of even rank with positive definite quadratic form. There is a natural isomorphism

\[
K^0_{\pi^*W_3(V)}(M) \cong K^0_{\pi^*\sigma}(V), \quad K^1_{\pi^*W_3(V)}(M) \cong K^1_{\pi^*\sigma}(V).
\]

Suppose that \( \pi : V \to M \) is an oriented real vector bundle over \( M \) of odd rank equipped with a positive definite quadratic form. Then there is a natural isomorphism

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\]

We conclude by checking that we get the expected result for the Atiyah-Singer index theory for Dirac operators on Clifford modules and we explain how our result is relevant to string theoretic considerations.

2. **Review of Twisted K-theory**

We use \([2]\) as our primary reference for twisted K-theory. We now review the definition and some basic properties.

As noticed in \([2]\), on the one hand, for a smooth fiber bundle \( f : X \to Y \) as appearing in the push-forward map in the Introduction, the Hilbert bundle over \( Y \) whose fiber is the space \( \mathcal{H} \) of \( L^2 \)-half densities along the fiber of \( f \) does not admit \( U(\mathcal{H}) \) with norm topology as structure group. On the other hand for a compact Lie group, as in the definition of equivariant twisted K-theory, the regular representation \( G \to U(L^2(G)) \) is not norm-continuous, so for a stable \( G \)-Hilbert space \( \mathcal{H}_G \cong \mathcal{H}_G \otimes L^2(G) \), the action of \( G \) on the space of Fredholm operators is not
norm-continuous. For these two reasons, we use the compact-open topology developed in the Appendix 1 in [2] on the unitary group and the projective unitary group of a Hilbert space.

Fix a standard $\mathbb{Z}_2$ graded Hilbert space
\[ \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \]
such that both $\mathcal{H}^+$ and $\mathcal{H}^-$ are infinite dimensional. Let $\text{Fred}^{(0)}(\mathcal{H})$ be the space of self-adjoint Fredholm operators $\tilde{A}$ in $\mathcal{H}$ of degree 1 with respect to the grading such that $\tilde{A}^2$ differs from the identity by a compact operator. The topology comes from the embedding $\tilde{A} \mapsto (\tilde{A}, \tilde{A}^2 - 1)$ in $\mathcal{B}(\mathcal{H}) \times K(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the space of bounded operators in $\mathcal{H}$ with the compact-open topology and $K(\mathcal{H})$ is the Banach algebra of compact operators with norm topology. Then as shown in [2], $\text{Fred}^{(0)}(\mathcal{H})$ is a representing space for $K$-theory, and admits a continuous conjugation action of the projective unitary group $\text{PU}(\mathcal{H})$ equipped with the compact-open topology.

Note also that $\text{PU}(\mathcal{H})$ with the compact-open topology acts continuously by conjugation on the Banach space $K(\mathcal{H})$ of compact operators.

Given a twisting $\sigma \in H^1(M, \mathbb{Z}_2) \oplus H^3(M, \mathbb{Z})$, there is a unique isomorphism class of locally trivial bundles over $M$ with fibre an infinite dimensional, separable, complex projective Hilbert space in which a unitary involution is given in each fibre
\[ \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- . \]
The structure group is the graded projective unitary group $\text{PU}_{gr}(\mathcal{H})$ with the compact-open topology, where $\text{PU}_{gr}(\mathcal{H}) = U_{gr}(\mathcal{H})/U(1)$ and $U_{gr}(\mathcal{H})$ has two connected components with the identity component
\[ U_{gr}^0(\mathcal{H}) = U(\mathcal{H}^+) \oplus U(\mathcal{H}^-) . \]
It was proved in [22][23] that the grading preserving isomorphism classes of graded projective Hilbert bundles are classified by
\[ H^1(M, \text{PU}_{gr}(\mathcal{H})) \cong H^1(M, \mathbb{Z}_2) \oplus H^3(M, \mathbb{Z}) . \]

Given a twisting $\sigma \in H^3(M, \mathbb{Z})$, there is an associated isomorphism class of projective Hilbert space bundles (see [2] and [6]). The natural way to get a projective Hilbert space bundle with involution is by introducing the $\mathbb{Z}_2$ graded space
\[ \mathcal{H} = \mathcal{H} \oplus \mathcal{H} = \mathcal{H} \otimes \mathbb{C}^2 , \]
such that the structure group reduces to $U_{gr}^0(\mathcal{H})/U(1)$, hence the corresponding twisting has trivial components in $H^1(M, \mathbb{Z}_2)$. For simplicity, throughout this paper, we only consider twistings in $H^3(M, \mathbb{Z})$.

Fix a choice of such a projective Hilbert bundle $P_\sigma$ with involution associated to $\sigma \in H^3(M, \mathbb{Z})$ such that there exists a good cover $\{U_\alpha\}$ of $M$ satisfying
\[ P_\sigma|_{U_\alpha} \cong U_\alpha \times \mathbb{P}(\mathcal{H}) \]
and such that the transition between two such trivializations is given by a continuous function
\[ g_{\alpha\beta} : U_\alpha \cap U_\beta \to PU(\mathcal{H}) , \]
for $PU(\mathcal{H})$ with either the norm topology or the compact-open topology, Cf. Proposition 2.1 in [2].

Denote by $\hat{P}_\sigma$ the graded tensor product
$$\hat{P}_\sigma = P_\sigma \otimes \mathcal{F} \mathcal{H}.$$
With respect to the local trivialization (2.1), the transition functions are given by
$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow PU(\mathcal{H} \otimes \hat{\mathcal{H}}).$$
Denote by $\mathcal{P}_\sigma$ the associated principal bundle over $M$, and $\text{Fred}^{(0)}(\hat{P}_\sigma)$ the associated bundle of Fredholm operators with fibre $\text{Fred}^{(0)}(\mathcal{H} \otimes \hat{\mathcal{H}})$.

**Definition 2.1.** If $M$ is compact, then the twisted $K$-group $K^0_{\mathcal{P}_\sigma}(M)$ is defined to be the space of homotopy classes of sections of $\text{Fred}^{(0)}(\hat{P}_\sigma)$, that is,
$$K^0_{\mathcal{P}_\sigma}(M) = [M, \text{Fred}^{(0)}(\hat{P}_\sigma)] \cong [\hat{P}_\sigma, \text{Fred}^{(0)}(\hat{\mathcal{H}})]_{PU(\mathcal{H})},$$
where $[\hat{P}_\sigma, \text{Fred}^{(0)}(\hat{\mathcal{H}})]_{PU(\mathcal{H})}$ is the group of homotopy classes of equivariant maps. If $M$ is locally compact, then the twisted $K$-group $K^0_{\mathcal{P}_\sigma}(M)$ is defined to be the space of homotopy classes of admissible sections of $\text{Fred}^{(0)}(\hat{P}_\sigma)$, where a section $s$ is called admissible if there is a compact set $K \subset M$ such that $s|_{M-K}$ is an invertible section of $K^0_{\mathcal{P}_\sigma}(M-K)$. If $M_1$ is a closed subset of $M$, we define the relative twisted $K$-group
$$K^0_{\mathcal{P}_\sigma}(M, M_0) = K^0_{\mathcal{P}_\sigma}(M - M_0).$$

In terms of local trivializations (2.1) and the corresponding transition functions, an element in $K^0_{\mathcal{P}_\sigma}(M)$ can be represented by a twisted family of local sections:
$$s_\alpha : U_\alpha \longrightarrow \text{Fred}^{(0)}(\hat{\mathcal{H}}),$$
such that $s_\beta = g_{\alpha\beta} s_\alpha g_{\alpha\beta}^{-1}$. From the definition, we know that if $\sigma$ is trivial, then $K^0_{\mathcal{P}_\sigma}(M)$ is isomorphic to the ordinary topological $K$-group $K^0(M)$.

As discussed in [2], there is an additive structure in $K^0_{\mathcal{P}_\sigma}(M)$ given by the operation of fiberwise direct sum after a choice of an isomorphism between $\hat{\mathcal{H}} \otimes \mathbb{C}^2$ and $\hat{\mathcal{H}}$ (the space of these isomorphisms is a contractible, hence, this is well-defined). The group $K^0_{\mathcal{P}_\sigma}(M)$ is functorial with respect to the pair $(M, P)$ and an isomorphism of projective Hilbert bundles induces an isomorphism on their twisted $K$-cohomology groups. In particular, $K^0_{\mathcal{P}_\sigma}(M)$ admits a natural action of $\text{Aut}(P_\sigma) \cong H^2(M, \mathbb{Z})$. Up to a canonical isomorphism, the twisted $K$-group $K^0_{\mathcal{P}_\sigma}(M)$ depends only on the Dixmier-Douady class $\sigma$. With these understood, we sometimes denote the twisted $K$-group $K^0_{\mathcal{P}_\sigma}(M)$ by $K^0(\sigma)$.

In [2], twisted groups $K^0_{\mathcal{P}_\sigma}(M)$ for all $n \in \mathbb{Z}$ are also defined. The bundle $\text{Fred}^{(0)}(\hat{P}_\sigma)$ has a base point on each fiber, represented by a chosen fiberwise identification $(\hat{P}_\sigma)_x^+ \cong (\hat{P}_\sigma)_x^-$. Thus the fiberwise iterated based loop space forms a bundle
$$\Omega^\times \text{Fred}^{(0)}(\hat{P}_\sigma),$$
whose fiber at $x$ is $\Omega^n \text{Fred}^{(0)}((\hat{P}_\sigma)_x)$. Let the homotopy classes of sections of $\Omega^\times \text{Fred}^{(0)}(\hat{P}_\sigma)$ be denoted $K^{-n}_{\sigma}(M)$. As shown in [2], there exists a fiberwise homotopy equivalence
$$\text{Fred}^{(0)}(\hat{\mathcal{H}}) \longrightarrow \Omega^2 \text{Fred}^{(0)}(\hat{\mathcal{H}})$$
which is $PU(\hat{\mathcal{H}})$-equivariant. Therefore, these groups \( \{K_{\mathcal{P}}^{-n}(M)\}_{n \leq 0} \) are periodic in \( n \) with period 2. This periodicity can be used to define twisted groups $K_{\mathcal{P}}^{n}(M)$ for all $n \in \mathbb{Z}$ such that the twisted K-theory forms a periodic cohomology theory of period 2 on the category of manifolds equipped with a projective Hilbert bundle. In particular, an element in $K_{\mathcal{P}}^{1}$ can be represented by a based $S^1$-family of local twisted sections

\[
\tilde{s}_\alpha : U_\alpha \times S^1 \longrightarrow Fred^{(0)}(\hat{\mathcal{H}})
\]

such that $\tilde{s}_\beta = g_{\alpha \beta} \tilde{s}_\alpha g_{\alpha \beta}^{-1}$.

For a manifold $M \times \mathbb{R}$ with a twisting given by the pull-back of a twisting $\sigma$ on $M$, then

\[
K_{\mathcal{P}}^{0}(M \times \mathbb{R}) \cong K_{\mathcal{P}}^{1}(M),
\]

and

\[
K_{\mathcal{P}}^{1}(M \times \mathbb{R}) \cong K_{\mathcal{P}}^{-1}(M \times \mathbb{R}) \cong K_{\mathcal{P}}^{-2}(M) \cong K_{\mathcal{P}}^{0}(M).
\]

Twisted K-theory satisfies the following basic properties [2].

1. Given two projective Hilbert bundles $P_1$ and $P_2$ with involution over $M$ with their twisting in

\[
H^1(M, \mathbb{Z}_2) \oplus H^3(M, \mathbb{Z}),
\]

denote by $P_1 \otimes P_2$ the graded tensor product of $P_1$ and $P_2$. There is a cup product homomorphism

\[
K_{P_1}^i(M) \times K_{P_2}^j(M) \longrightarrow K_{P_1 \otimes P_2}^{i+j}(M),
\]

coming from the map $(A, A') \mapsto A \otimes \sigma + 1 \otimes A'$ (the graded tensor product) defined on the space of self-adjoint degree 1 Fredholm operators. In particular, $K_{0}^{0}(M)$ is a module over the untwisted K-group $K^{0}(M)$, extending the action of $H^2(M, \mathbb{Z}) \subset K^{0}(M)$.

2. For any proper continuous map $f : M \to N$ there exists a natural pull-back homomorphism, which is a homotopy invariant of $f$,

\[
f^* : K_{f^*P_\mathcal{P}}^i(N) \longrightarrow K_{P_\mathcal{P}}^i(M),
\]

where $f^*P_\mathcal{P}$ is the pull-back bundle of $P_\mathcal{P}$. In particular, if $M$ is a closed subset of $N$ with the inclusion map $\iota : M \to N$, the pull-back map is just the restriction map.

3. If $M$ is covered by two closed subsets $M_1$ and $M_2$ whose interiors cover $M$, for a projective Hilbert bundle $P$ over $M$, there is a Mayer-Vietoris exact sequence

\[
\begin{array}{cccccc}
K_{P}^{0}(M) & \longrightarrow & K_{P_1}^{0}(M_1) \oplus K_{P_2}^{0}(M_2) & \longrightarrow & K_{P_{12}}^{0}(M_1 \cap M_2) & \longrightarrow \\
K_{P_{12}}^{1}(M_1 \cap M_2) & \hookrightarrow & K_{P_1}^{1}(M_1) \oplus K_{P_2}^{1}(M_2) & \hookrightarrow & K_{P}^{1}(M) & \\
\end{array}
\]

where $P_1$, $P_2$ and $P_{12}$ are the restrictions of $P$ to $M_1$, $M_2$ and $M_1 \cap M_2$ respectively.
(4) If $M$ is covered by two open subsets $U_1$ and $U_2$, for a projective Hilbert bundle $P$ over $M$, there is a Mayer-Vietoris exact sequence

\begin{equation}
K_0^0(P(M)) \longrightarrow K_1^1(U_1 \cap U_2) \longrightarrow K_1^1(U_1) \oplus K_1^1(U_2) \longrightarrow K_0^0(P_1(U_1) \oplus P_2(U_2)) \longrightarrow K_0^0(P_1(U_1) \cap P_2(U_2)) \longrightarrow K_1^1(P(M)) \end{equation}

where $P_1$, $P_2$ and $P_{12}$ are the restrictions of $P$ to $U_1$, $U_2$ and $U_1 \cap U_2$ respectively.

**Remark 2.2.**

1. The torsion subgroup of $H^3(M, \mathbb{Z})$ can be identified with the unitary Brauer group of $M$, the isomorphism classes of finite dimensional projective bundles over $M$. For each torsion class $\sigma$ in $H^3(M, \mathbb{Z})$, $K$-theory with local coefficients is defined in [12] to get a complete Thom isomorphism in ordinary $K$-theory. We shall see that this is a prototype for the Thom isomorphism in Section 3.

2. (Cf. [2] [25]) Using the continuous action of $PU(\hat{H})$ with the compact-open topology on $K(\hat{H})$, we associate to $P_\sigma$ the bundle $K_{P_\sigma}$ of non-unital algebras whose fiber is the compact operators $K(\hat{H})$. Then the twisted $K$-group $K_0^0(P_\sigma(M))$ is canonically isomorphic to the $K$-theory of the Banach algebra of sections of $K_{P_\sigma}$, if $M$ is only locally compact, sections of $K_{P_\sigma}$ are required to have compact support.

3. For an open embedding $\iota : U \subset M$, the natural extension defines a homomorphism

$$K_1^1(P_{|U})(U) \longrightarrow K_1^1(P(M),$$

which is the push-forward map $\iota_*$ defined in next Section. Moreover, there is the following exact sequence

\begin{equation}
K_0^0(P_{|U})(U) \longrightarrow K_0^0(P(M)) \longrightarrow K_0^0(P_{|M-U})(M-U) \longrightarrow K_1^1(P_{|M-U})(M-U) \longrightarrow K_1^1(P(M)) \longrightarrow K_1^1(P_{|U})(U) \end{equation}

from which the Mayer-Vietoris exact sequences (2.6) and (2.7) can deduced for two pairs of open embeddings $M_2 - (M_1 \cap M_2) \subset M_2$ and $M_2 - (M_1 \cap M_2) \subset M$ for two closed covering subsets $M_1$ and $M_2$.

Throughout this paper, we only deal with twisted $K$-theory over a locally compact space.

### 3. Thom isomorphism in twisted $K$-theory

#### 3.1. Twisted $K$-theory in the torsion case.

Given a torsion class $\sigma \in H^3(M, \mathbb{Z})$, denote by $P_\sigma$ the corresponding principal $PU(n)$-bundle over $M$ with Dixmier-Douady invariant equal to...
Let $\Gamma_\sigma$ be the lifting bundle gerbe $[20]$

\[
\begin{array}{c}
\Gamma_\sigma \\
\downarrow \\
\mathcal{P}^{[2]}_\sigma \xrightarrow{\pi_1} \mathcal{P}_\sigma \\
\downarrow \pi_2 \\
M
\end{array}
\]

over $M$ associated to $\mathcal{P}_\sigma$ and the central extension

\[
1 \rightarrow U(1) \rightarrow U(n) \rightarrow PU(n) \rightarrow 1.
\]

Note that $\Gamma_\sigma$ is the natural groupoid $U(1)$-extension of the groupoid $\mathcal{P}^{[2]}_\sigma = \mathcal{P}_\sigma \times_M \mathcal{P}_\sigma$ with the source map given by $\pi_1 : (y_1, y_2) \mapsto y_2$ and the target map given by $\pi_2 : (y_1, y_2) \mapsto y_1$.

**Definition 3.1.** A bundle gerbe module $E$ of $\Gamma_\sigma$ (defined in [6]), also called a $\Gamma_\sigma$-module in [21], is really a complex vector bundle $E$ over $\mathcal{P}_\sigma$ with a groupoid action of $\Gamma_\sigma$, i.e., an isomorphism

\[
\phi : \Gamma_\sigma \times_{(s, \pi)} E \rightarrow E
\]

where $\Gamma_\sigma \times_{(s, \pi)} E$ is the fiber product of the source $s : \Gamma_\sigma \rightarrow \mathcal{P}_\sigma$ and $\pi : E \rightarrow \mathcal{P}_\sigma$ such that

1. $\pi \circ \phi(g, v) = t(g)$ for $(g, v) \in \Gamma_\sigma \times_{(s, \pi)} E$, and $t$ is the target map of $\Gamma_\sigma$.
2. $\phi$ is compatible with the bundle gerbe multiplication $m : \Gamma_\sigma \times_{(s, t)} \Gamma_\sigma \rightarrow \Gamma_\sigma$, which means

\[
\phi \circ (id \times \phi) = \phi \circ (m \times id).
\]

Using the source map $r : \Gamma_\sigma \rightarrow \mathcal{P}_\sigma$, it is easy to see that $\Gamma_\sigma$ is a topologically trivial $U(n)$-bundle over $\mathcal{P}_\sigma$. If we think of $U(n)$ as acting on $\mathcal{P}_\sigma$ via the $PU(n)$-action (and using the quotient map $U(n) \rightarrow PU(n)$), then we know that [6] any $U(n)$-module $V$ gives rise to a bundle gerbe module

\[
\mathcal{V} = \mathcal{P}_\sigma \times V
\]

with $U(n)$-action $g \cdot (p, v) = (pg^{-1}, gv)$.

The weight of a $U(n)$-equivariant vector bundle is defined to be the weight of the induced $U(1)$ action. Then the additive category of isomorphism classes of $\Gamma_\sigma$-modules $M$ is equivalent to the additive category of $U(n)$-equivariant vector bundles over $\mathcal{P}_\sigma$ of weight $1 \mod(k)$, where $k$ is the order of $\sigma$. Therefore, we conclude that the K-group of bundle gerbe modules $K_{bg}(M, \mathcal{P}_\sigma)$, defined to be the Grothendieck group of the additive category of $\Gamma_\sigma$-modules as in [6], is isomorphic to the submodule $K_{U(n)}^0(P_{\mathcal{P}_\sigma})$ of weight $1 \mod(k)$ in the equivariant K-group $K_{U(n)}^0(\mathcal{P}_{\mathcal{P}_\sigma})$ regarded as an $R(U(1))$-module, where $R(U(1))$ denotes the complex representation ring of $U(1)$.

Here we recall that $K_{U(n)}^0(\mathcal{P}_{\mathcal{P}_\sigma})$ is the Grothendieck group of the additive category of $U(n)$-vector bundles over the $U(n)$-manifold $\mathcal{P}_\sigma$. The tensor product of $U(n)$-vector bundles induces a commutative ring structure in $K_{U(n)}^0(\mathcal{P}_{\mathcal{P}_\sigma})$. The natural morphism from the representation ring $R(U(n))$ to $K_{U(n)}^0(\mathcal{P}_{\mathcal{P}_\sigma})$ defines a $R(U(n))$-module structure on $K_{U(n)}^0(\mathcal{P}_{\mathcal{P}_\sigma})$. The $R(U(1))$-module structure on $K_{U(n)}^0(\mathcal{P}_{\mathcal{P}_\sigma})$ is defined by the action of $U(1)$ on $U(n)$-vector bundles.
The following proposition gives a correct proof of Proposition 6.4 of [6]. The discussion in [6] does not handle $U(n)$-equivariant contractibility of the unitary group of Hilbert space correctly (and the argument below fixes this using the appendix to [2]), and the index map

$$\text{ind} : [\mathcal{P}_\sigma, \text{Fred}(C^n \otimes \mathcal{H})|_{U(n)}] \rightarrow K_{bg}(M, \Gamma_\sigma),$$

corrected in section 6.2 of [6] is neither injective nor surjective, contrary to the claim in the proof of Proposition 6.4 of [6].

**Proposition 3.2.** For a torsion class $\sigma \in H^3(M, \mathbb{Z})$ of order $k$, let $\mathcal{P}_\sigma$ be a principal $PU(n)$-bundle whose Dixmier-Douady class is $\sigma$, then the twisted $K$-group $K^0_{\mathcal{P}_\sigma}(M)$ is isomorphic to $K^0_{U(n), (1)}(\mathcal{P}_\sigma)$, hence

$$K^0_{\mathcal{P}_\sigma}(M) = K_{bg}(M, \Gamma_\sigma).$$

**Proof.** Using the representing space of $K^0_{U(n)}$ developed in Appendix 3 of [2], the proof is rather straightforward. Let $\mathcal{H}_U(n)$ be a stable $U(n)$-Hilbert space, then the closed subspace of $U(n)$-continuous Fredholm operators in $\text{Fred}(\mathcal{H}_U(n))$

$$\text{Fred}_{cts}(\mathcal{H}_U(n)) = \{ D \in \text{Fred}(\mathcal{H}_U(n)) : g \mapsto gDg^{-1} \text{ is continuous for } g \in U(n) \},$$

with norm topology, is a representing space of $K^0_{U(n)}$. That means,

$$K^0_{U(n)}(\mathcal{P}_\sigma) = [\mathcal{P}_\sigma, \text{Fred}_{cts}(\mathcal{H}_U(n))]_{U(n)},$$

the group of all homotopy classes of equivariant maps. As a $R(U(1))$-module, the submodule of weight 1 $mod(k)$

$$K^0_{U(n), (1)}(\mathcal{P}_\sigma) = [\mathcal{P}_\sigma, \text{Fred}^{(1)}(\mathcal{H}_U(n), (1))]_{U(n)}$$

where $\text{Fred}_{cts}(\mathcal{H}_U(n), (1))$ is the $U(n)$-Hilbert subspace of $\mathcal{H}_U(n)$ of weight 1 $mod(k)$ under the action of $U(1)$. By the results in Appendix 3 in [2], we obtain that

$$K^0_{U(n), (1)}(\mathcal{P}_\sigma) = [\mathcal{P}_\sigma, \text{Fred}^{(0)}(\mathcal{H}_U(n), (1))]_{U(n)}.$$

From the definition of the twisted $K$-group, applying the action of $U(n)$ on $C^n \otimes \hat{\mathcal{H}}$ defined by $g \otimes 1$, we know that

$$K^0_{\mathcal{P}_\sigma}(M) = [\mathcal{P}_\sigma \times_{PU(n)} PU(C^n \otimes \hat{\mathcal{H}}), \text{Fred}^{(0)}(C^n \otimes \hat{\mathcal{H}})]_{PU(C^n \otimes \hat{\mathcal{H}})}$$

$$\cong [\mathcal{P}_\sigma \times_{PU(n)} PU(\mathcal{H}_U(n), (1)), \text{Fred}^{(0)}(\mathcal{H}_U(n), (1))]_{PU(\mathcal{H}_U(n), (1))}$$

$$\cong [\mathcal{P}_\sigma, \text{Fred}^{(0)}(\mathcal{H}_U(n), (1))]_{PU(n)}$$

$$\cong K^0_{U(n), (1)}(\mathcal{P}_\sigma)$$

$$\cong K_{bg}(M, \Gamma_\sigma).$$

Let $\mathcal{E}_{bg}(M, \Gamma_\sigma)$ be the additive category of $\Gamma_\sigma$-modules. Let $\mathcal{A}_\sigma$ be the Azumaya bundle over $M$ with Dixmier-Douady class $\sigma$

$$\mathcal{A}_\sigma = \mathcal{P}_\sigma \times_{Ad} M_n(\mathbb{C}),$$
and where $Ad$ is the adjoint representation of $PU(n)$ on $M_n(\mathbb{C})$ by conjugation. An $A_\sigma$-module is a complex vector bundle over $M$ with a fiberwise $A_\sigma$ action

$$A_\sigma \times_M E \longrightarrow E.$$ 

Applying the natural representation of $U(n)$ on $\mathbb{C}^n$, then $C^n \cong \mathbb{C}^n \times \mathcal{P}_\sigma$ is a $\Gamma_\sigma$-module. Tensoring with $C^n\star$, there is a natural equivalence of categories between $\mathcal{E}_{bg}(M, \Gamma_\sigma)$ and $\mathcal{E}^{A_\sigma}(M)$, the category of $A_\sigma$-modules. Therefore, we obtain

$$K^0_{P_\sigma}(M) \cong K(\mathcal{E}^{A_\sigma}(M)), \tag{3.2}$$

the Grothendieck group of the category of $A_\sigma$-modules.

### 3.2. Thom isomorphism in twisted K-theory for the torsion case.

Consider an oriented real vector bundle $V$ of even rank over $M$ with a positive definite quadratic form $Q_V$. Denote by $\mathcal{F}r$ the frame bundle of $V$, the principal $SO(2n)$-bundle of positively oriented orthonormal frames, i.e.,

$$V = \mathcal{F}r \times_{\rho_{2n}} \mathbb{R}^{2n},$$

where $\rho_n$ is the standard representation of $SO(2n)$ on $\mathbb{R}^n$. The lifting bundle gerbe (Cf. [20]) associated to the frame bundle and the central extension

$$1 \rightarrow U(1) \longrightarrow Spin^c(2n) \longrightarrow SO(2n) \rightarrow 1$$

has its Dixmier-Douady class given by the integral third Stiefel-Whitney class $W_3(V) \in H^3(M, \mathbb{Z})$. We denote this lifting bundle gerbe by $\Gamma_{W_3(V)}$

$$\begin{array}{ccc}
\Gamma_{W_3(V)} & \overset{\lambda_1}{\longrightarrow} & \mathcal{F}r \\
\mathcal{F}r^{[2]} & \overset{\lambda_2}{\longrightarrow} & \mathcal{F}r \\
\downarrow & & \downarrow \\
M & & M.
\end{array}$$

Note that $\Gamma_{W_3(V)} \cong \Gamma_{spin} \times_{\mathbb{Z}_2} U(1)$ where $\Gamma_{spin}$ is a $\mathbb{Z}_2$-bundle gerbe associate to $\mathcal{F}r \rightarrow M$ and the central extension

$$1 \rightarrow \mathbb{Z}_2 \longrightarrow Spin(2n) \longrightarrow SO(2n) \rightarrow 1.$$ 

The Clifford bundle $Cl(V)$ of $V$ ([17]) is defined to be

$$Cl(V) = \mathcal{F}r \times_{cl(\rho_{2n})} Cl(\mathbb{R}^{2n}),$$

where $Cl(\mathbb{R}^{2n})$ is the Clifford algebra of $\mathbb{R}^n$ with the standard positive definite quadratic form, and $cl(\rho_{2n})$ is the induced representation of $SO(2n)$ on $Cl(\mathbb{R}^{2n})$. The fiberwise Clifford multiplication in $Cl(V)$, as a continuous map,

$$Cl(V) \times_M Cl(V) \longrightarrow Cl(V)$$
defines an algebra structure on $\text{Cl}(V)$. Hence $\text{Cl}(V)$ is called the bundle of Clifford algebras. A complex vector bundle $E$ over $M$ is called a complex $C(V)$-module if there is a continuous map, called the Clifford action,

$$\text{Cl}(V) \times_M E \longrightarrow E$$

such that each fiber $E_x$ ($x \in M$) is a complex $\text{Cl}(V_x)$-module. Denote by $\text{Cl}_C(V) = \text{Cl}(V) \otimes \mathbb{C}$ the complexified Clifford bundle. Note that a complex $C(V)$-module is a $\text{Cl}_C(V)$-module.

Let $\mathcal{E}^V(M)$ be the category of complex $C(V)$-modules whose morphisms are vector bundle morphisms which commute with the $\text{Cl}(V_x)$-actions over each point $x \in M$. Let $\mathcal{E}(M, \Gamma_{W_3(V)})$ be the category of $\Gamma_{W_3(V)}$-modules whose morphisms are vector bundle morphisms which commute with the groupoid action of $\Gamma_{W_3(V)}$. Under the Whitney sum of vector bundles, $\mathcal{E}^V(M)$ and $\mathcal{E}(M, \Gamma_{W_3(V)})$ are additive categories. As observed in [21], we now show that these two categories are in fact equivalent.

**Lemma 3.3.** There is a natural functor

$$\Psi : \mathcal{E}^V(M) \longrightarrow \mathcal{E}(M, \Gamma_{W_3(V)}),$$

which is a category equivalence. In particular, the Grothendieck group of $\Gamma_{W_3(V)}$-modules is isomorphic to the Grothendieck group of $\mathcal{E}^V(M)$ which is denoted by $K(\mathcal{E}^V(M))$

$$K_{bg}(M, \Gamma_{W_3(V)}) \cong K(\mathcal{E}^V(M)).$$

**Proof.** There exists a distinguished $\Gamma_{W_3(V)}$-module given by restricting the irreducible complex representation of $\text{Cl}(\mathbb{R}^{2n})$ to $\text{Spin}(2n) \subset \text{Cl}(\mathbb{R}^{2n})$. Denote the unique irreducible complex representation of $\text{Cl}(\mathbb{R}^{2n})$ by

$$\Delta_{2n}^c : \text{Spin}(2n) \rightarrow GL_C(S_{2n}).$$

Applying the groupoid structure of $\Gamma_{\text{spin}}$, the representation $\Delta_{2n}^c$ defines a natural $\Gamma_{\text{spin}}$-module structure on

$$S_{2n} = S_{2n} \times \mathcal{F}r,$$

see the discussion before Proposition 3.2. As $\text{Spin}^c(2n) = \text{Spin}(2n) \times \mathbb{Z}_2 U(1)$, $S_{2n}$ is a $\text{Spin}^c(2n)$-module with $U(1)$-action given by scalar multiplication of $U(1) \subset \mathbb{C}$. This implies that $S_{2n}$ is a $\Gamma_{W_3(V)}$-module. Note that the dual bundle $S_{2n}^*$ is also a $\Gamma_{W_3(V)}$-module.

Let $\pi : \mathcal{F}r \rightarrow M$ be the frame bundle associated to $(V, Q_V)$. Then $\text{Cl}(\pi^*V)$ is the Clifford bundle over $\mathcal{F}r$ associated to $(\pi^*V, \pi^*Q_V)$. Given a complex $C(V)$-module $E$, $S_{2n}^* \otimes C(\pi^*V) \pi^*E$ is a vector bundle over $\mathcal{F}r$ which admits a $\Gamma_{W_3(V)}$-module structure coming from $\text{Spin}^c(2n)$-action on $S_{2n}$. This defines a natural functor

$$\Psi : \mathcal{E}^V(M) \longrightarrow \mathcal{E}(M, \Gamma_{W_3(V)}),$$

$$E \quad \mapsto \quad W_E = S_{2n}^* \otimes C(\pi^*V) \pi^*E,$$

such that $\text{Hom}_{\text{Cl}(V)}(E, F) \cong \text{Hom}_{\Gamma_{W_3(V)}}(W_E, W_F)$ as $\text{Cl}_C(\pi^*V) \cong \text{End}(S_{2n})$.

Now we show that $\Psi$ is a functor setting up a category equivalence. We only need to construct an inverse to $\Psi$, up to isomorphism. Given a $\Gamma_{W_3(V)}$-module $W$, $W$ is a $\text{Spin}(2n)$-equivariant vector bundle over $\mathcal{F}r$. Then $W \otimes S_{2n}^*$ is an $SO(2n)$-equivariant vector bundle over $\mathcal{F}r$, hence, descends to a complex vector bundle over $M$, denoted by $E_W$. The Clifford action
of $\text{Cl}(\mathbb{R}^{2n})$-action on $S_{2n}$ defines a complex $\text{Cl}(V)$-module structure on $E_W$. Using the isomorphism $\text{Cl}_c(\pi^*V) \cong \text{End}(S_{2n})$ again, we see that $\Psi(E_W) = W$. Morphisms in $\mathcal{E}_{bg}(M, \Gamma_{W_3(V)})$ and $\mathcal{E}^V(M)$ can also be identified.

This identification of $\mathcal{E}^V(M)$ with $\mathcal{E}(M, \Gamma_{W_3(V)})$ is very important, as we can obtain some nice properties of $\mathcal{E}^V(M)$ from $\mathcal{E}(M, \Gamma_{W_3(V)})$. Using the Peter-Weyl theorem as in the proof of Proposition 2.4 in [26], for any $\Gamma_{W_3(V)}$-module $W$, there is a $\Gamma_{W_3(V)}$-module $W^\perp$ such that

\[(3.3) \quad W \oplus W^\perp \cong \mathbb{C}^N = \mathbb{C}^N \times \mathcal{F}_T\]

for a $\text{Spin}(2n)$-module $\mathbb{C}^N$. With this property, we can identify $K(\mathcal{E}^V(M))$ with the Grothendieck group, denoted by $K^V(X)$, of the following forgetful functor (see Ch IV.5 in [16])

\[\mathcal{E}^V \oplus \mathbb{R}(M) \longrightarrow \mathcal{E}^V(M)\]

where $\mathbb{R} = M \times \mathbb{R}$ is the trivial bundle of rank one with the standard quadratic form. Let $B(V)$ (resp. $S(V)$) be the ball bundle (resp. the sphere bundle) of $V$. Define

\[K^V_n(M) = K^V(M \times B^n, M \times S^{n-1}).\]

The following Thom isomorphism theorem is established in [16].

**Theorem 3.4.** *(Theorems IV.5.11 and IV.6.21 [16])* There is a natural isomorphism

\[K^V_n(M) \cong K_n(B(V), S(V)) = K_n(V)\]

which generalizes the standard Thom isomorphism (i.e., tensoring with the Thom class of $V$) for $V$ admitting a $\text{Spin}^c$-structure (i.e., $W_3(V) = 0$).

Applying Theorem III.4.12 in [16] to the category $\mathcal{E}^V(M)$ which is a pseudo-abelian category (see page 28 in [16]) by (3.3), we know that $K(\mathcal{E}^V(M))$ is isomorphic to the Grothendieck group of the following forgetful functor

\[\left(\mathcal{E}^V(M)\right)^{\text{Cl}(\mathbb{R})} \longrightarrow \mathcal{E}^V(M),\]

where $\left(\mathcal{E}^V(M)\right)^{\text{Cl}(\mathbb{R})}$ is a category whose objects are pairs $(E, \rho)$, where $E \in \mathcal{E}^V(M)$ and $\rho$ is an $\mathbb{R}$-linear bundle homomorphism

\[\rho : \text{Cl}(\mathbb{R}) \longrightarrow \text{End}_{\text{Cl}(V)}(E).\]

As $\left(\mathcal{E}^V(M)\right)^{\text{Cl}(\mathbb{R})} \cong \mathcal{E}^V \oplus \mathbb{R}(M)$, we obtain

\[(3.4) \quad K(\mathcal{E}^V(M)) \cong K^V(M) \cong K(V).\]

By substitution $M \mapsto (M \times B^n, M \times S^{n-1})$, we have

\[K_n(\mathcal{E}^V(M)) \cong K_n(V).\]

Together with Proposition 3.2, we have obtained the following isomorphisms:

\[K^0_{W_3(V)}(M) \cong K^0_{bg}(M, \Gamma_{W_3(V)}) \cong K(\mathcal{E}^V(M)) \cong K^V(M) \cong K(V).\]
This is the Thom isomorphism in K-theory for any real oriented vector bundle established by Donovan-Karoubi in [12] using K-theory with local coefficients.

Given a torsion element \( \sigma \in H^3(M, \mathbb{Z}) \), let \( \mathcal{P}_\sigma \) be the associated principal \( PU(n) \)-bundle and \( \Gamma_\sigma \) be the corresponding lifting bundle gerbe given by (3.1). We now describe the Thom isomorphism in twisted K-theory for the torsion case.

**Theorem 3.5.** Suppose that \( \pi : V \to M \) be an oriented real vector bundle over \( M \) of even rank with positive definite quadratic form. There is a natural isomorphism

\[
K^0_{\sigma^+ W_3(V)}(M) \cong K^0_{\pi^* \sigma}(V), \quad K^1_{\sigma^+ W_3(V)}(M) \cong K^1_{\pi^* \sigma}(V).
\]

Suppose that \( \pi : V \to M \) be an oriented real vector bundle over \( M \) of odd rank with positive definite quadratic form. There is a natural isomorphism

\[
K^0_{\sigma^+ W_3(V)}(M) \cong K^1_{\pi^* \sigma}(V), \quad K^1_{\sigma^+ W_3(V)}(M) \cong K^0_{\pi^* \sigma}(V).
\]

**Proof.** We know that \( K_{\sigma^+ W_3(V)}(M) \cong K(\mathcal{E}(\sigma^+ \Gamma W_3(V)), \) the Grothendieck group of the additive category of \( \Gamma \sigma \otimes \Gamma W_3(V) \)-modules (complex vector bundles over \( \mathcal{F}_r \times M \mathcal{P}_\sigma \) with the groupoid action of \( \Gamma \sigma \otimes \Gamma W_3(V) \)). Then the category of \( \Gamma \sigma \otimes \Gamma W_3(V) \)-modules is isomorphic to \( (\mathcal{E}^V(M))^{A_\sigma} \), the category of \( Cl(V) \)-modules admitting an \( A_\sigma \)-action. As \( K(\mathcal{E}^V(M)) \cong K(V) \), it is straightforward to see that \( K((\mathcal{E}^V(M))^{A_\sigma}) \cong K_{\pi^* \sigma}(V) \). We just apply the Thom isomorphism to \( (\mathcal{E}^V(M))^{A_\sigma} \), then \( \pi^*(E) \in \mathcal{E}^V(\pi^* A_\sigma) \). This implies that

\[
K_{\sigma^+ W_3(V)}(M) \cong K_{\pi^* \sigma}(V).
\]

As \( K^1_{\sigma^+ W_3(V)}(M) \cong K_{\sigma^+ W_3(V)}(M \times \mathbb{R}) \), and \( K_{\sigma^+ W_3(V)}(M \times \mathbb{R}) \cong K^0_{\pi^* \sigma}(V \times \mathbb{R}) \), we obtain

\[
K^1_{\sigma^+ W_3(V)}(M) \cong K^1_{\pi^* \sigma}(V).
\]

For an oriented real vector bundle \( V \) over \( M \) of odd rank with positive definite quadratic form, we apply the Thom isomorphism to \( V \oplus \mathbb{R} \). We have

\[
K_{\sigma^+ W_3(V)}(M) \cong K_{\pi^* \sigma}(V \oplus \mathbb{R}), \quad K^1_{\sigma^+ W_3(V)}(M) \cong K^1_{\pi^* \sigma}(V \oplus \mathbb{R}).
\]

Now we can think of \( V \oplus \mathbb{R} \) as a rank one real vector bundle over \( V \), and apply the Thom isomorphism again to get

\[
K_{\pi^* \sigma}(V \oplus \mathbb{R}) \cong K^1_{\pi^* \sigma}(V), \quad K^1_{\pi^* \sigma}(V \oplus \mathbb{R}) \cong K_{\pi^* \sigma}(V).
\]

Put (3.5) and (3.6) together, we have

\[
K_{\sigma^+ W_3(V)}(M) \cong K^1_{\pi^* \sigma}(V), \quad K^1_{\sigma^+ W_3(V)}(M) \cong K^0_{\pi^* \sigma}(V).
\]

\[\square\]
3.3. Thom isomorphism in twisted K-theory for the non-torsion case. The virtue of our approach in the torsion case is that it may be adapted to the non-torsion case. For a non-torsion twisting $\sigma \in H^3(M, \mathbb{Z})$, let $\Gamma_\sigma$ be the lifting bundle gerbe associated to a $PU(\mathcal{H})$-principal bundle

$$\pi_\sigma : \mathcal{P}_\sigma \to M,$$

where $\mathcal{H}$ is the standard $\mathbb{Z}_2$ graded separable complex Hilbert space. Note that $\Gamma_\sigma$ has a groupoid structure with the space of objects given by $\mathcal{P}_\sigma$, it is a groupoid $U(1)$-extension of the natural groupoid $P^2[\sigma] = \mathcal{P}_\sigma \times_M \mathcal{P}_\sigma$.

Let $U_K$ be the normal subgroup of $U(\mathcal{H})$ of unitary operators of the form $1 +$ compact, and $U_2$ be the normal subgroup of $U(\mathcal{H})$ of unitary operators of the form $1 +$ Hilbert-Schmidt operator.

**Definition 3.6.** Denote by $\mathcal{E}^{U_2}_{bg}(M, \Gamma_\sigma)$ the additive category of $\Gamma_\sigma$-modules with $U_2$ structure group. Here a $\Gamma_\sigma$-module $W$ with $U_2$ structure is a Hilbert bundle $W$ over $\mathcal{P}_\sigma$ with structure group $U_2$ and an action of the groupoid $\Gamma_\sigma$, i.e., an isomorphism $\phi : \Gamma_\sigma \times_{(s, \pi)} W \to W$ where $\Gamma_\sigma \times_{(s, \pi)} W$ is the fiber product of the source $s : \Gamma_\sigma \to \mathcal{P}_\sigma$ and $\pi : W \to \mathcal{P}_\sigma$ such that

1. $\pi \circ \phi(g, v) = t(g)$ for $(g, v) \in \Gamma_\sigma \times_{(s, \pi)} E$, and $t$ is the target map of $\Gamma_\sigma$.
2. $\phi$ is compatible with the bundle gerbe multiplication $m : \Gamma_\sigma \times_{(s, t)} \Gamma_\sigma \to \Gamma_\sigma$, which means $\phi \circ (id \times \phi) = \phi \circ (m \times id)$.

In [6], it was shown that

$$K_\sigma(M) \cong [\mathcal{P}_\sigma, Fred(0)(\mathcal{H})]^{PU(\mathcal{H})} \cong K(\mathcal{E}^{U_2}_{bg}(M, \Gamma_\sigma)).$$

To see this, we need a $PU(\mathcal{H})$-equivariant homotopy equivalence between $Fred(0)(\mathcal{H})$ and $\mathbb{Z} \times BU_K$ where $BU_K = U(\mathcal{H})/U_K$ with the conjugation action of $PU(\mathcal{H})$. We apply Bott periodicity:

$$U \sim \Omega(\mathbb{Z} \times BU)$$

$$\mathbb{Z} \times BU \sim \Omega U$$

where $U = U_K$, or $U_2$. Then use the $PU(\mathcal{H})$-equivariant homotopy equivalence between $U_K$ and $U_2$, to obtain a $PU(\mathcal{H})$-equivariant homotopy equivalence between $BU_K$ and $BU_2$, where we choose a homotopy model of $BU_2$ as the component of the identity in $\Omega U_2$, on which $PU(\mathcal{H})$ acts by conjugation. This implies that

$$K_\sigma(M) \cong [\mathcal{P}_\sigma, BU_2]^{PU(\mathcal{H})},$$

that is, the twisted K-group of $K_\sigma(M)$ can be described by isomorphism classes of $\Gamma_\sigma$-modules with $U_2$ structure group (Cf. Proposition 7.2 in [6])

Recall that the unitary group $U(\mathcal{H})$ with the compact-open topology is equivariantly contractible and acts continuously by conjugation on the Banach space $K(\mathcal{H})$ of compact operators in $\mathcal{H}$, and also on the Hilbert space

$$End_{HS}(\mathcal{H}) = \mathcal{H} \otimes \mathcal{H}^*$$

of Hilbert-Schmidt operators (Cf. Appendix in [2]).
Definition 3.7. Let $A_\sigma$ be the bundle of Hilbert-Schmidt operators associated to $P_\sigma$,

$$A_\sigma = P_\sigma \times_{PU(H)} \text{End}_{HS}(H),$$

where $PU(H)$ acts on the space of Hilbert-Schmidt operators $\text{End}_{HS}(H)$ by conjugation. Let $\mathcal{E}^{A_\sigma}_{U_2}(M)$ be the category of $A_\sigma$-modules with $U_2$ structure group (where a $A_\sigma$-module $E$ with $U_2$ structure group is a Hilbert bundle $E$ over $M$ with structure group $U_2$ and an action

$$A_\sigma \times_M E \to E.$$  

This bundle of Hilbert-Schmidt operators plays the role that the Azumaya bundle plays for the torsion case. Just as there exists an equivalence functor between the category of bundle gerbe modules and the category of modules of the Azumaya bundle, we have the corresponding equivalence of categories between the category of $\Gamma_\sigma$-modules with $U_2$ structure group and the category of $A_\sigma$-modules with $U_2$ structure group.

Lemma 3.8. There is a natural functor $\Psi : \mathcal{E}^{U_2}_{bg}(M, \Gamma_\sigma) \to \mathcal{E}^{A_\sigma}_{U_2}(M)$ which defines an equivalence of categories. In particular,

$$K_\sigma(M) \cong K(\mathcal{E}^{U_2}_{bg}(M, \Gamma_\sigma)) \cong K(\mathcal{E}^{A_\sigma}_{U_2}(M)).$$

Proof. There exists a distinguished element

$$\mathcal{H} = H \times P_\sigma$$

in $\mathcal{E}^{U_2}_{bg}(M, \Gamma_\sigma)$ and the dual bundle $\mathcal{H}^*$ is a $\Gamma_\sigma^*$-module with $U_2$ structure group, where $\Gamma_\sigma^*$ is the dual bundle gerbe associated to $\Gamma_\sigma$. Given a $\Gamma_\sigma$-module $W$ with $U_2$ structure group, then

$$\mathcal{H}^* \otimes W$$

is a Hilbert bundle over $P_\sigma \times P_{-\sigma} \cong M \times PU(H)$, whose structure group has a reduction to $U_2$, see Section 4.2 in [19] for the similar argument for $U_K$. Notice that $\mathcal{H}^* \otimes W$ is a $PU(H)$-equivariant bundle over $P_\sigma$ and hence descends to a Hilbert bundle over $M$ with structure group $U_2$. Denote this bundle by $\Psi(W)$. Then $\Psi(W)$ admits a natural action of $A_\sigma$, fiberwisely defined by the action of $\text{End}_{HS}(H)$ on $H$. Hence, $\Psi(W)$ is a $A_\sigma$-module with $U_2$ structure group.

Conversely, given a $A_\sigma$-module $E$ with $U_2$ structure group, then

$$\mathcal{H} \otimes_{\pi_\sigma^*A_\sigma} \pi_\sigma^* E \cong \text{Hom}_{\pi_\sigma^*A_\sigma}(\mathcal{H}^*, \pi_\sigma^* E)$$

is a $\Gamma_\sigma$-module with $U_2$ structure group satisfying $\Psi(\mathcal{H} \otimes_{\pi_\sigma^*A_\sigma} \pi_\sigma^* E) = E$. Here we use

$$\pi_\sigma^* A_\sigma \cong \mathcal{H}^* \otimes \mathcal{H},$$

and $\pi_\sigma^* E$ is a $\pi_\sigma^* A_\sigma$-module with $U_2$ structure group. Morphisms in $\mathcal{E}^{U_2}_{bg}(M, \Gamma_\sigma)$ and $\mathcal{E}^{A_\sigma}_{U_2}(M)$ can also be identified. Therefore, $\Psi$ defines a category equivalence. □

Now we can prove the Thom isomorphism in twisted K-theory for the general non-torsion case.
Theorem 3.9. Let \( \sigma \in H^3(M, \mathbb{Z}) \) be a non-torsion element. Suppose that \( \pi : V \rightarrow M \) is an oriented real vector bundle over \( M \) of even rank with positive definite quadratic form. There is a natural isomorphism

\[
K_0^{\sigma+W_3(V)}(M) \cong K_{\pi^*\sigma}(V), \quad K_1^{\sigma+W_3(V)}(M) \cong K_{\pi^*\sigma}(V).
\]

Suppose that \( \pi : V \rightarrow M \) is an oriented real vector bundle over \( M \) of odd rank with positive definite quadratic form. There is a natural isomorphism

\[
K_0^{\sigma+W_3(V)}(M) \cong K_{\pi^*\sigma}(V), \quad K_1^{\sigma+W_3(V)}(M) \cong K_{\pi^*\sigma}(V).
\]

Proof. Let \( \Gamma_\sigma \) and \( \Gamma_{W_3(V)} \) be lifting bundle gerbes associated to the corresponding \( PU(H) \)-bundles \( \pi_\sigma : cP_\sigma \rightarrow M \) and \( \pi : F_r \rightarrow M \). Then

\[
K^{\sigma+W_3(V)}(M) \cong K(E_{U_2}^{bg}(M, \Gamma_\sigma \otimes \Gamma_{W_3(V)}))
\]

\[
\cong K(E_{U_2}^{A_\sigma \otimes Cl}(V))
\]

\[
\cong K(E_{U_2}^{\pi^*\sigma}(V)).
\]

The Thom isomorphisms for other cases can be obtained by the same arguments as in the proof of Theorem 3.5. \( \square \)

Remark 3.10. For a locally compact and non-compact manifold \( M \), some care is required as twisted K-theory is defined with compact support. We always assume that the isomorphism (3.7) should be understood with respect to compactly supported \( PU(H) \)-equivariant homotopy equivalent classes, and \( \Gamma_\sigma \)-modules in \( E_{bg}^{U_2}(M, \Gamma_\sigma) \) should be isomorphic to \( \mathcal{H} \) away from compact sets.

4. Push-forward map in twisted K-theory

In this section, we will establish the main theorem of this paper: the existence of the push-forward map in twisted K-theory for any differentiable map \( f : X \rightarrow Y \). We equip \( X \) and \( Y \) with Riemannian metrics. So the Clifford bundles associated with their tangent bundles are well-defined.

Denote by \( W_3(f) \) the image of \( w_2(X) - f^*(w_2(Y)) \in H^2(X, \mathbb{Z}_2) \) under the Bockstein homomorphism

\[
\beta : H^2(X, \mathbb{Z}_2) \rightarrow H^3(X, \mathbb{Z}).
\]

For \( f : X \rightarrow pt \), \( W_3(f) \) is the third integer Stiefel-Whitney class \( W_3(X) \) of \( X \).

For a twisting \( \sigma \in H^3(Y, \mathbb{Z}) \), we choose a \( PU(H) \)-principal bundle

\[
\pi_\sigma : P_\sigma \rightarrow Y.
\]

Let \( A_\sigma \) be the bundle of Hilbert-Schmidt operators associated to \( P_\sigma \),

\[
A_\sigma = P_\sigma \times_{PU(H)} End_{HS}(H),
\]

where \( PU(H) \) acts on the space of Hilbert-Schmidt operators \( End_{HS}(H) \) by conjugation.
The twisted K-group $K^*_T(Y)$ is defined with respect to this $PU(H)$-principal bundle $P_\sigma$ using Definition 2.1. Equivalently, $K^*_T(Y)$ can be defined as the Grothendieck group of the category of $A_\sigma$-modules with $U_2$-structure (see Lemma 3.8).

Using the pull-back bundle $f^*A_\sigma$ tensored with the Clifford bundle $Cl(TX \oplus f^*TY)$ associated with

$$TX \oplus f^*TY,$$

we get a bundle $A_{f^*\sigma+W_3(f)}$ of Hilbert-Schmidt operators over $X$ with the twisting

$$f^*\sigma + W_3(f) \in H^3(X, \mathbb{Z}),$$

here the Hilbert space is the graded tensor product of $H$ with the unique $\mathbb{Z}_2$-graded irreducible module of the Clifford algebra. Then the twisted K-group $K^*_T(f^*\sigma+W_3(f))$ is defined to be the Grothendieck group of the category of $A_{f^*\sigma+W_3(f)}$-modules with $U_2$-structure.

**Theorem 4.1.** Let $f : X \to Y$ be a differentiable map and $\sigma \in H^3(Y, \mathbb{Z})$. There exists a natural push-forward map

$$f_! : K^*_T(f^*\sigma+W_3(f))(X) \to K^*_T(Y)$$

with grading shifted by $\dim X - \dim Y (\text{mod } 2)$.

We hope that the following remark will clarify some confusions in the current literature about the role of automorphisms of twists and their action on twisted K-theroy.

**Remark 4.2.**

1. Automorphisms of the $PU(H)$-principal bundle $P_\sigma$ result in a natural action of the Picard group $Pic(Y)$ on the twisted K-group $K^*_T(Y)$. Note that $Pic(Y)$ acts naturally on $K^*_T(f^*\sigma+W_3(f))$ via the pull-back construction. Our push-forward map in the theorem is defined with a fixed choice of a $PU(H)$-principal bundle $P_\sigma$ over $Y$, and compatible with the action of the automorphisms of $P_\sigma$ via the natural action of $Pic(Y)$

$$f_!([f^*L] \cdot \xi) = [L] \cdot f_!(\xi),$$

for $[L] \in Pic(Y)$ and $\xi \in K^*_T(f^*\sigma+W_3(f))(X)$.

2. If $f$ is K-oriented, that is, $W_3(f) = 0$, our choice of the Clifford bundle $Cl(TX \oplus f^*TY)$ determines a canonical choice of $Spin^c$ structure on $TX \oplus f^*TY$ with the associated first Chern class $c_1 \in H^2(X, \mathbb{Z})$ of the determinant bundle such that

$$c_1 \equiv w_2(X) + f^*w_2(Y).$$

Hence, we can denote $f_!$ by $f_!^{c_1}$. Then for any other choice of $Spin^c$ structure corresponding to $c_1 + 2c$ for $c \in H^2(X, \mathbb{Z})$, the associated push-forward map

$$f_!^{c_1+2c} : K^*_T(X) \to K^*_T(Y),$$

is given by $f_!^{c_1+2c}(\xi) = f_!^{c_1}([L_c] \cdot \xi)$, where $\xi \in K^*_T(X)$, and $[L_c] \in K^0(X)$ is the K-element defined by the equivalence class of the line bundle over $X$ with first Chern class $c \in H^2(X, \mathbb{Z})$. 

Assume that $\dim X - \dim Y = 0 (\mod 2)$. The proof will be divided into two parts. We can choose an embedding $\iota : X \to S^{2n}$, where $S^{2n}$ is an even dimensional sphere. Let $g = (f, \iota) : X \to Y \times S^{2n}$ and $\pi$ be the projection $p : Y \times S^{2n} \to Y$. Then $f = \pi \circ g$

In the first part, we show that the push-forward map

$\pi^! : K^f \sigma + W_3(f) \to K^\pi \sigma (Y \times S^{2n})$,

exists for the smooth embedding $g$. Note that $f^* \sigma = g^*(\pi^* \sigma)$ and $W_3(g) = W_3(f)$. We will give a construction of the push-forward map for any smooth embedding in the first subsection. In the second part, we will establish the push-forward map for any $\text{Spin}^c$ fibration which includes the projection $p : Y \times S^{2n} \to Y$. This will be done in the second subsection.

As twisted K-theory is homotopy invariant, the push-forward map is independent of the choice of the embedding $\iota : X \to S^{2n}$ for a fixed $n$. We will show that the push-forward map is independent of the dimension of the sphere $S^{2n}$ when we establish the functoriality for push-forward maps in the third subsection.

Replacing $f : X \to Y$ by $f \times \text{Id} : X \times \mathbb{R} \to Y \times \mathbb{R}$, the push-forward map of the differentiable map $f \times \text{Id}$,

$(f \times \text{Id}) : K^f_{f^* \sigma + W_3(f)}(X \times \mathbb{R}) \to K^\pi(X \times \mathbb{R})$,

induces the push-forward map for the odd degree twisted K-group

$f_! : K^f_{f^* \sigma + W_3(f)}(X) \to K^\pi(Y)$.

If $\dim X - \dim Y = 1 (\mod 2)$, we can think $f : X \to Y$ as the composition of $f \times \text{pt} : X \to Y \times S^1$ and the projection $p : Y \times S^1 \to Y$.

then we can define the push-forward map $f_!$ as the following composition

$$f_! : K^f_{f^* \sigma + W_3(f)}(X) \xrightarrow{(f \times \text{pt})^*} K^{\pi}_{\pi^* \sigma} (Y \times S^1) \xrightarrow{p^*} K^\pi_{\sigma^* + 1}(Y).$$

**Corollary 4.1.** If $f : X \to Y$ is K-oriented, i.e., $W_3(f) = 0$, with $c_1 \in H^2(X, \mathbb{Z})$ satisfying

$$c_1 \equiv w_2(X) - f^* w_2(Y) \quad (\mod 2),$$

there exists a natural push-forward map

$$f_!^{c_1} : K^f_{f^* \sigma}(X) \to K^\pi_{\sigma}(Y)$$

which agrees with the push-forward map for the ordinary K-theory when $\sigma$ is trivial.
Corollary 4.2. Let $f : X \to Y$ be a fibration with a closed oriented fiber, then the topological index in twisted $K$-theory is given by

$$K_{\pi^*f^*}(T(X/Y)) \xrightarrow{\pi} K_{f^*+W_3(f)}(X) \xrightarrow{f} K^*_\sigma(Y),$$

where $\pi : T(X/Y) \to X$ is the vertical tangent bundle of $f : X \to Y$.

4.1. Push-forward map for embeddings. Let $f : X \to Y$ be any differentiable embedding and $\sigma \in H^3(Y, \mathbb{Z})$. We will show that there is a natural homomorphism

$$f_! : K_{f^*+W_3(f)}(X) \to K_\sigma(Y).$$

Identifying the normal bundle $V_X$ with a neighborhood of $X$ in $Y$, so we have an embedding $\iota : V_X \to Y$. Then we have

As $TV_X \oplus \iota^*TY$ admits a $Spin^c$ structure, hence, $W_3(\iota) = 0$.

Lemma 4.3. $\iota^* \circ \iota^* \sigma = f^* \sigma$, $W_3(\iota) = 0$ and $W_3(\iota^*) = W_3(f)$.

Proof. $\iota^* \circ \iota^* \sigma = f^* \sigma$ follows from $\iota \circ \iota = f$. Let us calculate $W_3(\iota)$:

$$W_3(\iota) = W_3(TX \oplus \iota^*T(V_X)) = W_3(TX \oplus \iota^*(\pi^*V_X) \oplus \iota^*(\pi^*TX)) = W_3(TX \oplus TX \oplus V_X) = W_3(V_X) = W_3(f).$$

As $TV_X \oplus \iota^*TY$ admits a $Spin^c$ structure, hence, $W_3(\iota) = 0$. □

We define the push-forward

$$\iota : K_{f^*+W_3(f)}(X) \to K_\iota^* \sigma(V_X) \cong K_{\iota^* f^* \sigma}(V),$$

to be the Thom isomorphism for the oriented real vector bundle $V_X$ equipped with a positive definite quadratic form.

As $\iota : V_X \to Y$ is an open embedding, from the definition of twisted $K$-theory, we know that there is a natural extension map

$$\iota_! : K_\iota^* \sigma(V_X) \to K_{\sigma}(Y).$$

This can be seen as follows: any element in $K_\iota^* \sigma(V_X)$ is represented by a section of $P_\sigma|_{V_X} \times PU(\mathcal{H}) Fred^0(\mathcal{H}) \to V_X$ such that away from a compact set, it is invertible, where $P_\sigma$ is the principal $PU(\mathcal{H})$-bundle over $Y$ with Dixmier-Douady class $\sigma$. By homotopy, such a section can be chosen so that away from a compact set, it is the identity operator, hence, naturally defines a section of $P_\sigma \times PU(\mathcal{H}) Fred^0(\mathcal{H}) \to Y$.

We define the push-forward map for $f : X \to Y$ to be

$$(4.2) \quad f_! = \iota_! \circ \iota_!. $$
Applying the homotopy invariance of twisted K-theory, we know that \( f \) doesn’t depend on the choice of the identification of the normal bundle \( V_X \) with a tubular neighborhood \( X \) in \( Y \).

4.2. **Push-forward map for \( \text{Spin}^c \) fibrations.** Given a smooth fibration \( \pi : X \to Y \) over a closed smooth manifold \( Y \), whose fibers are diffeomorphic to an even dimensional compact, oriented, closed \( \text{Spin}^c \) manifold \( M \), we will define a push-forward map on twisted K-theory using a twisted family of Fredholm operators coupled to Dirac operators along the fibre of \( \pi \).

In fact, from the algebraic definition of the twisted K-theory, one sees that the push-forward map

\[
\pi_! : K^0_{\pi^* \sigma}(X) \to K^0_{\sigma}(Y),
\]

is defined via the Kasparov product with the longitudinal Dirac element, which is the element in \( KK(C_0(X), C_0(Y)) \) defined by the fiberwise Dirac operator (Cf. [11]). Let us illustrate this idea using the following projection

\[
p : Y \times S^{2n} \to Y,
\]

which is enough for our construction of the push-forward map (Cf. Diagram 4.1).

Recall the algebraic definition of the twisted K-theory (Cf. [2] [25]). Let \( \mathcal{H} \) be an infinite dimensional, complex and separable Hilbert space, and \( K(\mathcal{H}) \) be the C*-algebra of compact operators on \( \mathcal{H} \). Given a class \( \sigma \in H^3(Y, \mathbb{Z}) \) on a locally compact Hausdorff space \( Y \), we can associate a principal \( \text{PU}(\mathcal{H}) \)-bundle \( P_\sigma \) over \( Y \) with its Dixmier-Douady class given by \( \sigma \).

Applying the natural identification \( \text{PU}(\mathcal{H}) \cong \text{Aut}(K(\mathcal{H})) \) (the *-automorphism group of \( K(\mathcal{H}) \)), one can has a bundle of compact operators

\[
P_\sigma \times \text{PU}(\mathcal{H}) K(\mathcal{H}),
\]

of which the space of continuous sections vanishing at infinity is denoted by \( \Lambda_\sigma(Y) \).

**Definition 4.4.** The algebraic definition of twisted K-theory is given by the K-theory of the corresponding continuous trace C*-algebra \( \Lambda_\sigma(Y) \)

\[
K^i_\sigma(Y) := K^i_{P_\sigma}(Y) = KK^i(C, \Lambda_\sigma(Y)),
\]

for \( i = 0, 1 \).

**Proposition 4.5.** For the natural projection \( p : Y \times S^{2n} \to Y \), the push-forward map

\[
p_! : K^0_{P^* \sigma}(Y \times S^{2n}) \to K^0_{\sigma}(Y)
\]

is given by the Kasparov product with the Dirac element \([\delta_{S^{2n}}] \in KK(C(S^{2n}), \mathbb{C})\).

**Proof.** Denote by \( p^* \sigma \in H^3(Y \times S^{2n}, \mathbb{Z}) \) the pull-back of \( \sigma \) via \( p : Y \times S^{2n} \to Y \), then we have

\[
\Lambda_{p^* \sigma}(Y \times S^{2n}) \cong C(S^{2n}) \otimes \Lambda_\sigma(Y).
\]

It is well-known that the Dirac operator on \( S^{2n} \) with standard metric defines a canonical analytic K-cycle:

\[
[\delta_{S^{2n}}] \in KK(C(S^{2n}), \mathbb{C}).
\]

Applying the Kasparov product,

\[
KK(C, C(S^{2n}) \otimes \Lambda_\sigma(Y)) \otimes KK(C(S^{2n}), \mathbb{C}) \to KK(C, \Lambda_\sigma(Y)),
\]

\[
\]
we get the push-forward map associated to \( p : Y \times S^{2n} \to Y \),
\[
p_! = \otimes [\delta_{S^{2n}}] : K^0_p(Y \times S^{2n}) \cong KK(\mathbb{C}, \delta_{p^{-1}}(Y \times S^{2n})) \to K^0_p(Y).
\]
\( \square \)

As we use the model of twisted Fredholm operators to define the twisted K-theory, we now provide a geometric proof of the existence of the push-forward map, which is merely the obvious translation of the Kasparov product to the differential geometry set-up. For simplicity, we also assume that \( X \) and \( Y \) are compact manifolds.

Let \( g^{X/Y} \) be a metric on the relative tangent bundle \( T(X/Y) \) (the vertical tangent subbundle of \( TX \)). The spinor bundle associated to \( (T(X/Y), g^{X/Y}) \) is denoted by \( S^{X/Y} = S^+_X \oplus S^-_X \).

Let \( T^H X \) be a smooth vector subbundle of \( TX \), (a horizontal vector subbundle of \( TX \)), such that
\[
TX = T^H X \oplus T(X/Y).
\]
Denote by \( P^v \) the projection of \( TX \) onto the vertical tangent bundle under the decomposition (4.3). As shown in Theorem 1.9 of [5], \( (T^H X, g^{X/Y}) \) determines a canonical Euclidean connection \( \nabla^{X/Y} \) as follows. Choose a metric \( g^X \) on \( TX \) such that \( T^H X \) is orthogonal to \( T(X/Y) \) and such that \( g^{X/Y} \) is the restriction of \( g^X \) to \( TX \). Then
\[
\nabla^{X/Y} = P^v \circ \nabla^X
\]
where \( \nabla^X \) is the Levi-Civita connection on \( (TX, g^X) \).

The Clifford multiplication of \( T^*(X/Y) \) on \( S_{X/Y} \) and the Euclidean connection \( \nabla^{X/Y} \) defines a fiberwise Dirac operator acting on \( C^\infty(X_y, S_{X/Y}) \), which can be completed to a Hilbert space, denoted by \( \mathcal{H}_y \). For any \( y \in Y \), let \( \mathcal{D}_y \) be the Dirac operator along the fiber \( \pi^{-1}(y) \). Then \( \{\mathcal{D}_y\}_{y \in Y} \) is a family of elliptic operators over \( Y \) acting on an infinite dimensional Hilbert bundle \( \mathcal{H}_{X/Y} \), whose fiber at \( y \) is \( \mathcal{H}_y \). Note that \( \mathcal{H}_{X/Y} \) is a \( \mathbb{Z}_2 \)-graded Hilbert bundle over \( Y \) for the \( Spin^c \) fibration with even dimensional fibres.

The partial isometry part in the polar decomposition of \( \mathcal{D}_y \) is not a continuous family of bounded Fredholm operators parametrized by \( y \in Y \). Modulo compact operators, there is a continuous family of bounded Fredholm operators parametrized by \( y \in Y \), denoted by
\[
V_y = \mathcal{D}_y (\mathcal{D}_y^* \mathcal{D}_y + 1)^{-1/2},
\]
under the usual assumption that \( \mathcal{D}_y \) is a smooth family of Dirac operators parametrized by \( y \in Y \).

Take a trivialization of \( P_\sigma \) for a cover \( \{U_\alpha\} \) of \( Y \):
\[
P_\sigma|U_\alpha \cong U_\alpha \times \mathcal{P}(\hat{\mathcal{H}}),
\]
with the transition functions given by \( \gamma_{\alpha\beta} : U_{\alpha\beta} \to PU(\hat{\mathcal{H}}) \). We assume that the fibration \( \pi : X \to Y \) is also trivialized over \( U_\alpha \), i.e.,
\[
\pi^{-1}(U_\alpha) \cong U_\alpha \times M.
\]
Then there is a trivialization of $P_{\pi^*\sigma} = \pi^* P_{\pi}$ with respect to the cover $\{\pi^{-1}(U_\alpha)\}$ of $X$ and the transition functions given by $\pi^* g_{\alpha\beta} = g_{\alpha\beta} \circ \pi$.

Given an element $T \in K_{\hat{P}_{\pi^*\sigma}}^0(X)$, we can represent it by a section of $P_{\pi^*} \times_{PU(\hat{H})} Fred^0(\hat{H})$, written as

$$T_\alpha : \pi^{-1}(U_\alpha) \rightarrow Fred^0(\hat{H})$$

satisfying $T_\beta = (\pi^* g_{\alpha\beta})T_\alpha (\pi^* g_{\alpha\beta})^{-1}$. We can couple the fiberwise Fredholm operators $\{V_y\}$ in (4.4) with $T_\alpha$ to get a section of

$$Fred^0(P_\sigma) = P_\sigma \times_{PU(\hat{H})} Fred^0(\hat{H} \otimes \mathcal{H}_{X/Y}),$$

by graded tensor product

$$\pi_1(T_\alpha) = V \otimes I + I \otimes T_\alpha : U_\alpha \rightarrow Fred^0(\hat{H} \otimes \mathcal{H}_{X/Y}),$$

as these local sections $\{\pi_1(T_\alpha)\}$ satisfy

$$\pi_1(T_\beta) = (g_{\alpha\beta} \times Id) \pi_1(T_\alpha) (g_{\alpha\beta} \otimes Id)^{-1}.$$

Fix a graded isomorphism $\phi : \hat{H} \otimes \mathcal{H}_{X/Y} \rightarrow \hat{H}$. We have a principal $PU(\hat{H})$-bundle $\tilde{P}_{\sigma}$ whose transition functions with respect to the cover $\{U_\alpha\}$ of $Y$ are given by

$$\tilde{g}_{\alpha\beta} = \phi \circ (g_{\alpha\beta} \times Id) \circ \phi^{-1} : U_{\alpha\beta} \rightarrow PU(\hat{H}).$$

Hence, $\tilde{P}_{\sigma} \cong P_{\sigma}$. Then the above local section $\{\pi_1(T_\alpha)\}$ induces a section of

$$Fred^0(\tilde{P}_{\sigma}) = \tilde{P}_{\sigma} \times_{PU(\hat{H})} Fred^0(\hat{H}).$$

Notice that a homotopy equivalent section of $Fred^0(P_{\pi^*\sigma})$, under the above procedure, produces a homotopy equivalent section of $Fred^0(\tilde{P}_{\sigma})$.

A different choice of the graded isomorphism $\phi : \hat{H} \otimes \mathcal{H}_{X/Y} \rightarrow \hat{H}$ gives rise to an isomorphism on the resulting twisted $K$-group. Note that $Aut(P_{\sigma}) \cong H^2(Y, \mathbb{Z})$ acts on $K_{P_{\sigma}}^0(Y)$ by natural isomorphisms. This finishes the definition of the push-forward map, up to the natural actions of $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$,

$$\pi_1 : K_{\pi^*\sigma}^0(X) \rightarrow K_\sigma^0(Y).$$

The push-forward map on the odd degree $K$-group

$$\pi_1 : K_{\pi^*\sigma}^1(X) \rightarrow K_\sigma^1(Y)$$

can also be defined using a based $S^1$-family of local twisted sections.

**Remark 4.6.** For a $Spin^c$ fibration $\pi : X \rightarrow Y$ with odd dimensional fibers, with the help of isomorphisms (2.3) and (2.4), we obtain the corresponding push-forward maps:

$$\pi_1 : K^0_{\hat{P}_{\pi^*\sigma}}(X) \rightarrow K^0_{P_{\sigma}}(Y),$$

$$\pi_1 : K^1_{\hat{P}_{\pi^*\sigma}}(X) \rightarrow K^1_{P_{\sigma}}(Y).$$
4.3. **Functoriality for push-forward maps.** In this subsection, we will establish the wrong way functoriality

\[(g \circ f)! = g! \circ f_! : K^*_{(g \circ f)!}(X) \rightarrow K^*_{g!}(Y) \rightarrow K^*_f(Z)\]

for two differentiable maps \(f : X \rightarrow Y, \ g : Y \rightarrow Z\) and \(\sigma \in H^3(Z, \mathbb{Z})\).

We first prove the wrong way functoriality for two embeddings.

**Lemma 4.7.** Let \(f : X \rightarrow Y\) and \(g : Y \rightarrow Z\) be two smooth embeddings, then \((g \circ f)! = g! \circ f_!\).

**Proof.** For a smooth embedding, the corresponding push-forward map is given by the composition of Thom isomorphism and the natural extension for open embeddings. Denote by \(V_{X \subset Y}\) and \(V_{Y \subset Z}\) the normal bundles over \(X\) and \(Y\), together with fixed identifications to the tubular neighbourhoods of \(f(X)\) and \(g(Y)\). Denote by \(\emptyset_X\) and \(\emptyset_Y\) the corresponding zero sections of \(V_{X \subset Y}\) and \(V_{Y \subset Z}\) respectively, then we have the following commutative diagram

\[
\begin{array}{ccc}
V_{X \subset Y} & \xrightarrow{\emptyset} & V_{Y \subset Z} \\
X & \xrightarrow{f} & Y \xrightarrow{g} Z,
\end{array}
\]

where \(i^*_1(V_{Y \subset Z})\) is the restriction of \(V_{Y \subset Z}\) to the open tubular neighbourhood \(V_{X \subset Y}\), and \(i_1\), \(i_2\) and \(i_3\) are open embeddings. As \(i^*_1(V_{Y \subset Z})\) is also a normal bundle over \(X\), with a fixed identification to the tubular neighbourhood of \((g \circ f)(X)\) in \(Z\), then \((g \circ f)! = g! \circ f_!\) follows from the definition of the push-forward map. \(\square\)

Now we show functoriality for general smooth maps \(f : X \rightarrow Y\) and \(g : Y \rightarrow Z\). Choosing two embeddings \(\iota_1 : X \rightarrow S^{2m}\) and \(\iota_2 : Y \rightarrow S^{2n}\), we have the following commutative diagram:

\[
(4.6)
\]

where \(\pi_5\) is the natural projection \(Z \times S^{2n} \times S^{2m} \rightarrow Z \times S^{2m}\). Then from the definition of the push-forward map, we know that

\[
(g_! \circ f_!)(\iota_1) = (g_! \circ f_!)(\iota_1) = \pi_3 \circ ((g \circ f) \times \iota_1)_!.
\]

The proof of \((g \circ f)! = g! \circ f_!\) is given in the following sequence of identities.
Lemma 4.8. \((g \times \iota_2) \triangleright (\pi_1)_! = (\pi_4)_! \circ (g \times \iota_2 \times \text{Id})\).

Proof. The proof of this lemma follows directly from the definition of the push-forward map for embeddings and the following commutative diagram:

\[
\begin{array}{ccccccccc}
Y \times S^{2m} & \xrightarrow{\iota_1} & V_Y & \xrightarrow{\pi_4} & Z \times S^{2n} \\
\downarrow{\pi_1} & & \downarrow{\pi_4} & & \downarrow{\pi_5} \\
Y & \xrightarrow{g \times \iota_2 \times \text{Id}} & Z \times S^{2n} & \xrightarrow{\text{Id} \times \iota_1} & Z \times S^{2N} \\
\end{array}
\]

where the normal bundle \(V_Y\) is identified with the tubular neighbourhood of \(Y\) in \(Z \times S^{2n}\), with zero section \(\iota_1: Y \to V_Y\), and the normal bundle \(V_Y \times S^{2m}\) is identified with the tubular neighbourhood of \(Y \times S^{2m}\) in \(Z \times S^{2m} \times S^{2n}\).

The following commutative diagram with the obvious projections

\[
\begin{array}{ccccccc}
Z \times S^{2n} \times S^{2m} & \xrightarrow{\pi_4} & Z \times S^{2n} \\
\downarrow{\pi_5} & & \downarrow{\pi_2} \\
Z \times S^{2m} & \xrightarrow{\pi_3} & Z \\
\end{array}
\]

implies that \((\pi_2)_! \circ (\pi_4)_! = (\pi_3)_! \circ (\pi_5)_!\), as the Dirac element on \(S^{2m} \times S^{2n}\) is given by the external product of the Dirac elements on \(S^{2m}\) and \(S^{2n}\).

Putting together the above identities, we know that \((g \circ f)_! = g_! \circ f_!\) follows, if we can show that

\[
(\pi_3)_! \circ ((g \circ f) \times \iota_1)_! = (\pi_3)_! \circ (\pi_5)_! \circ (g \times \iota_2 \times \text{Id})_! \circ (f \times \iota_1)_!.
\]

Choose an isometric embedding \(\iota: S^{2n} \times S^{2m} \to S^{2N}\) where these spheres are equipped with the natural round metrics. Consider the following commutative diagram, where maps in the top row are all smooth embeddings,

\[
\begin{array}{ccccccccc}
X & \xrightarrow{f \times \iota_1} & Y \times S^{2m} & \xrightarrow{g \times \iota_2 \times \text{Id}} & Z \times S^{2n} \times S^{2m} & \xrightarrow{\text{Id} \times \iota_1} & Z \times S^{2N} \\
\downarrow{(g \circ f) \times \iota_1} & & \downarrow{\pi_5} & & \downarrow{\pi_6} \\
Z \times S^{2m} & \xrightarrow{\pi_3} & Z. \\
\end{array}
\]

Notice that the right square in the above diagram tells us that

\[(\pi_6)_! \circ (\text{Id} \times \iota)_! = (\pi_3)_! \circ (\pi_5)_!\].
Here we use functoriality in topological index theory

\[ S^{2m} \times S^{2n} \xrightarrow{\iota} S^{2N} \]

which reads, in terms of Dirac elements on the round spheres,

\[ [\delta_{S^{2m} \times S^{2n}}] = \pi! \otimes [\delta_{S^{2n}}], \]

under the Kasparov product (Cf. [11]). Here \( \pi! \) is the canonical element in KK-theory

\[ KK(C(S^{2m} \times S^{2n}), C(S^{2N})) \]

defined in [11] by Connes and Skandalis. Hence, the required equation (4.8) becomes

\[ (\pi_3)! \circ (g \circ f) \times \iota_1 \] \( = \) \( (\pi_6)! \circ (Id \times \iota)! \circ (g \times \iota_2 \times Id)! \circ (f \times \iota_1)!. \]

Apply functoriality for embedding maps (Cf. Lemma 4.7), so we know that

\[ (Id \times \iota)! \circ (g \times \iota_2 \times Id)! \circ (f \times \iota_1) = ((g \circ f) \times \iota), \]

where \((g \circ f) \times \iota : X \to Z \times S^{2N}\) is the induced embedding from \( \iota_1, \iota_2 \) and \( \iota \).

It remains to show that

\[ (\pi_3)! \circ (g \circ f) \times \iota_1 = (\pi_6)! \circ (g \circ f) \times \iota. \]

Equivalently, the push-forward map \((g \circ f)\)! is independent of the choice of sphere in its definition.

We establish this fact for a general smooth map \( f : X \to Z \), with a slight change of notations for simplicity. For two different embeddings \( \iota_1 : X \to S^{2m} \) and \( \iota_2 : X \to S^{2n} \), we can choose smooth embeddings \( \iota' : S^{2m} \to S^{2N} \) and \( \iota'' : S^{2n} \to S^{2N} \). Then the proof follows from the following commutative diagram:

\[
\begin{array}{ccc}
Y \times S^{2m} & \xrightarrow{1d \times \iota'} & Y \times S^{2N} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
Y \times S^{2n} & \xleftarrow{1d \times \iota''} & Y \times S^{2N}
\end{array}
\]

Here we consecutively apply functoriality in topological index theory for the following diagrams:

\[
\begin{array}{ccc}
S^{2m} & \xrightarrow{\iota'} & S^{2N} \\
\downarrow & & \downarrow \\
pt & \xleftarrow{\iota''} & S^{2n}
\end{array}
\]

i.e., \([\delta_{S^{2m}}] = (\iota')! \otimes [\delta_{S^{2N}}], \] \([\delta_{S^{2n}}] = (\iota'')! \otimes [\delta_{S^{2N}}], \) the functoriality for embedding maps (Lemma 4.7)

\[ (Id \times \iota') \circ f \times \iota_1 = f \times (\iota' \circ \iota_1), \quad (Id \times \iota'') \circ f \times \iota_1 = f \times (\iota'' \circ \iota_2) \]

and the homotopy invariance of twisted K-theory for two embeddings \( \iota' \circ \iota_1 \) and \( \iota'' \circ \iota_2 \).
5. Some applications

5.1. Index of Dirac operators for Clifford modules. According to the Atiyah-Singer index theory for a Dirac operator on a Clifford module $E$ over a compact oriented even dimensional manifold $X$, see Theorem 4.3 [4], the index is given by

\begin{equation}
\int_X \hat{A}(X) Ch(E/S),
\end{equation}

where $Ch(E/S)$ is the relative Chern character form defined by a Clifford connection on $E$.

Let $Cl(X)$ be the bundle of Clifford algebra associated to the tangent bundle equipped with the Riemannian metric. From Lemma 3.3, we know that the category of $Cl(X)$-modules is isomorphic to the category $E_{bg}(X, \Gamma_{W_3(X)})$ of $\Gamma_{W_3(X)}$-modules. We can express the Atiyah-Singer index (5.1) as the topological index:

\[ t - \text{index} : K_{W_3(X)}(X) \longrightarrow \mathbb{Z}. \]

By the Thom isomorphism for the tangent bundle equipped with a Riemannian metric (Cf. Theorem 3.9), we have

\[ K_{W_3(X)}(X) \cong K(TX). \]

The topological index is given by

\begin{equation}
K(TX) \longrightarrow \mathbb{Z}.
\end{equation}

Recall the definition of (5.2) for a compact manifold $X$ by choose an embedding $\iota : X \rightarrow \mathbb{R}^m$, then the tangent map, also denoted by

\[ \iota : TX \rightarrow T\mathbb{R}^m \cong \mathbb{C}^m, \]

is K-oriented. Then the corresponding push forward map in ordinary K-theory and Bott periodicity give rise to

\[ \iota_! : K(TX) \longrightarrow K(T\mathbb{R}^m) \cong K(\mathbb{C}^m) \cong \mathbb{Z}. \]

This proves that the Atiyah-Singer index (5.1) is the push-forward map in twisted K-theory for $f : X \rightarrow pt$.

In [21], generalized Dirac operators are introduced for any even dimensional Riemannian manifold $M$ in terms of bundle gerbe modules for the lifting bundle gerbe $\Gamma_{W_3(X)}$. The index of Murray-Singer’s generalized Dirac operators defines a map:

\[ \text{Ind} : \mathcal{E}_{bg}(X, \Gamma_{W_3(X)}) \longrightarrow \mathbb{Z}, \]

which agrees with Atiyah-Singer index (5.1) under the category equivalence

\[ \mathcal{E}_{bg}(X, \Gamma_{W_3(X)}) \cong \mathcal{E}^{TX}(X). \]

Descending to the twisted K-group $K_{W_3(X)}(X)$, the Murray-Singer index is the push-forward map in twisted K-theory for $f : X \rightarrow pt$. 
5.2. D-brane charges. In type IIA/IIB superstring theory with topologically trivial $B$-field, a $D$-brane (see [28]) in an oriented 10-dimensional $\text{Spin}$ manifold $X$, is a $\text{Spin}^c$ submanifold $\iota : Q \to X$, together with a Chan-Paton bundle and a superconnection, defined by an element $\xi \in K(Q)$. Using the push-forward map in ordinary K-theory,
\[
\iota : K(Q) \longrightarrow K^*(X)
\]
we see that such $D$-branes are classified by elements in $K(X)$, and the Ramond-Ramond charge of $(Q, \xi)$ is given by
\[
\text{Ch}(\iota! \xi) \sqrt{\hat{A}(X)} \in H^*(X, \mathbb{R}).
\]

When the $B$-field is non-trivial and its characteristic class $\sigma$ lies in $H^3(M, \mathbb{Z})$ with the curvature denoted by $H$, there is a constraint. A submanifold $\iota : Q \to X$, has associated to it a well defined action when it satisfies the Freed-Witten anomaly cancellation condition (Cf. [14]):
\[
\iota^* \sigma + W_3(Q) = 0,
\]
in $H^3(Q, \mathbb{Z})$. It was proposed in [29][15][7] that $D$-brane charges should be classified by the twisted K-group $K^*_\sigma(X)$, yet a rigorous formulation of such $D$-brane charges hasn’t been found.

Now as an application of our push-forward map in twisted K-theory, we can associate a canonical element in $K^*_\sigma(X)$, which can be called the $D$-brane charge of the underlying $D$-branes. As $X$ is spin, $W_3(Q)$ agrees with $W_3(V_Q)$, the third Stiefel-Whitney class of a normal bundle of $Q$ in $X$. Hence, $W_3(Q) = W_3(\iota)$. For a submanifold $\iota : Q \to X$ satisfying the Freed-Witten anomaly cancellation condition $\iota^* \sigma + W_3(Q) = 0$, in $H^3(Q, \mathbb{Z})$. We apply the push-forward map for $\iota$ to get
\[
\iota : K(Q) \cong K^*_{\iota^*\sigma + W_3(Q)}(Q) \longrightarrow K^*_\sigma(X),
\]
hence, for any $D$-brane wrapping on $Q$ determined by an element $\xi \in K(Q)$, we can define its charge as
\[
\iota(\xi) \in K^*_\sigma(X).
\]

References

(Alan L. Carey) Mathematical Sciences Institute, Australian National University, Canberra ACT 0200, Australia
E-mail address: acarey@maths.anu.edu.au

(Bai-Ling Wang) Mathematical Sciences Institute, Australian National University, Canberra ACT 0200, Australia
E-mail address: wangb@maths.anu.edu.au