Exact triangles in Seiberg-Witten-Floer theory.
Part III: proof of exactness

Matilde Marcolli, Bai-Ling Wang

Contents

1 Introduction

In the previous parts of this work [6], [17], we considered a homology 3-sphere Y and an embedded knot K. We considered manifolds Y_1 and Y_0 obtained by 1-surgery and 0-surgery on K, respectively. We proved that the oriented moduli space of solutions of the Seiberg–Witten equations on Y can be described as the union

$$\mathcal{M}_{Y,\mu} \cong \mathcal{M}_{Y_1} \cup \bigcup_{\mathfrak{s}_k} \mathcal{M}_{Y_0}(\mathfrak{s}_k), \tag{1}$$

where μ is a suitable perturbation that simulates the effect of surgery, and the $\mathfrak{s}_k$ are the Spin^c structures on Y_0 that reduce to the trivial structure when restricted on the tubular neighbourhood of the knot $\nu(K)$ and on the knot complement V. The irreducible components are similarly related

$$\mathcal{M}^*_{Y,\mu} \cong \mathcal{M}^*_{Y_1} \cup \bigcup_{\mathfrak{s}_k} \mathcal{M}_{Y_0}(\mathfrak{s}_k),$$

as shown in [6].

We also proved that there is a way of assigning compatible choices of the relative grading, so that (1) gives an exact sequence *of abelian groups*

$$0 \to C_q(Y_1) \stackrel{j_q}{\to} C_q(Y,\mu) \stackrel{\pi_q}{\to} \bigoplus_{\mathfrak{s}_k} C_{(q)}(Y_0,\mathfrak{s}_k) \to 0. \tag{2}$$

Recall that the grading of $C_{(q)}(Y_0,\mathfrak{s}_k)$ denotes a lifting to a $\mathbb{Z}$-graded complex of the $\mathbb{Z}_{2i_k}$-graded complex. This lifting is determined by a compatible choice of grading on (1). This $\mathbb{Z}$-graded Floer complex is analyzed in detail in [18]. The maps j_* and π_* are induced by the inclusion

$$j : \mathcal{M}_{Y_1} \hookrightarrow \mathcal{M}_{Y,\mu},$$

and the projection on the quotient, generated by the elements of

$$\cup_{\mathfrak{s}_k} \mathcal{M}_{Y_0}(\mathfrak{s}_k).$$

The analysis of the splitting and gluing of the moduli spaces of flow lines in [17] shows that, in general, one should not expect (2) to be an exact sequence *of chain complexes*: the maps j_* and π_* need not commute with the boundary operators of the Floer complexes. Thus, on the algebraic point of view, the existence of the decomposition (1) simply signifies that the existence of an exact sequence is possible: in fact, the decomposition (2) shows that the ranks are compatible with the existence of the desired exact sequence. The maps that provide the exact sequence are derived from the surgery cobordisms.

In this paper we introduce maps w^1_* and w^0_*, induced by the surgery cobordisms W_1 and W_0 connecting Y_1 and Y, and Y and Y_0, respectively. The maps w^1_* and w^0_* are defined by a suitable choice of the Spin^c structures $\mathfrak{s}_\ell$ and $\mathfrak{s}_k$ on the 4-manifolds W_1 and W_0, that restrict to the assigned Spin^c structure at the two ends of the cobordism.

We show that w^1_* and w^0_* are chain homomorphisms,

$$C_q(Y_1) \stackrel{w^1_q}{\to} C_q(Y,\mu) \tag{3}$$

and

$$C_q(Y,\mu) \stackrel{w^0_q}{\to} \bigoplus_{\mathfrak{s}_k} C_{(q)}(Y_0,\mathfrak{s}_k). \tag{4}$$

We then show that the map w^0_* is surjective and the map w^1_* is injective. The main technique is similar to the technique developed in [17] in order to study the behavior of flow lines under the splitting $Y = V \cup_{T^2} \nu(K)$. Here we consider punctured surgery cobordisms $W_1 \backslash \{x_1\}$ and $W_0 \backslash \{x_0\}$ and we stretch product regions $T^2 \times [-r,r] \times \mathbb{R}$ inside these punctured cobordisms, thus also stretching product regions $T^2 \times [-r,r] \times [0,\infty)$ in the $S^3 \times [0,\infty)$ end near the puncture x_i. We have geometric limits and a gluing theorem as in [17] and we can show that, when the parameter ϵ in the surgery perturbation is small enough, the solutions on W_1 and W_0 have the following behavior. If we consider solutions on W_1 with asymptotic values a_1 in $\mathcal{M}_{Y_1}$ and $j(a_1)$ in $j(\mathcal{M}_{Y_1}) \subset \mathcal{M}_{Y,\mu}$, and solutions on W_0 with asymptotic values $a \in \mathcal{M}_{Y,\mu} \backslash j(\mathcal{M}_{Y_1})$ and $\pi(a)$ in $\cup_k \mathcal{M}_{Y_0}(\mathfrak{s}_k)$, then the corresponding components of the maps w^1_* and w^0_* agree with those of the maps j_* and π_* in (2). In other words, we prove the following relations:

$$\langle j(a_1'), w^1_*(a_1) \rangle = \delta_{a_1', a_1}, \tag{5}$$

for all a_1' and a_1 in $\mathcal{M}_{Y_1}$, and

$$\langle a_0, w^0_*(a) \rangle = \delta_{a_0, \pi(a)}, \tag{6}$$

for all $a \in \mathcal{M}_{Y,\mu} \backslash j(\mathcal{M}_{Y_1})$ and for all $a_0 \in \cup_k \mathcal{M}_{Y_0}(\mathfrak{s}_k)$.

We then prove the relation $w^0_* \circ w^1_* = 0$. Again, we follow the same technique. We compare the geometric limits of zero-dimensional moduli spaces

$$\mathcal{M}^{W_1}_{\ell}(a_1, a)$$

and

$$\mathcal{M}^{W_0}_{k}(j(a_1), \pi(a)),$$

for $a \in \mathcal{M}_{Y,\mu} \backslash j(\mathcal{M}_{Y_1})$, regarded as subsets of the moduli space

$$\mathcal{M}^{W}_{\ell,k}(a_1, \pi(a))$$

on the composite cobordism $W = W_1 \cup_Y W_0$. We obtain an orientation reversing diffeomorphism

$$\mathcal{M}^{W_1}_{\ell}(a_1, a) \cong \mathcal{M}^{W_0}_{k}(j(a_1), \pi(a)),$$

which proves the relation $w^0_* \circ w^1_* = 0$. The main technique consists of identifying the moduli spaces of solutions on the cobordisms with certain pre-gluing data obtained out of the explicit description of the geometric limits developed in Part II [17]. The set of pre-gluing data and the resulting moduli space can be identified up to a diffeomorphism given by the gluing map. The set of pre-gluing data consists of moduli spaces of finite energy monopoles on $V \times \mathbb{R}$ together with holomorphic triangles in a covering of the character variety $\chi(T^2)$ of flat $U(1)$-connections on T^2, with boundary along arcs of Lagrangians ℓ, ℓ_1, ℓ^*_μ determined by the asymptotic values $\partial_\infty \mathcal{M}^*_V$, and by the flat connections on $\nu(K)$, with or without surgery perturbation. In order to use dimensional arguments, we compare the formulae for the Maslov index in the splitting of the spectral flow, with the formulae for the dimension of the moduli space of such holomorphic triangles.

Then we can complete the proof of the exactness of the sequence

$$0 \to C_q(Y_1) \stackrel{w^1_q}{\to} C_q(Y,\mu) \stackrel{w^0_q}{\to} \bigoplus_{\mathfrak{s}_k} C_{(q)}(Y_0, \mathfrak{s}_k) \to 0. \tag{7}$$

It is enough to show that (5) and (6), together with the inclusion $Im(w^1_*) \subset Ker(w^0_*)$ determine enough relations among the coefficients of the chain maps that force the reverse inclusion $Ker(w^0_*) \subset Im(w^1_*)$ to hold as well.

In the last Section we analyze the connecting homomorphism in the long exact sequence

$$\cdots \to HF_q(Y_1) \to HF_q(Y,\mu) \to \oplus_k HF_{(q)}(Y_0, \mathfrak{s}_k) \stackrel{\Delta_q}{\to} HF_{q-1}(Y_1) \to \cdots.$$

We first show that the coefficients of Δ are given by the component of the boundary on Y that counts flow lines connecting critical points of $\mathcal{M}_{Y,\mu}\backslash j(\mathcal{M}_{Y_1})$ to critical points of $j(\mathcal{M}_{Y_1})$, of relative index one. We then proceed to identify this counting with the counting of zero dimensional moduli spaces on another cobordism $\bar{W}_2$ connecting Y_0 and Y_1, satisfying the relation

$$\bar{W} = \bar{W}_2 \# \mathbb{C}P^2,$$

where $\bar{W}$ is the composite cobordism $\bar{W} = \bar{W}_0 \cup_Y \bar{W}_1$. Thus, we obtain the result that the exact triangle for Seiberg-Witten Floer homology is a surgery triangle, that is, the connecting homomorphism in the exact sequence is determined by a chain map $\bar{w}^2_*$ induced by the surgery cobordism $\bar{W}_2$, and the resulting diagram

$$C_*(Y_1) \stackrel{w^1_*}{\to} C_*(Y,\mu) \stackrel{w^0_*}{\to} \oplus_k C_{(*)}(Y_0,\mathfrak{s}_k) \stackrel{\bar{w}^2_*}{\to} C_*(Y_1)[-1]$$

is a distinguished triangle.

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2 The cobordisms

We describe briefly the topology of the cobordisms and then introduce the appropriate perturbed Seiberg–Witten equations, and the corresponding moduli spaces.

The cobordism W_1 is obtained by removing from the trivial cobordism $Y_1 \times I$ an $S^1 \times D \cong \nu(K) \times \{1\}$, where D is a disk, and $\nu(K)$ is the tubular neighbourhood of the knot in Y_1, and then attaching a 2-handle with framing -1. We denote by D_1 the core disk of the 2-handle in W_1. Similarly, the cobordism W_0 is obtained by removing from the trivial cobordism $Y_0 \times I$ an $S^1 \times D \cong \nu(K) \times \{0\}$ and attaching a 2-handle with framing zero. We denote by D_0 the core disk of the 2-handle in W_0. Attaching the two-handle has the effect of modifying the boundary component $Y_1 \times \{1\}$ in the trivial cobordism to the boundary component $Y \times \{1\}$ in the non-trivial cobordism W_1, or, respectively, the boundary component $Y_0 \times \{0\}$ in the trivial cobordism to the boundary component $Y \times \{0\}$ in W_0.

Lemma 2.1 *The cobordisms W_1 and W_0 have the following topology:*

$$\begin{array}{llll} b_1(W_1)=0 & b_2(W_1)=1 & b_2^+(W_1)=0 & b_2^-(W_1)=1 \\ b_1(W_0)=0 & b_2(W_0)=1 & b_2^+(W_0)=0 & b_2^-(W_0)=0. \end{array}$$

The composite cobordism $W = W_0 \cup_Y W_1$, connecting Y_1 and Y_0, can be written as a blow up $W = W_2 \# \overline{\mathbb{C}P^2}$, where W_2 satisfies

$$b_1(W_2)=0 \quad b_2(W_2)=1 \quad b_2^+(W_2)=0 \quad b_2^-(W_2)=0.$$

Proof. Let σ be a Seifert surface for the knot K in the homology sphere Y, $\partial\sigma = K$. In the cobordism W_1 consider the surface Σ_1 obtained by attaching the Seifert surface and the core disk D_1 along the knot K,

$$\Sigma_1 = \sigma \cup_K D_1.$$

The homology $H_2(W_1,\mathbb{Z})$ is generated by the class $[\Sigma_1]$ with self intersection -1. Similarly, consider the surface Σ_0 in W_0, obtained by attaching along K the Seifert surface and the core disk D_0,

$$\Sigma_0 = \sigma \cup_K D_0.$$

The homology $H_2(W_0,\mathbb{Z})$ is generated by the class $[\Sigma_0]$ with self intersection zero. The surface

$$\Sigma = D_1 \cup_K \overline{D_0}$$

in the composite cobordism W has self intersection -1. The homology $H_2(W,\mathbb{Z})$ is generated by $[\Sigma] = [\Sigma_1] - [\Sigma_0]$ and $[\Sigma_0]$. The class $[\Sigma]$ represents the exceptional divisor E in the blowup. Thus, the blown down cobordism W_2 has homology $H_2(W_2,\mathbb{Z})$ generated by a class, which we still write $[\Sigma_0]$, with self intersection zero.
$\diamond$

Notice that the surface Σ_0 is homologous in W_0 to the generator $\sigma \cup_K D$ of $H_2(Y_0,\mathbb{Z})$. The following simple calculation is useful in classifying the possible Spin^c structures on the cobordisms.

Lemma 2.2 *We have $H_2(W_0, Y \cup Y_0, \mathbb{Z}) = \mathbb{Z}$ generated by a surface $\tilde{\Sigma}_0$ in W_0 with $\partial\tilde{\Sigma}_0 = \gamma$, with $[\gamma]$ the generator of $H_1(Y_0,\mathbb{Z})$. The class $[\Sigma_0]$ introduced above, which generates $H_2(W_0,\mathbb{Z})$ is trivial in $H_2(W_0, Y \cup Y_0)$.*

Moreover, we have $H_2(W_1, Y_1 \cup Y, \mathbb{Z}) = \mathbb{Z}$ generated by the image of the class $[\Sigma_1]$ which generates $H_2(W_1,\mathbb{Z})$, and $H_2(W, Y_1 \cup Y_0, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, *generated by the class E of the exceptional divisor of the blow-up and by $[\tilde{\Sigma}_0]$. Again, the class $[\Sigma_0]$, which is non-trivial in $H_2(W,\mathbb{Z})$, is mapped trivially to $H_2(W, Y_1 \cup Y_0, \mathbb{Z})$. Finally, we have $H_2(W_2, Y_1 \cup Y_0, \mathbb{Z}) = \mathbb{Z}$ generated by $[\tilde{\Sigma}_0]$.*

Proof. The results simply follow from the exact sequence in homology. We have

$$H_3(W_0, Y \cup Y_0, \mathbb{Z}) \cong H_1(W_0) = 0,$$

hence we have

$$0 \to H_2(Y_0, \mathbb{Z}) \xrightarrow{\cong} H_2(W_0, \mathbb{Z}) \xrightarrow{0} H_2(W_0, Y \cup Y_0, \mathbb{Z}) \longrightarrow H_1(Y_0, \mathbb{Z}) \to 0.$$

Similarly, we have

$$0 \to H_2(Y_0, \mathbb{Z}) \longrightarrow H_2(W, \mathbb{Z}) \longrightarrow H_2(W, Y_1 \cup Y_0, \mathbb{Z}) \longrightarrow H_1(Y_0, \mathbb{Z}) \to 0$$

which is of the form

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0 \oplus I} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0.$$

$\diamond$

Thus, we obtain the following result.

Lemma 2.3 *We have an identification of the* Spin^c*-structures on W_0 and W_1 given by*

$$\mathcal{S}(W_0) = \{\mathfrak{s}_k\}_{k \in \mathbb{Z}},$$

with

$$\mathfrak{s}_k \mapsto c_1(\det W^+_{\mathfrak{s}_k}) = 2k \ e_0 = 2k PD_{W_0}[\tilde{\Sigma}_0],$$

with $[\tilde{\Sigma}_0]$ the generator of $H_2(W_0, Y \cup Y_0, \mathbb{Z})$. Similarly, we have

$$\mathcal{S}(W_1) = \{\mathfrak{s}_\ell\}_{\ell \in \mathbb{Z}},$$

with

$$\mathfrak{s}_\ell \mapsto c_1(\det W^+_{\mathfrak{s}_\ell}) = (2\ell + 1) \ e_1 = (2\ell + 1) PD_{W_1}[\Sigma_1],$$

with $[\Sigma_1]$ the generator of $H_2(W_1, Y_1 \cup Y, \mathbb{Z})$. Moreover, we have

$$\mathcal{S}(W) = \{\mathfrak{s}_{\ell,k}\}_{\ell,k \in \mathbb{Z}},$$

with

$$\mathfrak{s}_{\ell,k} \mapsto c_1(\det W^+_{\mathfrak{s}_{\ell,k}}) = (2\ell + 1) \ e + 2k \ e_0,$$

where we have $e = PD_W(E)$, and E is the image of the class $[\Sigma_1 - \Sigma_0]$ in $H_2(W, Y_1 \cup Y, \mathbb{Z})$. Similarly, we have

$$\mathcal{S}(W_2) = \{\mathfrak{s}_k\}_{k \in \mathbb{Z}}.$$

Proof. The result follows from the previous Lemma with the additional observation that in W_1 we have Σ_1 with self intersection $\Sigma_1^2 = -1$, hence the Spin^c structures have odd Chern class $(2\ell + 1)PD_{W_1}[\Sigma_1]$. $\diamond$

In the case of W_0, W_2, and W, the Spin^c structure $\mathfrak{s}_k$ or $\mathfrak{s}_{\ell,k}$ restricts to the end $Y_0 \times [T_0, \infty)$ of the cobordism to the pullback of the Spin^c structure $\mathfrak{s}_k$ in $\mathcal{S}(W_0)$. All the Spin^c structures of Lemma 2.3 restrict to the trivial Spin^c structure on the cylindrical ends modeled on Y or Y_1.

2.1 Splitting of the cobordisms

In the following, we shall introduce moduli spaces of Seiberg-Witten equations on the cobordisms. Our purpose is to apply to the moduli spaces on the cobordisms the same techniques we developed in [17], in the study of the moduli spaces of flowlines, that is, of monopoles on the trivial cobordisms. Thus, it is convenient to consider the manifolds W_1 and W_0 endowed with infinite cylindrical ends $Y_1 \times (-\infty, -T_0]$ and $Y \times [T_0, \infty)$, and $Y_0 \times [T_0, \infty)$ and $Y \times (-\infty, -T_0]$, respectively, with metrics $g_Y + dt^2$ and $g_{Y_i} + dt^2$. Moreover, we shall assume that the 3-manifolds Y_1, Y, and Y_0 are endowed with metrics with a long cylinder $T^2 \times [-r, r]$, as specified in [6].

We can then think of the cobordisms as endowed with a metric which restricts to the flat product metric on the region $T^2 \times [-r, r] \times \mathbb{R}$. Moreover, we can identify in the cobordisms W_i a product region $V \times \mathbb{R}$, on the complement of a tubular neighbourhood of the knot, where the cobordism is trivial. Thus, we can decompose the cobordisms W_i as

$$W_i = V \times \mathbb{R} \cup_{T^2 \times \mathbb{R}} T^2 \times [-r, r] \times \mathbb{R} \cup_{T^2 \times \mathbb{R}} W_i(\nu(K)). \tag{8}$$

The non-compact region $W_i(\nu(K))$ has the following property. There is a compact set $\mathcal{K}$ in W_i such that the intersection $\mathcal{K} \cap W_i(\nu(K))$ is obtained by attaching a 2-handle $D \times D$ to the product $\nu(K) \times [-T_0, T_0]$, and, outside of $\mathcal{K}$, the region $\mathcal{K}^c \cap W_i(\nu(K))$ consists of product regions $\nu(K) \times [T_0, \infty)$ and $\nu(K) \times (-\infty, -T_0]$, and $T^2 \times [r_0, r] \times [-T_0, T_0]$.

In the cobordism W_i consider an interior point x_i contained in the core disk of the 2-handle, $x_i \in D_i$. As in [2], we denote by $\hat{W}_i$ the punctured cobordism $\hat{W}_i = W_i \backslash \{x_i\}$. Similarly, we can consider the punctured manifold

$$\hat{W}_i(\nu(K)) = W_i(\nu(K)) \backslash \{x_i\}.$$

In the manifolds $\hat{W}_i(\nu(K))$ we can identify a product region

$$\mathcal{V} = \nu(K)_{r_0} \times \mathbb{R} \cong D \times (D_i \backslash \{x_i\}). \tag{9}$$

This corresponds to endowing the manifold $\hat{W}_i$ with an extra asymptotic end of the form $S^3 \times [0, \infty)$ at the puncture. Thus, we identify the manifold W_i with a connected sum

$$W_i = \hat{W}_i \# Q_i,$$

with a long cylindrical neck $S^3 \times [-T(r), T(r)]$, and with Q_i a 4-ball, as in [2].

Consider the sphere S^3 decomposed as the union of two solid tori in the standard way, $S^3 = \nu \cup \tilde{\nu}$, with $\nu \cong \tilde{\nu} \cong D \times S^1$. Then the product region $\mathcal{V}$ of (9) in W_i identifies the standard solid torus ν in S^3 with the neighborhood $\nu(K)$ of the knot K in Y, and, similarly, the other solid torus $\tilde{\nu}$ in S^3 is identified with the tubular neighbourhood $\nu(K)$ in Y_i, after the surgery. This is illustrated in Figure 1.

2.2 Metrics and perturbations on the cobordism

The results of this subsection are based on the metric deformation Lemma that Liviu Nicolaescu kindly communicated to us, [23], and that we enclosed in Part I [6].

Lemma 2.4 *Let A be an element in $SL(2, \mathbb{Z})$. Suppose given $\epsilon > 0$ sufficiently small. Consider the metric on T^2 given by*

$$g_0 = A^* g,$$

where g is the standard flat metric as before. There exists a constant $\delta > 0$ and a smooth path $g(s)$ of flat metrics on T^2 with the following properties:

(i) $g(s) \equiv \frac{1}{\delta^2} g_0$, for all $s \leq \epsilon$;

(ii) $g_1 = g(1)$ is a metric of the form

$$g_1 = k_1 du^2 + k_2 dv^2$$

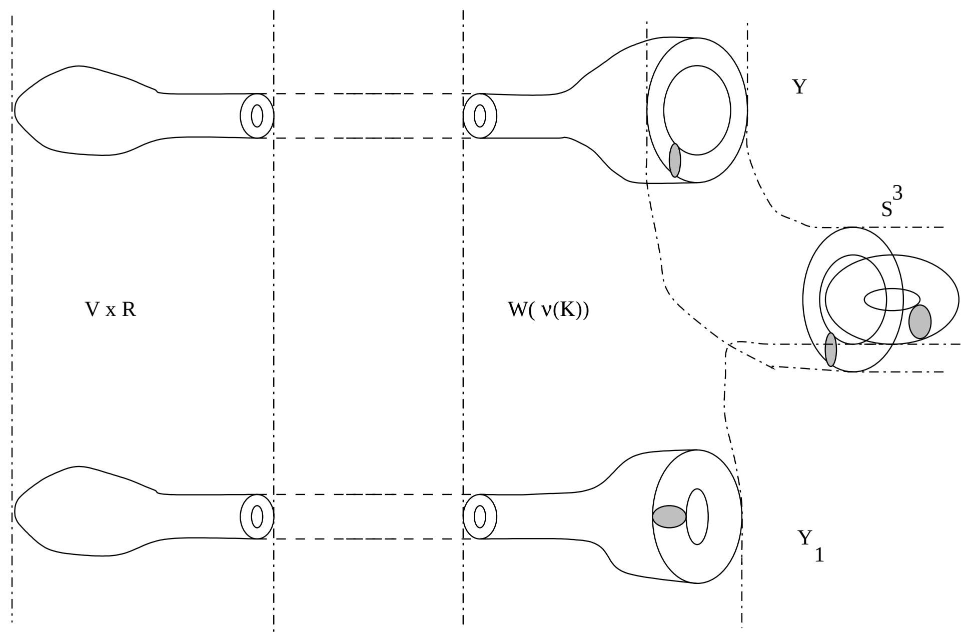

Figure 1: The decomposition of the punctured cobordism $\hat{W}_1$

for k_i positive constants;

(iii) $g(s) = g_1$ for all $s \geq 1 - \epsilon$;

(iv) The scalar curvature of the metric $\hat{g} = ds^2 + g(s)$ on $T^2 \times \mathbb{R}$ is non-negative.

(v) The metric g_1 can be extended to a metric inside the solid torus $\nu(K)$, which we still denote g_1, that has non-negative scalar curvature.

The constant δ is given by $\delta^2 = g_0(\partial_u, \partial_u)$.

Using this result, when we construct the manifold Y_1 from Y, by removing a tubular neighborhood $\nu(K)$ and gluing it back along T^2 with the matrix $A \in SL(2,\mathbb{Z})$ prescribed by the surgery, we can consider the same metric on the knot complement V, with an end isometric to $T^2 \times [0, \infty)$, with the metric $g + ds^2$. On the other hand, on $\nu(K)$ we can consider the metric $\delta^2(g(s) + ds^2)$, with $g(s)$ constructed as above, with the parameterization chosen in such a way that we have

$$g(s) \equiv \frac{1}{\delta^2} A^* g$$

on the end $T^2 \times [\delta^{-2} r_0, \infty)$ and $g(s) \equiv g_1$ near $T^2 \times \{0\}$, extended to a positive scalar curvature metric inside the solid torus.

With this choice of metrics, we still have the decomposition of the moduli spaces of critical points of the CSD functional, as proved in Part I, [6]. Moreover, this particular choice of metrics allows us to describe the choice of metrics on the cobordisms.

Consider first the trivial cobordism $Y_1 \times \mathbb{R}$. In the limit $r \to \infty$ this splits as $V \times \mathbb{R}$, with an end of the form $T^2 \times \mathbb{R}^+ \times \mathbb{R}$ with metric $g + ds^2 + dt^2$, and $\nu(K) \times \mathbb{R}$, with an end of the form $T^2 \times \mathbb{R}^+ \times \mathbb{R}$ with the metric $\delta^2(g(s) + ds^2) + dt^2$, as described above. Now consider the punctured cobordism $\hat{W}_1$. This contains a product region $\nu \times \mathbb{R}$ which connects the solid torus $\nu(K) \subset Y$ with a solid torus $\nu \subset S^3$ at the puncture. On this product region we consider the metric $G := g + \delta^2 ds^2 + dt^2$, with g the standard flat metric on T^2 extended to a non-negative scalar curvature metric inside the solid torus as described in [6]. The other product region $\tilde{\nu} \times \mathbb{R}$ connecting the solid torus $\nu(K) \subset Y_1$ with a solid torus $\tilde{\nu}$ inside S^3, is glued in the punctured 2-handle, along a region $T^2 \times \mathbb{R}$, with framing one. Thus, on this product region we can consider the metric $G(s) := \delta^2(g(s) + ds^2) + dt^2$ described above. These regions are illustrated schematically in Figure 2, with a lower dimensional picture of the punctured handle.

Our purpose is to define chain maps between the Floer complexes of the 3-manifolds using the Seiberg–Witten equations on the cobordisms, and to adapt the techniques of [17] to analyze these chain maps, by understanding their asymptotic limits under the splitting of the cobordisms illustrated in the previous section. Thus, we need to introduce a suitable perturbation of the Seiberg–Witten equations on W_i which is compatible with the perturbations of the Chern–Simons–Dirac functional on the manifolds Y_i and Y, described in [6].

Recall that on Y_1 we have perturbed flow equations of the form

$$\left\{ \begin{array}{lcl} \dfrac{\partial A}{\partial t} & = & -*F_A + \sigma(\psi,\psi) + \sum_{j=1}^N \dfrac{\partial U_1}{\partial \tau_j} \mu_j^{(1)} \\ \dfrac{\partial \psi}{\partial t} & = & -\partial_A \psi - \sum_{i=1}^K \dfrac{\partial V_1}{\partial \zeta_i} \nu_i^{(1)}.\psi, \end{array} \right. \tag{10}$$

where (U_1, V_1) is a pair of functions in the class $\mathcal{P}_\delta$ described in [6], that is, it becomes exponentially small along the cylinder $T^2 \times [-r, r]$ inside Y_1, and is exponentially small on the solid torus $\nu(K) \subset Y_1$. All the notation we use here follows [6].

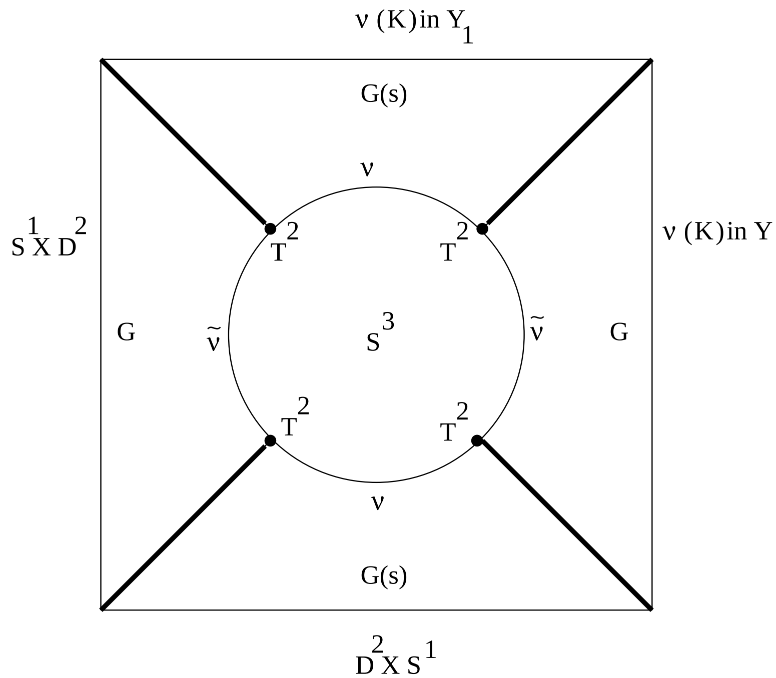

Figure 2: Product regions and metrics on the punctured 2-handle

The equations (10) can be written equivalently as the four dimensional equations on $Y_1 \times \mathbb{R}$,

$$\left\{ \begin{array}{l} F^+_{\mathcal{A}} = \tau(\Psi,\Psi) + P^{Y_1}_1(\mathcal{A},\Psi) \\ D_{\mathcal{A}}\Psi = P^{Y_1}_2(\mathcal{A},\Psi), \end{array} \right. \tag{11}$$

with

$$P^{Y_1}_1(\mathcal{A},\Psi) = \sum_{j=1}^N \frac{\partial U_1}{\partial \tau_j}(\mu^{(1)}_j \wedge dt + *_3\mu^{(1)}_j), \tag{12}$$

$$P^{Y_1}_2(\mathcal{A},\Psi) = \sum_{i=1}^K \frac{\partial V_1}{\partial \zeta_i}\nu^{(1)}_i.\Psi. \tag{13}$$

As in [6], we consider similarly perturbed equations on Y, with the additional surgery perturbation on the solid torus $\nu(K) \subset Y$. Thus, on Y we have flow equations

$$
\left\{ \begin{array}{lcl} \dfrac{\partial A}{\partial t} & = & -*F_A + \sigma(\psi,\psi) + \sum_{j=1}^{N} \dfrac{\partial U}{\partial \tau_j}\mu_j + f'(T_{A(t)}) * \mu \\ \dfrac{\partial \psi}{\partial t} & = & -\partial_A \psi - \sum_{i=1}^{K} \dfrac{\partial V}{\partial \zeta_i}\nu_i.\psi, \end{array} \right. \tag{14}
$$

with

$$
T_A(z) = -i \int_{\{z\in D^2\}\times S^1} (A - A_0).
$$

Here the function f', which depends on the choice of a small parameter $0 < \epsilon \leq \epsilon_0$, is constructed as in [6], μ is a compactly supported 2-form, and (U, V) is a perturbation in the class $\mathcal{P}_\delta$ on Y.

The equation (14) can be written equivalently as

$$
\left\{ \begin{array}{l} F^+_{\mathcal{A}} = \tau(\Psi,\Psi) + P^Y_1(\mathcal{A},\Psi) + f'(T_{\mathcal{A}})\mu^+ \\ D_{\mathcal{A}}\Psi = P^Y_2(\mathcal{A},\Psi), \end{array} \right. \tag{15}
$$

where μ^+ is the self-dual part of the pullback of μ along the projection $\nu(K)\times\mathbb{R}\to\nu(K)$.

We discuss a choice of perturbation on W_i which behaves nicely under the splitting of the cobordisms and restricts to the prescribed perturbations on the asymptotic ends. Consider the case of W_1. The case of W_0 is analogous.

The manifold W_1 has a cylindrical end $Y \times [T_0,\infty)$ and a cylindrical end $Y_1 \times (-\infty, -T_0]$. Consider a cutoff function $\chi(t)$ which is $\chi(t) \equiv 1$ for $t \geq T_0$ and $\chi(t) \equiv 0$ for $t \leq T_0 - 1$. In the following we shall use the notation $\hat\chi(t) = \chi(-t)$. Consider, on the cylindrical ends $Y \times [T_0,\infty)$ and $Y_1 \times (-\infty, -T_0]$ of W_1, the equations (11) and (15), respectively. Now consider the manifold $V \times \mathbb{R}$, endowed with an infinite end of the form $T^2 \times [0,\infty) \times \mathbb{R}$. Notice that on this region inside the manifold $V \times \mathbb{R}$ it makes sense to define the temporal gauge condition for pairs $(\mathcal{A},\Psi)$. We denote by $(A(t),\psi(t))$ a temporal gauge representative of the gauge class of $(\mathcal{A},\Psi)$.

On V we can also introduce perturbations in the class $\mathcal{P}_\delta$, as described in Section 3 of[6].

This gives a choice of perturbation on $V_r \times \mathbb{R}$ inside W_1, for large $r \geq r_0$. Namely, we consider on $V_r \times \mathbb{R}$ the equations

$$\begin{cases} F^+_{\mathcal{A}} = & \tau(\Psi,\Psi) + (1-\hat{\chi}-\chi)P_1^V(\mathcal{A},\Psi) \\ & +\hat{\chi}P_1^{Y_1}(\mathcal{A},\Psi) + \chi P_1^Y(\mathcal{A},\Psi) \\ D_{\mathcal{A}}\Psi = & (1-\hat{\chi}-\chi)P_2^V(\mathcal{A},\Psi) \\ & +\hat{\chi}P_2^{Y_1}(\mathcal{A},\Psi) + \chi P_2^Y(\mathcal{A},\Psi). \end{cases} \tag{16}$$

This matches the perturbations (11) and (15) on $V \times [T_0,\infty)$ and $V \times (-\infty,-T_0]$ in W_1.

On the product region $\mathcal{V}$ of (9), which connects $\nu(K) \subset Y$ to the solid torus $\nu \subset S^3$, we consider the Seiberg–Witten equations with the surgery perturbation, namely the equations

$$\begin{cases} F^+_{\mathcal{A}} = \tau(\Psi,\Psi) + f'(T_{\mathcal{A}})\mu^+ + \chi P_1^Y(\mathcal{A},\Psi) \\ D_{\mathcal{A}}\Psi = \chi P_2^Y(\mathcal{A},\Psi). \end{cases} \tag{17}$$

On the product region $\tilde{\mathcal{V}}$ inside $\hat{W}_1(\nu(K))$ that connects the solid torus $\tilde{\nu}$ in S^3 to $\nu(K)$ in Y_1, we consider equations

$$\begin{cases} F^+_{\mathcal{A}} = \tau(\Psi,\Psi) + \hat{\chi}P_1^{Y_1}(\mathcal{A},\Psi) \\ D_{\mathcal{A}}\Psi = \hat{\chi}P_2^{Y_1}(\mathcal{A},\Psi). \end{cases} \tag{18}$$

In the case of W_0, if we consider Y_0 with the trivial Spin^c-structure, the perturbation along the cylindrical end $Y_0 \times [T_0,\infty)$ includes a perturbation ρ_0 with $[*\rho_0] = \eta PD_{Y_0}(m)$, with respect to the $*$-operator on Y_0, with $\eta > 0$ as in [6] and m the generator of the first homology group of Y_0.

Throughout the paper, when we consider finite energy solutions of the Seiberg–Witten equations on the cobordisms W_i, we will mean finite energy solutions of the Seiberg–Witten equations on the punctured cobordisms $\hat{W}_i$, as in Definition 2.5 below, with the perturbations introduced here, and with a removable singularity at the puncture x_i, that is, such that they extend to solutions on W_i.

In the following, we shall analyze the behavior of finite energy solutions on W_i, when stretching $r \to \infty$ in the cobordism

$$\hat{W}_i(r) = V_r \times \mathbb{R} \cup \mathcal{V}(r) \cup \tilde{\mathcal{V}}(r), \tag{19}$$

with $\mathcal{V}(r) \cong \nu_r \times \mathbb{R}$ and $\tilde{\mathcal{V}}(r) \cong \tilde{\nu}_r \times \mathbb{R}$, the product regions inside $\hat{W}_i(\nu(K))_r$.

In particular, with the choice of perturbation discussed here, a finite energy solution on W_1 will have an asymptotic value

$$a_1 = [A_1, \psi_1] \in \mathcal{M}_{Y_1}$$

on the Y_1 end, and

$$a = [A, \psi] \in \mathcal{M}_{Y,\mu},$$

on the other end of the cobordism, with the moduli spaces of solutions of the perturbed equations on Y and Y_1 as in [6]. Similarly, a finite energy solution on the cobordism W_0 will have asymptotic values

$$a = [A, \psi] \in \mathcal{M}_{Y,\mu},$$

$$a_0 = [A_0, \psi_0] \in \mathcal{M}_{Y_0}(\mathfrak{s}_k).$$

Thus we can define moduli spaces $\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1, a)$ and $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a, a_0)$ as follows.

2.3 Finite energy monopoles and virtual dimension

We consider finite energy solutions of the perturbed equations introduced above, on the punctured cobordisms $\hat{W}_i$, with a removable singularity at the puncture x_i, that is, such that they extend to solutions on W_i. More precisely, we have the following.

Definition 2.5 *Consider the manifold $\hat{W}_i$, as a complete Riemannian manifold with infinite cylindrical ends. Consider a fixed choice of the* Spin^c *structure. Write C for any of the cylindrical ends of the manifold $\hat{W}_i$. A solution $(\mathcal{A}, \Psi)$ of the Seiberg–Witten equations on the manifold $\hat{W}_i$ is of finite energy, with a removable singularity at the point x_i, iff on any of the cylindrical ends C the solutions $(\mathcal{A}, \Psi)$ in a temporal gauge satisfies the condition*

$$\int_C |\partial_t A(t)|^2 + |\partial_t \psi(t)|^2 < \infty.$$

The analysis in Section 3 of [19] of the asymptotics of finite energy solutions on a trivial cobordism $Y \times \mathbb{R}$ carries over to the present case and shows that finite energy solutions on the manifolds $\hat{W}_i$ decay along the cylindrical ends to asymptotic values (A, ψ) satisfying the 3-dimensional Seiberg–Witten equations on the boundary 3-manifolds. Moreover, if the asymptotic value is an irreducible critical point, then the rate of decay is exponential, with the exponent determined by the first eigenvalue of the Hessian at the critical point.

Thus, we define configuration spaces $\mathcal{A}_{k,\delta}(W_i)$ and the group of gauge transformations $\mathcal{G}_{k+1,\delta}(W_i)$ and consider finite energy solutions in the quotient space. The configuration space $\mathcal{A}_{k,\delta}(W_i)$ consists of pairs $(\mathcal{A}, \Psi)$ on $\hat{W}_i$ that are of finite energy, with a removable singularity at x_i, and with a rate of decay with exponent δ along the cylindrical ends modeled on Y and Y_1. Since the asymptotic limit along the end $S^3 \times [0, \infty)$ at the puncture x_i is the unique reducible solution θ_{S^3} on S^3, we can define moduli spaces

$$\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1, a) \quad \text{and} \quad \mathcal{M}^{W_0}_{\mathfrak{s}_k}(a, a_0)$$

of solutions in $\mathcal{A}_{k,\delta}(W_i)$ modulo gauge action, which depend only on the asymptotic limits at the two ends of W_i and on the Spin^c structure on W_i. We can estimate the virtual dimension of these moduli spaces of solutions.

Lemma 2.6 *Consider the linearization $\mathcal{D}_{(\mathcal{A},\Psi)}$ of the Seiberg–Witten equations at a solution $(\mathcal{A}, \Psi)$ on $(W_1, \mathfrak{s}_\ell)$, with asymptotic values $a_1 = [A_1, \psi_1]$ and $a = [A, \psi]$ on the boundary three-manifolds Y_1 and Y. The index of $\mathcal{D}_{(\mathcal{A},\Psi)}$ is given by*

$$Ind(\mathcal{D}_{(\mathcal{A},\Psi)}) = \frac{1}{4}(c_1(\det(\mathfrak{s}_\ell))^2 - 2\chi(W_1) - 3\sigma(W_1)) + \frac{\rho}{2}(Y_1, a_1) - \frac{\rho}{2}(Y, a),$$

where the last two summands are the APS ρ-invariants [1] of the extended Hessian operators $H_{(A_1,\psi_1)}$ and $H_{(A,\psi)}$. The grading analyzed in [6] satisfies

$$\deg_{Y_1}(a_1) = \frac{1}{8\pi^2} CSD_{Y_1}(a_1) + \frac{\rho}{2}(Y_1, a_1),$$

up to redefining the CSD functional by a global additive constant.

Proof. In the case of a solution $(\mathcal{A}, \Psi)$ on the trivial cobordism $Y \times I$, with asymptotic values $a = [A_a, \psi_a]$ and $b = [A_b, \psi_b]$ on Y, we have

$$\deg_Y(a) - \deg_Y(b) = Ind(\mathcal{D}_{(\mathcal{A},\Psi)}) = \frac{c_1(\det(\mathfrak{s}))^2}{4} + \frac{\rho}{2}(Y, a) - \frac{\rho}{2}(Y, b)$$

$$= \frac{-1}{16\pi^2} \int_{Y \times \mathbb{R}} F_{\mathcal{A}} \wedge F_{\mathcal{A}} + \frac{\rho}{2}(Y, a) - \frac{\rho}{2}(Y, b)$$

$$= \frac{-1}{8\pi^2}(CSD(A_b, \psi_b) - CSD(A_a, \psi_a)) + \frac{\rho}{2}(Y, a) - \frac{\rho}{2}(Y, b).$$

$\diamond$

See the energy estimates in [17] (cf. [1], [21]). This virtual dimension of Seiberg–Witten moduli spaces on four-manifolds with boundary is computed explicitly in [22], in the case of Seifert fibered spaces, where a particular choice of metric makes it possible to compute $\rho(Y, a)$ explicitly.

By the results of [6], the choice of the grading on Y_1 determines uniquely the grading $\deg_{Y,\mu}$ on Y, up to changing the functional CSD by a global additive constant. The grading $\deg_{Y,\mu}$ also determines [6] the choice of the grading $\deg_{Y_0}$ on Y_0. For the properties of the Floer complex on Y_0 see also [18].

We define the expression

$$\iota(\mathfrak{s}_\ell, W_1, a_1, a) = \frac{1}{4}(c_1(\det(\mathfrak{s}_\ell))^2 - 2\chi(W_1) - 3\sigma(W_1)) + \frac{\rho}{2}(Y_1, a_1) - \frac{\rho}{2}(Y, a)$$

in the case of W_1. Similarly, in the case of W_0, we define

$$\iota(\mathfrak{s}_k, W_0, a, a_0) = \frac{1}{4}(\int_{W_0} c_1(\det(\mathfrak{s}_k))^2 - L(\hat{\nabla}^0)) - \frac{1}{2}(\chi(W_0) + \sigma(W_0))$$
$$+ \frac{\rho}{2}(Y, a) - \frac{\rho}{2}(Y_0, a_0),$$

where $\hat{\nabla}^0$ is the metric compatible connection, which along the end $Y_0 \times [T_0, \infty)$ has the form $dt \otimes \partial_t + \nabla^0$.(This is analogous to the case of [22], but in our setting the connection ∇^0 is simply the metric connection with no adiabatic rescaling.) The expressions

$$\iota(\mathfrak{s}_\ell, W_1, a_1, a) \text{ and } \iota(\mathfrak{s}_k, W_0, a, a_0)$$

compute the virtual dimensions of the moduli spaces

$$\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1, a) \quad \text{and} \quad \mathcal{M}^{W_0}_{\mathfrak{s}_k}(a, a_0).$$

Notice that, in the case of W_0, the Spin^c-structure may be non-trivial along the end $Y_0 \times [T_0, \infty)$. This has the important consequence that the virtual dimension $\iota(\mathfrak{s}_k, W_0, a, a_0)$ is only defined modulo the integer $d(\mathfrak{s}_k)$, with $d(\mathfrak{s}_k)$ satisfying

$$c_1(\det(\mathfrak{s}_k))(H_2(Y_0, \mathbb{Z})) = d(\mathfrak{s}_k)\mathbb{Z},$$

that is, in this case, $d(\mathfrak{s}_k) = 2k$. This ambiguity corresponds to different components of the moduli space of solutions of the Seiberg-Witten equations on W_0, with different energies. The minimal energy component corresponds to the minimal non-negative value of $\iota(\mathfrak{s}_k, W_0, a, a_0)$.

2.4 Compactification and invariants

Under a generic choice of the perturbation, we can assume that all the moduli spaces $\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1,a)$ and $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a,a_0)$ are cut out transversely, of dimension $\iota(\mathfrak{s}_\ell,W_1,a_1,a)$ and $\iota(\mathfrak{s}_k,W_0,a,a_0)$, respectively. Unless otherwise stated, when we write $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a,a_0)$, we only consider the component of minimal energy, with dimension $\iota(\mathfrak{s}_k,W_0,a,a_0)$.

The following description of the compactification of $\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1,a)$ and $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a,a_0)$ follows from the main gluing theorem proved in [19], together with the results of the previous subsections.

Proposition 2.7 *Suppose given a non-empty moduli space $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a,a_0)$ of dimension*

$$\iota(\mathfrak{s}_k,W_0,a,a_0)=n>0,$$

with $a\in\mathcal{M}^*_{Y,\mu}$. Then $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a,a_0)$ admits a compactification to a manifold with corners, where the codimension 1 boundary strata consist of

$$\begin{array}{c}\bigcup_{c\in\mathcal{M}^*_{Y,\mu}}\hat{\mathcal{M}}_{Y,\mu}(a,c)\times\mathcal{M}^{W_0}_{\mathfrak{s}_k}(c,a_0)\\ \bigcup_{c_0\in\mathcal{M}_{Y_0}(\mathfrak{s}_k)}\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a,c_0)\times\hat{\mathcal{M}}^{(0)}_{Y_0,\mathfrak{s}_k}(c_0,a_0),\end{array}\tag{20}$$

and $\hat{\mathcal{M}}^{(0)}_{Y_0,\mathfrak{s}_k}(c_0,a_0)$ is the minimal energy component of the moduli space $\hat{\mathcal{M}}_{Y_0,\mathfrak{s}_k}(c_0,a_0)$, as discussed in [18]. If the reducible point $c=\theta$ satisfies

$$\deg_{Y,\mu}(a)>\deg_{Y,\mu}(\theta),$$

then we also have an extra component in (20) of the form

$$\hat{\mathcal{M}}_{Y,\mu}(a,\theta)\times U(1)\times\mathcal{M}^{W_0}_{\mathfrak{s}_k}(\theta,a_0)\tag{21}$$

with a $U(1)$ gluing parameter.

We have a similar compactification of $\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1,a)$ of dimension

$$\iota(\ell,W_1,a_1,a)=n>0,$$

for $a_1\in\mathcal{M}^*_{Y_1}$ and $a\in\mathcal{M}^*_{Y,\mu}$, with codimension 1 boundary strata of the form

$$\begin{array}{c}\bigcup_{c\in\mathcal{M}^*_{Y,\mu}}\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1,c)\times\hat{\mathcal{M}}_{Y,\mu}(c,a)\\ \bigcup_{c_1\in\mathcal{M}^*_{Y_1}}\hat{\mathcal{M}}_{Y_1}(a_1,c_1)\times\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(c_1,a),\end{array}\tag{22}$$

and with extra components

$$\begin{aligned}&\hat{\mathcal{M}}_{Y_1}(a_1,\theta_1)\times U(1)\times \mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(\theta_1,a)\\&\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1,\theta)\times U(1)\times \hat{\mathcal{M}}_{Y,\mu}(\theta,a),\end{aligned}\tag{23}$$

when splitting through the reducibles.

The Proposition follows from the main gluing theorem of [19]. The fact that only the minimal energy component, among the components of $\hat{\mathcal{M}}_{Y_0}(\lambda c_0, a_0)$, occurs in the compactification of $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(c,a_0)$ is explained in the following, in the proof of Lemma 4.3.

For later use, we also need the following.

Corollary 2.8 *If we have $\iota(\mathfrak{s}_\ell, W_1, a_1, a) = 1$, or $\iota(\mathfrak{s}_k, W_0, a, a_0) = 1$, with $a_1 \in \mathcal{M}^*_{Y_1}$, $a \in \mathcal{M}^*_{Y,\mu}$, and $a_0 \in \mathcal{M}_{Y_0}(\mathfrak{s}_k)$, then the boundary strata are given by (22) and (20), as in the compactification of Proposition 2.7, with c_1 and c irreducible.*

Proof. We need to show that the components (21) and (23) do not occur in the compactification. This follows by dimensional arguments. $\diamond$

We can define numerical invariants associated to the zero-dimensional components of the moduli spaces $\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1,a)$, and $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a,a_0)$, with $a_1 \in \mathcal{M}^*_{Y_1}$, $a \in \mathcal{M}^*_{Y,\mu}$, and $a_0 \in \cup_k \mathcal{M}_{Y_0}(\mathfrak{s}_k)$.

Recall that these moduli spaces come endowed with an orientation, given by the trivialization of the determinant line bundle of the linearization of the Seiberg–Witten equations. Some care is needed in defining the orientation in the case of non-compact 4-manifolds with infinite cylindrical ends. The necessary details can be found in [22]. The orientation is compatible with the compactification of Proposition 2.7. Throughout this discussion we shall always assume that the perturbations are chosen so that all the moduli spaces are cut out transversely by the equations.

According to Proposition 2.7 of the previous section, in the case of $\iota(\mathfrak{s}_\ell, W_1, a_1, a) = 0$, or $\iota(\mathfrak{s}_k, W_0, a, a_0) = 0$, the moduli spaces $\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1,a)$ and $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a,a_0)$ consist of a finite set of points with an attached sign given by the orientation. Thus, we can define

$$N^{W_1}_{\mathfrak{s}_\ell}(a_1,a) \quad \text{or} \quad N^{W_0}_{\mathfrak{s}_k}(a,a_0)$$

as the algebraic sum of the points in $\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1,a)$ and $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a,a_0)$, respectively. If we have $\iota(\mathfrak{s}_\ell, W_1, a_1, a) < 0$ and either a or a_1 is irreducible, or if we have $\iota(\mathfrak{s}_k, W_0, a, a_0) < 0$, then the corresponding moduli space is generically empty, so we just set the corresponding invariant equal to zero.

3 Geometric limits

In this section we describe the geometric limits of finite energy solutions on W_i when stretching $r \to \infty$ inside the cobordism

$$\hat{W}_i(r) = V_r \times \mathbb{R} \cup \mathcal{V}(r) \cup \tilde{\mathcal{V}}(r).$$

The analysis is very similar to the analysis of the geometric limits of flow lines in [17].

We give the general description of the geometric limits. We describe the case of the cobordism W_1. Simple modifications adapt the argument to W_0. We shall omit here the parts of the argument which are completely analogous to the case discussed in [17]. We assume that the metric is chosen as discussed previously.

Proposition 3.1 *Consider a family $[\mathcal{A}_r, \Psi_r]$ of finite energy solutions of the Seiberg–Witten equations on the cobordism $W_1(r)$, with the choices of perturbation as discussed previously. Assume that the $[\mathcal{A}_r, \Psi_r]$ have asymptotic values $a_1 = [A_1, \psi_1]$ and $a = [A, \psi]$ on the ends modeled on Y_1 and Y. By [6], these asymptotic limits can be written as*

$$(A_1, \psi_1) = (A_1', \psi_1') \#_{a^-} (a^-, 0),$$

and

$$(A, \psi) = (A', \psi') \#_{a^+} (a^+, 0),$$

for large $r \geq r_0$.

Let ϑ_1 in $\chi_0(T^2, Y_1)$ denote the intersection point between the lines $\{v = 0\}$ and $\{v - u = 1\}$.

We have the following types of geometric limits of $[\mathcal{A}_r, \Psi_r]$, as $r \to \infty$.

(a) A finite energy solution $(\mathcal{A}', \Psi')$ of the perturbed equations (11) on $V \times \mathbb{R}$. In radial gauge, this solution decays in the radial direction to a flat connection a'_∞ on T^2,

$$a'_\infty \in \chi_0(T^2, V).$$

In a temporal gauge on $V \times \mathbb{R}$, $(\mathcal{A}', \Psi')$ converges to elements $[A, \psi]$ and $[\tilde{A}, \tilde{\psi}]$ in $\mathcal{M}_V^*$ as $t \to \pm\infty$, with

$$\partial_\infty [A, \psi] = a'_\infty = \partial_\infty [\tilde{A}, \tilde{\psi}].$$

(b) Non-uniform limits on $V \times \mathbb{R}$ given by paths $[A(t), \psi(t)]$ in $\mathcal{M}_V^*$ connecting $[A, \psi]$ to $[A_1', \psi_1']$ and $[\tilde{A}, \tilde{\psi}]$ to $[A', \psi']$, and by a function a_V :

$D^+ \to \chi_0(T^2, V)$, holomorphic on some neighbourhood of the half disk D^+, which agrees on the subset $\theta \in \{-\pi/2, \pi/2\}$ of the boundary of D^+ with the asymptotic values $\partial_\infty[A(t), \psi(t)] =: a(t)$ of the paths in $\mathcal{M}^*_V$.

(c) A flat connection a''_∞ on T^2, obtained as a finite energy solution of the equation

$$F^+_{\mathcal{A}} = f'(T_{\mathcal{A}})\mu^+,$$

on the product region $\mathcal{V} = \nu(K) \times \mathbb{R}$ with an infinite end $T^2 \times [0, \infty) \times \mathbb{R}$.

(d) A flat connection $\tilde{a}''_\infty$ on T^2, obtained as a finite energy solution of the unperturbed equations (18) on the region

$$\hat{W}_1(\nu(K)) \backslash \mathcal{V}$$

with an infinite end $T^2 \times [0, \infty) \times \mathbb{R}$.

(e) Non-uniform limits on the ends of $W_1(\nu(K))$ given by a path $\tilde{a}''(\tau)$ in

$$\mathcal{M}^{red}_{\nu(K)} = \{u - v = 1\},$$

for $\tau \in [-1, 1]$, with $\tilde{a}''(-1) = a^-$, $\tilde{a}''(0) = \tilde{a}''_\infty$, and $\tilde{a}''(1) = \vartheta_1$, and a path $a''(\tau)$ in the perturbed

$$\mathcal{M}^{red}_{\nu(K),\mu} = \{v = f'(u)\},$$

for $\tau \in [-1, 1]$, with $a''(-1) = \vartheta_1$, $a''(0) = a''_\infty$ and $a''(1) = a^+$.

(f) We also have a map $a_{\nu(K)} : D^+_\epsilon \to H^1(T^2, \mathbb{R})$, holomorphic on some neighbourhood of the domain

$$D^+_\epsilon = \{\rho \in [\log \epsilon, 0], \theta \in [-\pi/2, \pi/2]\}.$$

Upon identifying $\tau = e^\rho \sin \theta$, this map agrees with the path $\tilde{a}''(\tau)$, for $\tau \in [-1, -\epsilon]$ and with $a''(\tau)$ for $\tau \in [\epsilon, 1]$, on the subset

$$\theta \in \{-\pi/2, \pi/2\} \quad \text{and} \quad \rho \in [\log \epsilon, 0]$$

of the boundary of D^+_ϵ.

(g) A "thin holomorphic triangle" $\Delta(a''_\infty, \tilde{a}''_\infty, \vartheta_1)$ in the character variety $\chi_0(T^2, V)$ (cf. [2] pg.234), with vertices $\{a''_\infty, \tilde{a}''_\infty, \vartheta_1\}$ and with two sides along the lines $\{v - u = 1\}$ and $\{v = f'(u)\}$, with the parameterization $\tilde{a}''(\tau)$ for $\tau \in [0, 1]$ and $a''(\tau)$ for $\tau \in [-1, 0]$.

(h) A limit on compact sets in the region $T^2 \times [-r, r] \times \mathbb{R}$, given by a flat connection a_∞ on T^2, and a non-uniform limit after rescaling, given by a map $\hat{a} : D \to H^1(T^2, i\mathbb{R})$, holomorphic up to the boundary, matching the values of a_V and $a_{\nu(K)}$, as described in [17].

Proof. The result follows from the analysis of the convergence of flow lines in [17]. The limits (a), (b), (f) and (h) are derived exactly as the analogous cases in [17]. The case (c) describes the limit of the solutions $(\mathcal{A}_r, \Psi_r)$, uniformly on compact sets, in the product region $\mathcal{V}(r)$ as $r \to \infty$. Up to gauge transformations, and up to passing to a subsequence, the solutions $(\mathcal{A}_r, \Psi_r)$ converge smoothly on compact sets in $\mathcal{V}(r)$ to a finite energy solution of the perturbed abelian ASD equation

$$F_{\mathcal{A}}^{+} = f'(T_{\mathcal{A}})\mu^{+}$$

on the strip $\mathcal{V}$ with an infinite end $T^2 \times [0, \infty) \times \mathbb{R}$. By the analysis of [17], this is (up to gauge) a constant flat connection a''_∞ on T^2, with holonomies satisfying $v = f'(u)$. Similarly, the case (d) describes the uniform convergence in $\tilde{\mathcal{V}}(r)$. The solutions $(\mathcal{A}_r, \Psi_r)$ converge smoothly on compact sets in $\tilde{\mathcal{V}}(r)$ to a finite energy solution of the unperturbed abelian ASD equation on the region $\tilde{\mathcal{V}}$ with an infinite end $T^2 \times [0, \infty) \times \mathbb{R}$. Again, by the analysis of [17] we see that this is up to gauge a constant flat connection $\tilde{a}''_\infty$ on T^2 contained in the line $\{v - u = 1\}$ in $\chi_0(T^2, Y_1)$. Case (e) describes the non-uniform limits in the regions $\tilde{\mathcal{V}}(r)$ and $\mathcal{V}(r)$, away from compact sets, after suitable rescaling as described in [17]. We have $\tilde{\mathcal{V}} \cong \tilde{\nu} \times \mathbb{R}$ and $\mathcal{V} \cong \nu \times \mathbb{R}$, connecting the two solid tori ν and $\tilde{\nu}$ in the standard Heegaard splitting of S^3 to the solid tori $\nu(K)$ in Y_1 and Y, respectively. Thus, we can adapt the analysis used in [17] for non-uniform limits on $\nu(K) \times \mathbb{R}$. After a suitable rescaling, we resulting non-uniform limits in the region $\tilde{\mathcal{V}}$ consist of a path $\tilde{a}''(\tau)$ along the line $\{u - v = 1\}$ in $\chi_0(T^2, V)$ and a map $\tilde{a}_\nu : D^+ \to H^1(T^2, i\mathbb{R})$, holomorphic in a neighbourhood of D^+, which agrees with $\tilde{a}''$ along the subset

$$\theta \in \{-\pi/2, \pi/2\} \quad \text{and} \quad \rho \in (-\infty, 0]$$

of the boundary of D^+. Similarly, the non-uniform limits on $\mathcal{V}$, after rescaling, consist of a path $a''(\tau)$ along the curve $v = f'(u)$ in $\chi_0(T^2, V)$ and a map $a_\nu : D^+ \to H^1(T^2, i\mathbb{R})$, holomorphic in a neighbourhood of D^+, which agrees with $a''(\tau)$ along the subset

$$\theta \in \{-\pi/2, \pi/2\} \quad \text{and} \quad \rho \in (-\infty, 0]$$

of the boundary of D^+.

The thin holomorphic triangle of Case (g) is obtained by the overlap of these two regions, with the vertex ϑ_1 at the flat connection on $T^2 = \partial\nu = \partial\tilde{\nu}$

which extends to both sides of the standard Heegaard splitting of S^3 to give the unique reducible solution on S^3. That is, we have $\theta_{S^3}|_{T^2} = \vartheta_1$. Notice that, in general, the flat connections a'_∞, a''_∞, $\tilde{a}''_\infty$, a_∞, a^+, and a^- on T^2 are all distinct.

$\diamond$

3.1 The holomorphic triangles

This subsection contains some observations on the holomorphic triangles that appear among the geometric limits of solutions on the cobordisms, as discussed in Proposition 3.1. A better understanding of these triangles will be very useful in analyzing the different properties of the coefficients of the chain maps defined by the cobordisms W_1, W_0, and W.

In the following, let ϵ be the parameter used in the construction of the surgery perturbation.

Definition 3.2 *Consider the unique holomorphic triangle Δ^ϵ with vertices $\{a^-, \vartheta_1, a^+\}$ and sides along the curves $\ell_1 = \{v - u = 1\}$, $\ell_\mu = \{v = f'(u)\}$ and $\ell = \partial_\infty(\mathcal{M}^*_V)$, defined by the geometric limits a_V, $a_{\nu(K)}$, $\hat{a}$, and $\Delta(\tilde{a}''_\infty, \vartheta_1, a''_\infty)$ of Proposition 3.1. We say that the triangle Δ^ϵ is* degenerate *if the holomorphic map obtained as a limit of the triangles Δ^ϵ, as $\epsilon \to 0$ is a disk Δ with boundary along arcs of ℓ_1 and ℓ connecting the vertices a^- and a^+. We say that Δ^ϵ is* non-degenerate *if the holomorphic map obtained as a limit is a triangle Δ with boundary along arcs of ℓ_1, $\cup_k \ell_k$, and ℓ, with $\cup_k \ell_k = \{u = 2k, k \in \mathbb{Z}\}$.*

Lemma 3.3 *Suppose given a_1 and $\tilde{a}_1$ in $\mathcal{M}_{Y_1}$, $a \in \mathcal{M}_{Y,\mu} \backslash j(\mathcal{M}_{Y_1})$, and $a_0 \in \mathcal{M}_{Y_0}(\mathfrak{s}_k)$. Then the holomorphic triangles that appear in the geometric limits of solutions in the zero-dimensional moduli spaces $\mathcal{M}^{W_1}_\ell(a_1, j(\tilde{a}_1))$ and $\mathcal{M}^{W_0}_k(a, a_0)$ are all degenerate in the limit $\epsilon \to 0$.*

Proof. The triangle Δ^ϵ has two sides along arcs of the lines $\{v - u = 1\}$ and $\{v = f'(u)\}$ connecting ϑ_1 and a^- and ϑ_1 and a^+, respectively, with the parameterization of Case (e) of Proposition 3.1. If for $\epsilon \to 0$ the points $\{\vartheta_1, a^-, a^+\}$ all lie on the same line $\{v - u = 1\}$, then the holomorphic map in the limit is a disk with one side along the arc in $\{v - u = 1\}$ connecting a^- and a^+ and the other side along an arc in $\partial_\infty(\mathcal{M}^*_V)$ connecting these same two points. Thus, the limit triangle is degenerate.

$\diamond$

Now observe, instead, that when we consider solutions on the cobordism that intertwine the generators of the Floer complex for Y_1 with those for Y_0, we may have holomorphic triangles that do not degenerate in the limit when $\epsilon \to 0$.

Lemma 3.4 *Suppose given, as before, critical points a_1 and $\tilde{a}_1$ in $\mathcal{M}^*_{Y_1}$, $a_0(\epsilon) \in \mathcal{M}_{Y,\mu} \backslash j(\mathcal{M}_{Y_1})$, and $a_0 \in \mathcal{M}_{Y_0}(\mathfrak{s}_k)$. Consider zero-dimensional moduli spaces of the form $\mathcal{M}^{W_1}_\ell(a_1, a_0(\epsilon))$ and $\mathcal{M}^{W_0}_k(\tilde{a}_1(\epsilon), a_0)$, with $\tilde{a}_1(\epsilon) = j(\tilde{a}_1)$. Then, in general, the limit holomorphic map will still be a triangle.*

Proof.

Suppose given a_1 in $\mathcal{M}_{Y_1}$ and $a_0(\epsilon) \in \mathcal{M}_{Y,\mu} \backslash j(\mathcal{M}_{Y_1})$, and consider the geometric limits of solutions in a zero dimensional moduli space $\mathcal{M}^{W_1}_\ell(a_1, a)$. We still may have holomorphic triangles that degenerate in the limit $\epsilon \to 0$. By the open mapping theorem, this happens whenever two sides of the boundary of the holomorphic triangle are mapped together as $\epsilon \to 0$. However, since the pair of points a_1 and $a_0 = \lim_\epsilon a_0(\epsilon)$, or $\tilde{a}_1$ and a_0, are now on two different lines in $H^1(T^2, \mathbb{R})$, we can also have non-degenerate holomorphic triangles.

$\diamond$

The existence of these non-degenerate holomorphic triangles, that are not "thin" for small ϵ, characterizes the difference between the chain maps induced by the cobordisms and the homomorphisms of abelian groups defined by inclusion and projection, as described in Part I [6], as we shall see when discussing injectivity and surjectivity of the maps in the exact sequence.

4 The chain homomorphisms

In this section we introduce the chain homomorphisms w^1_* and w^0_*.

First we observe that we have an analogue of the formula

$$N_X(-\mathfrak{s}) = (-1)^{(1+b_2^+(X)-b_1(X))/2} N_X(\mathfrak{s}) \tag{24}$$

that holds for compact 4-manifolds. The version we need is given by the following statement.

Lemma 4.1 *On the manifold W_1 we have*

$$N_\ell^{W_1}(a_1, a) = (-1)^{2\,Ind_{\mathbb{C}}(D_{\mathcal{A}})} N_{-\ell}^{W_1}(a_1, a).$$

Thus, if $\mathfrak{s} \in \mathcal{S}(W_1)$ satisfies

$$\iota(\mathfrak{s}, W_1, a_1, a) = 0,$$

for given $a_1 \in \mathcal{M}^*_{Y_1}$ and $a \in \mathcal{M}^*_{Y,\mu}$, then $-\mathfrak{s}$ is the unique other Spin^c *structure in $\mathcal{S}(W_1)$ which satisfies $\iota(-\mathfrak{s}, W_1, a_1, a) = 0$. The corresponding invariant satisfies*

$$N_{-\mathfrak{s}}^{W_1}(a_1, a) = N_{\mathfrak{s}}^{W_1}(a_1, a).$$

Proof. The proof for the closed manifold case [20] adapts to this context, with fixed asymptotic values, and compatible choice of admissible sections. In fact, the sign is given by the change of orientation, and the orientation is compatible with the boundary strata.

$\diamond$

Notice that the same argument does not extend to the case of the manifold W_0. In fact, in the case of W_1 changing $\mathfrak{s}$ to $-\mathfrak{s}$ does not change the Spin^c structure on the boundary $Y \cup Y_1$, whereas, in the case of W_0, changing $\mathfrak{s}$ to $-\mathfrak{s}$ amounts to changing the Spin^c structure on Y_0. On the other hand, on W_0 or W_2 we simply do not have solutions in $\mathcal{M}_k^{W_0}$ and in $\mathcal{M}_{-k}^{W_0}$ with the same asymptotic values.

Let $a_1 = [A_1, \psi_1]$ be a class in $\mathcal{M}^*_{Y_1}$, and $a = [A, \psi]$ a class in $\mathcal{M}^*_{Y,\mu}$, such that the gradings, assigned according to [6], coincide

$$\deg_{Y_1}(a_1) = \deg_{Y,\mu}(a).$$

Then we consider the zero dimensional moduli spaces $\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1, a)$ with $\mathfrak{s}_\ell$, $\ell \geq 0$, satisfying $\iota(\mathfrak{s}_\ell, W_1, a_1, a) = 0$. Similarly, for $a_0 = [A_0, \psi_0]$ in some $\mathcal{M}_{Y_0}(\mathfrak{s}_k)$ with compatible grading

$$\deg_{Y,\mu}(a) = \deg_{Y_0,\mathfrak{s}_k}(a_0),$$

we consider the zero dimensional components

$$\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a, a_0) \quad \text{with} \quad \iota(\mathfrak{s}_k, W_0, a, a_0) = 0.$$

Recall that we have invariants $N_\ell^{W_1}(a_1, a)$ and $N_k^{W_0}(a, a_0)$ defined by counting points with the orientation in $\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1, a)$ and $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a, a_0)$, respectively.

Definition 4.2 *We define the map* $w_*^1 : C_*(Y_1) \to C_*(Y,\mu)$ *with matrix elements*

$$\langle a, w_*^1(a_1)\rangle = N_\ell^{W_1}(a_1, a),$$

with ℓ *is the unique non-negative* Spin^c *structure satisfying* $\iota(\mathfrak{s}_\ell, W_1, a_1, a) = 0$.

We define the map $w_*^0 : C_*(Y,\mu) \to \oplus_k C_{(*)}(Y_0, \mathfrak{s}_k)$ *with the matrix coefficients*

$$\langle a_0, w_*^0(a)\rangle = N_k^{W_0}(a, a_0).$$

Thus, we choose to define the map w_*^1 using only the "positive" Spin^c structures, $c_1(\det(\mathfrak{s})) = (2\ell+1)e_1$, with $\ell \geq 0$.

Lemma 4.3 *The maps* w_*^i *are chain homomorphisms.*

Proof. Suppose given a Spin^c-structure $\mathfrak{s}_\ell \in \mathcal{S}_1$, with $\ell \geq 0$, satisfying

$$\iota(\mathfrak{s}_\ell, W_1, a_1, b) = 1.$$

We have a compactification of the moduli space $\mathcal{M}_{\mathfrak{s}_\ell}^{W_1}(a_1, b)$ by boundary strata of the form

$$\bigcup_{a \in \mathcal{M}^*_{Y,\mu}} \mathcal{M}_{\mathfrak{s}_\ell}^{W_1}(a_1, a) \times \hat{\mathcal{M}}_Y(a, b)$$

and

$$\bigcup_{b_1 \in \mathcal{M}^*_{Y_1}} \hat{\mathcal{M}}_{Y_1}(a_1, b_1) \times \mathcal{M}_{\mathfrak{s}_\ell}^{W_1}(b_1, b),$$

as in Proposition 2.7 and Corollary 2.8. Here a and b_1 satisfy

$$\deg_{Y,\mu}(a) - \deg_{Y,\mu}(b) = 1,$$

$$\deg_{Y_1}(a_1) - \deg_{Y_1}(b_1) = 1.$$

The moduli spaces $\hat{\mathcal{M}}_{Y_1}(a_1, b_1)$ and $\hat{\mathcal{M}}_Y(a, b)$ are gauge classes of flow lines on Y_1 and Y respectively, modulo the action of $\mathbb{R}$ by translation. The counting of boundary points of the 1-dimensional $\mathcal{M}_{\mathfrak{s}_\ell}^{W_1}(a_1, b)$, with the orientation, gives the relation

$$\sum_a N_{\mathfrak{s}_\ell}^{W_1}(a_1, a) n_Y(a, b) - \sum_{b_1} n_{Y_1}(a_1, b_1) N_{\mathfrak{s}_\ell}^{W_1}(b_1, b) = 0.$$

Notice that $\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(a_1, a)$ and $\mathcal{M}^{W_1}_{\mathfrak{s}_\ell}(b_1, b)$ are zero-dimensional, thus we have

$$\iota(\mathfrak{s}_\ell, W_1, a_1, a) = \iota(\mathfrak{s}_\ell, W_1, b_1, b) = 0.$$

Thus, counting points in these moduli spaces gives exactly the counting of the matrix elements of w^1_*. This proves the relation

$$w^1_* \circ \partial_{Y_1} = \partial_{Y,\mu} \circ w^1_*.$$

The result for W_0 is analogous, except for the fact that some care is needed in the case of the trivial Spin^c-structure $\mathfrak{s}_0$. Let us first consider the case of non-trivial $\mathfrak{s}_k$ first. Suppose given a 1-dimensional $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a, b_0)$. In particular, we have

$$\deg_{Y,\mu}(a) = \deg_{Y_0}(b_0) + 1,$$

where $\deg_{Y_0}$ is the $\mathbb{Z}$-lift of the $\mathbb{Z}_{2k}$-relative grading on $\mathcal{M}_{Y_0}(\mathfrak{s}_k)$, induced by the grading on $\mathcal{M}_{Y,\mu}$, cf.[6], [18].

The boundary strata in the compactification of $\mathcal{M}^{W_0}_{\mathfrak{s}_k}(a, b_0)$ are given by

$$\begin{array}{l} \bigcup_{a_0 \in \mathcal{M}_{Y_0}(\mathfrak{s}_k)} \mathcal{M}^{W_0}_{\mathfrak{s}_k}(a, a_0) \times \hat{\mathcal{M}}^{(0)}_{Y_0,\mathfrak{s}_k}(a_0, b_0) \\ \bigcup_{b \in \mathcal{M}^*_{Y,\mu}} \hat{\mathcal{M}}_{Y,\mu}(a, b) \times \mathcal{M}^{W_0}_{\mathfrak{s}_k}(b, b_0). \end{array} \tag{25}$$

Here b and a_0 satisfy

$$\deg_{Y,\mu}(a) - \deg_{Y,\mu}(b) = 1,$$

$$\deg_{Y_0}(a_0) - \deg_{Y_0}(b_0) = 1, \tag{26}$$

in the integer grading. Notice that, as in [18], (26) can be derived by observing that, in the geometric limits for $r \to \infty$, the flowlines that contribute to the compactification define a contractible path $[A''(t), 0]$ in $\mathcal{M}_{\nu(K)} \subset \chi_0(T^2, \nu(K))$. Thus, with the notation of [18], we pick only the minimal energy component $\mathcal{M}^{(0)}_{Y_0,\mathfrak{s}_k}(a_0, b_0)$ in the moduli space $\mathcal{M}_{Y_0,\mathfrak{s}_k}(a_0, b_0)$. This component of the moduli space is exactly the one which defines the boundary operator on the $\mathbb{Z}$-lift $C_{(*)}(Y_0, \mathfrak{s}_k)$ of the $\mathbb{Z}_{2k}$ graded complex $C_*(Y_0, \mathfrak{s}_k)$, as analyzed in [18].

The argument used here, based on the geometric limits of solutions, can be generalized to the case of the trivial Spin^c-structure $\mathfrak{s}_0$. Again, suppose given $\mathcal{M}^{W_0}_{\mathfrak{s}_0}(a, b_0)$, with

$$\iota(\mathfrak{s}_0, W_0, a, b_0) = 1.$$

In this case we want to show that the strata

$$\mathcal{M}^{W_0}_{\mathfrak{s}_0}(a, a_0) \times \hat{\mathcal{M}}_{Y_0,\mathfrak{s}_0}(a_0, b_0)$$

which occur in the compactification, with

$$\deg_{Y_0,\mathfrak{s}_0}(a_0) - \deg_{Y_0,\mathfrak{s}_0}(b_0) = 1,$$

only contain flow lines in the component of minimal energy of $\mathcal{M}_{Y_0,\mathfrak{s}_0}(a_0, b_0)$. Again, the result follows from the fact that the path $[A''(t), 0]$ in the geometric limits is contractible, for any flow-line in $\mathcal{M}_{Y_0,\mathfrak{s}_0}(a_0, b_0)$ that arises in the compactification of $\mathcal{M}^{W_0}_{\mathfrak{s}_0}(a, b_0)$. This precisely identifies the right component of the boundary operator in the Floer complex for $(Y_0, \mathfrak{s}_0)$.

Thus, (25) implies that the map

$$w^0_* : CF_*(Y, \mu) \to \oplus_k CF_{(*)}(Y_0, \mathfrak{s}_k)$$

is a chain homomorphism precisely when the complex $CF_{(*)}(Y_0, \mathfrak{s}_k)$ is endowed with the $\mathbb{Z}$-grading described in [18] and, in the case of $\mathfrak{s}_0$, the Floer homology of Y_0 is defined as

$$HF_{(*)}(Y_0, \mathfrak{s}_0, \mathbb{Z}[[t]])|_{t=0}$$

on $(Y_0, \mathfrak{s}_0)$, as described in [18].
$\diamond$

4.1 The composite map

Let $N^W_{\ell,k}(a_1, a_0)$ denote the invariant obtained by counting solutions in moduli space $\mathcal{M}^W_{\ell,k}(a_1, a_0)$, for the unique choice of the Spin^c structure $\mathfrak{s}_{\ell,k}$ such that $\mathcal{M}^W_{\ell,k}(a_1, a_0)$ is zero dimensional.

Lemma 4.4 *Suppose given a_1 in $\mathcal{M}^*_{Y_1}$ and a_0 in $\mathcal{M}_{Y_0}(\mathfrak{s}_k)$. Let $\ell \geq 0$ be the unique non-negative* Spin^c*-structure satisfying*

$$\iota(\mathfrak{s}_{\ell,k}, W, a_1, a_0) = 0.$$

Then the composite map $w^0_* \circ w^1_*$ is given by

$$\langle w^0_* \circ w^1_*(a_1), a_0 \rangle = N^W_{\ell,k}(a_1, a_0).$$

Proof.

The composite map $w^0_* \circ w^1_*$ has matrix elements

$$\sum_{a\in\Theta} N_\ell^{W_1}(a_1,a) N_k^{W_0}(a,a_0),$$

where the sum is over the set

$$\begin{aligned}\Theta = \quad & \{a \in \mathcal{M}^*_{Y,\mu} | \iota(\mathfrak{s}_\ell, W_1, a_1, a) = 0, \iota(\mathfrak{s}_k, W_0, a, a_0) = 0, \\ & \deg_{Y_1}(a_1) = \deg_{Y,\mu}(a) = \deg_{Y_0}(a_0)\}.\end{aligned}$$

On the other hand, we have

$$N^W_{\ell,k}(a_1,a_0) = \sum_{a\in\Theta_1} N_\ell^{W_1}(a_1,a) N_k^{W_0}(a,a_0),$$

where now the sum is over the set

$$\Theta_1 = \{a \in \mathcal{M}^*_{Y,\mu} | \iota(\mathfrak{s}_\ell, W_1, a_1, a) = 0, \iota(\mathfrak{s}_k, W_0, a, a_0) = 0\}.$$

Notice that, given any $\mathfrak{s}_{\ell,k}$ on W which satisfies

$$\iota(\mathfrak{s}_{\ell,k}, W, a_1, a_0) = 0,$$

and such that we have

$$\mathcal{M}^W_{\mathfrak{s}_{\ell,k}}(a_1,a_0) \neq \emptyset,$$

there exists some $a \in \mathcal{M}^*_{Y,\mu}$ such that we have

$$\iota(\mathfrak{s}_\ell, W_1, a_1, a) = 0 \quad \text{and} \quad \iota(\mathfrak{s}_k, W_0, a, a_0) = 0.$$

This follows by stretching the cylinder $Y \times [-T,T]$ in the composite cobordism: the condition

$$\mathcal{M}^W_{\mathfrak{s}_{\ell,k}}(a_1,a_0) \neq \emptyset$$

ensures the existence of a limiting translation invariant solution on $Y \times \mathbb{R}$. The argument is similar to the one used in [15].

Thus, we only need to prove that we have

$$\sum_{a\in\Theta_1\backslash\Theta} N_\ell^{W_1}(a_1,a) N_k^{W_0}(a,a_0) = 0.$$

Suppose not. Then we have an element $a \in \Theta_1\backslash\Theta$ such that the moduli spaces

$$\mathcal{M}_\ell^{W_1}(a_1,a) \quad \text{and} \quad \mathcal{M}_k^{W_0}(a,a_0)$$

are zero-dimensional and non-empty. The point a will be in $j(\mathcal{M}^*_{Y_1})$, or in $\mathcal{M}^*_{Y,\mu}\backslash j(\mathcal{M}^*_{Y_1})$. We consider the first case. The proof in the second case is completely analogous. We have $a = j(\tilde{a}_1)$ and $\deg_{Y_1}(a_1) \neq \deg_{Y_1}(\tilde{a}_1)$. Consider the geometric limits of the solutions in $\mathcal{M}^{W_1}_{\ell}(a_1, j(\tilde{a}_1))$. As we discuss in greater detail in Lemma 5.1 and Lemma 5.3, as we let $\epsilon \to 0$, the geometric limits of solutions in $\mathcal{M}^{W_1}_{\ell}(a_1, j(\tilde{a}_1))$ define geometric limits of solutions in $\mathcal{M}_{Y_1}(a_1, \tilde{a}_1)$. In particular, if the moduli space $\mathcal{M}^{W_1}_{\ell}(a_1, j(\tilde{a}_1))$ is non-empty and zero-dimensional, also the moduli space $\mathcal{M}_{Y_1}(a_1, \tilde{a}_1)$ will be non-empty and zero-dimensional, which contradicts the assumption that $\deg_{Y_1}(a_1) \neq \deg_{Y_1}(\tilde{a}_1)$.

This implies that we have an identification of the zero-dimensional components

$$\mathcal{M}^{W}_{\mathfrak{s}_{\ell,k}}(a_1, a_0) \cong \cup_{a\in\Theta} \mathcal{M}^{W_1}_{\mathfrak{s}_{\ell}}(a_1, a) \times \mathcal{M}^{W_0}_{\mathfrak{s}_k}(a, a_0), \tag{27}$$

for large $T \geq T_0$ in the cylinder $Y \times [-(T-T_0), (T-T_0)]$ in the composite cobordism W. This follows from the gluing theorem of [19], cf. [5] [18], together with the previous argument.
$\diamond$

5 Injectivity and Surjectivity

In the previous section we have constructed a sequence

$$0 \to C_*(Y_1) \stackrel{w^1_*}{\to} C_*(Y,\mu) \stackrel{w^0_*}{\to} \oplus_k C_{(*)}(Y_0, \mathfrak{s}_k) \to 0.$$

We now proceed to show exactness in the first and last place, namely injectivity of w^1_* and surjectivity of w^0_*.

We want to give a better description of the components

$$\langle j(a_1'), w^1_*(a_1)\rangle,$$

with a_1 and a_1' in $\mathcal{M}^*_{Y_1}$, and

$$\langle a_0, w^0_*(a)\rangle,$$

$a \in \mathcal{M}^*_{Y,\mu}\backslash j(\mathcal{M}^*_{Y_1})$, and $a_0 \in \cup_k \mathcal{M}_{Y_0}(\mathfrak{s}_k)$. For simplicity, let us introduce the following notation. Let $\{a^{(1)}_i\}_{i=1,\dots m}$ be the elements in $\mathcal{M}^*_{Y_1}$ and let $\{a^{(0)}_j\}_{i=m+1,\dots n}$ be the elements in $\cup_{\mathfrak{s}_{i_k}} \mathcal{M}_{Y_0}(\mathfrak{s}_{i_k})$. Then the n elements in

$\mathcal{M}^*_{Y,\mu}$ can be identified with the union of these two sets of points. More precisely, if $\epsilon > 0$ is the parameter used in the definition of the function f' in the construction of the surgery perturbation, then the moduli space $\mathcal{M}^*_{Y,\mu}$ can be identified with a collection of points

$$\mathcal{M}^*_{Y,\mu} = \{a_i^{(1)}(\epsilon)\}_{i=1,\dots m} \cup \{a_j^{(0)}(\epsilon)\}_{j=m+1,\dots n}.$$

For ϵ small enough, there is a bijection

$$\mathcal{M}^*_{Y,\mu} \cong \{a_i^{(1)}\}_{i=1,\dots m} \cup \{a_j^{(0)}\}_{j=m+1,\dots n}, \tag{28}$$

which is compatible with the grading [6].

In the following we shall use the notation (28) which identifies the elements in $\mathcal{M}_{Y,\mu}$ with elements in the other two moduli spaces.Whenever it is crucial to distinguish between these moduli spaces, we shall use the notation $a_1 \in \mathcal{M}^*_{Y_1}$, $a_0 \in \mathcal{M}_{Y_0}(\mathfrak{s}_k)$, and $a \in \mathcal{M}^*_{Y,\mu}$, with $a = j(a_1)$ or $\pi(a) = a_0$. We hope this will not cause any confusion.

We want to describe solutions in the moduli spaces

$$\mathcal{M}^{W_1}_\ell(a_i^{(1)}, a_q^{(1)})$$

and

$$\mathcal{M}^{W_0}_k(a_j^{(0)}, a_p^{(0)}),$$

by describing these moduli spaces as a gluing of solutions on the trivial cobordism $V \times \mathbb{R}$ and solutions on the regions $\hat{W}_i(\nu(K))$. The techniques involved in the splitting and gluing of solutions on the cobordisms are analogous to the ones developed in [17] to analyze the splitting and gluing of flow lines on the trivial cobordism.

We are assuming here that the asymptotic values also satisfy the condition

$$\begin{array}{c} \deg_{Y_1}(a_i^{(1)}) = \deg_{Y,\mu}(a_q^{(1)}), \\ \deg_{Y,\mu}(a_j^{(0)}) = \deg_{Y_0,\mathfrak{s}_k}(a_p^{(0)}). \end{array} \tag{29}$$

In particular, we are going to prove the fundamental relations

$$\langle a_i^{(1)}, w^1_*(a_q^{(1)}\rangle = \delta_{iq}$$

for all $a_i^{(1)}$ and $a_q^{(1)}$ in $\mathcal{M}^*_{Y_1}$ and

$$\langle a_j^{(0)}, w^0_*(a_p^{(0)})\rangle = \delta_{jp}$$

for all $a_j^{(0)}$ and $a_p^{(0)}$ in $\cup_k \mathcal{M}_{Y_0}(\mathfrak{s}_k)$.

In the following we shall discuss the case of the manifold W_1. The case of W_0 is analogous. Consider the manifold

$$\hat{W}_1(r) = (V_r \times \mathbb{R}) \cup \mathcal{V}(r) \cup \hat{\mathcal{V}}(r),$$

as in (19).

We use the description of the geometric limits of solutions on W_1 given in Proposition 3.1, together with the analysis of [17] of the geometric limits of flow lines, in order to analyst solutions in $\mathcal{M}_\ell^{W_1}(a_i^{(1)}, a_i^{(1)})$ and in $\mathcal{M}_k^{W_0}(a_j^{(0)}, a_j^{(0)})$.

Suppose given a_1, an element of $\mathcal{M}^*_{Y_1}$. For large $r \geq r_0$ we represent a_1 as in [6],

$$a_1(r) = [(\tilde{A}', \tilde{\psi}')\#^r_{\tilde{a}''_\infty}(\tilde{a}''_\infty, 0)].$$

Lemma 5.1 *There is a unique solution $[\mathcal{A}_r, \Psi_r]$ in the zero-dimensional moduli space*

$$\mathcal{M}_\ell^{W_1(r)}(a_1(r), j(a_1(r))),$$

for large $r \geq r_0$ and for small enough $\epsilon > 0$, where ϵ is the parameter used in the definition of the surgery perturbation.

Proof. We write $a_1(r) = [A(r), \psi(r)]$, with

$$[A(r), \psi(r)] = [(\tilde{A}', \tilde{\psi}')\#^r_{\tilde{a}''_\infty}(\tilde{a}''_\infty, 0)]$$

on $Y_1(r)$, for large $r \geq r_0$, and

$$j(a_1(r)) = j[A(r), \psi(r)] = [(A', \psi')\#^r_{a''_\infty}(a''_\infty, 0)].$$

In the limit $\epsilon \to 0$, the asymptotic values a''_∞ and $\tilde{a}''_\infty$ coincide, and the elements $[A', \psi']$ and $[\tilde{A}', \tilde{\psi}']$ in $\mathcal{M}^*_V$ also coincide, because of the result of Lemma 3.3, which shows that the holomorphic triangle degenerates. Thus, when $\epsilon \to 0$ there is a unique finite energy solution on $V \times \mathbb{R}$ with the same asymptotic value $[A', \psi']$ at $t \to \pm\infty$, given by the constant flow $[A', \psi']$.

In fact, if we had a non-constant finite energy solution $(\mathcal{A}', \Psi')$ on $V \times \mathbb{R}$ with the same limits at $t \to \pm\infty$, then, using the gluing theorem of [17], we could glue this $(\mathcal{A}', \Psi')$ along the flat connection a''_∞ to a solution on $Y_1 \times \mathbb{R}$, extending it as the reducible solution $(a''_\infty, 0)$ on $\nu(K) \times \mathbb{R}$. This solution would be a non-trivial flow line connecting the critical point a_1 to itself in the configuration space on Y_1, but the moduli space $\hat{\mathcal{M}}_{Y_1}(a_1, a_1)$ is generically empty for dimensional reasons.

Thus, in the limit $\epsilon \to 0$, there is a unique solution $[\mathcal{A}_r, \Psi_r]$ on $W_1(r)$ with the same asymptotic value at the two ends, obtained by gluing along the asymptotic value a''_∞ the constant flow $[A', \psi']$ with the unique perturbed ASD equation on $W_1(\nu(K))$ which extends a_∞.

Since the moduli space

$$\mathcal{M}_\ell^{W_1(r)}(a_1(r), j(a_1(r)))$$

is discrete for all $0 < \epsilon \leq \epsilon_0$, we obtain that, for ϵ small enough, there is a unique solution.
$\diamond$

Corollary 5.2 *We have*

$$\langle a_i^{(1)}, w_*^1(a_i^{(1)}) \rangle = 1$$

for all $a_i^{(1)}$ *in* $\mathcal{M}^*_{Y_1}$, *and the analogous*

$$\langle a_j^{(0)}, w_*^0(a_j^{(0)}) \rangle = 1$$

for all $a_j^{(0)}$ *in* $\cup_k \mathcal{M}_{Y_0}(\mathfrak{s}_k)$.

We also have the following result.

Lemma 5.3 *Consider the moduli space* $\mathcal{M}_\ell^{W_1}(\tilde{a}_1, j(a_1))$, *for large enough* $r \geq r_0$ *and with two critical points* $a_1 \neq \tilde{a}_1$ *in* $\mathcal{M}^*_{Y_1}$. *Assume as before that* $0 < \epsilon \leq \epsilon_0$ *is the parameter used in the construction of the surgery perturbation. Then, for* ϵ *small enough, we have*

$$\mathcal{M}_\ell^{W_1}(\tilde{a}_1, j(a_1)) = \emptyset$$

if $\deg_{Y_1}(a_1) = \deg_{Y_1}(\tilde{a}_1)$.

Proof. Suppose given a solution $[\mathcal{A}_r, \Psi_r]$ in

$$\mathcal{M}_\ell^{W_1}(\tilde{a}_1, j(a_1)),$$

with $a_1 \neq \tilde{a}_1$ in $\mathcal{M}^*_{Y_1}$ and

$$\deg_{Y_1}(a_1) = \deg_{Y_1}(\tilde{a}_1).$$

According to the result of Proposition 3.1, we have geometric limits as $r \to \infty$. We show that we can construct from these geometric limits a non-trivial flow line in the configuration space over Y_1 which connects the two points a_1 and $\tilde{a}_1$ in $\mathcal{M}^*_{Y_1}$. This will contradict the assumption that

$$\deg_{Y_1}(a_1) = \deg_{Y_1}(\tilde{a}_1).$$

In fact, this assumption implies that the moduli space of flow lines is generically empty for dimensional reasons.

We can write the endpoints a_1 and $\tilde{a}_1$ as

$$j(a_1) = [(A', \psi')\#^r_{a_\infty}(a_\infty, 0)],$$

and

$$\tilde{a}_1 = [(\tilde{A}', \tilde{\psi}')\#^r_{\tilde{a}_\infty}(\tilde{a}_\infty, 0)].$$

Here a_∞ is a flat connection on T^2 satisfying the relation $v = f'(u)$, with f' depending on the choice of a small $\epsilon > 0$, and $\tilde{a}_\infty$ is a flat connection on T^2 satisfying the relation $v - u = 1$.

Suppose that we have $\mathcal{M}^{W_1}_\ell(\tilde{a}_1, j(a_1)) \neq \emptyset$. Then, in the limit $\epsilon \to 0$, the geometric limits on $\hat{W}_1(\nu(K))$ consist of the cases (c), (d), and (e) of Proposition 3.1, where the connections a''_∞ and $\tilde{a}''_\infty$ coincide, because of the result of Lemma 3.3 on the degenerate holomorphic triangles. Therefore, these geometric limits determine a preglued solution on $\nu(K) \times \mathbb{R}$ inside $Y_1 \times \mathbb{R}$ which can be glued to the geometric limits on $V \times \mathbb{R}$ to determine a flow line on $Y_1(r) \times \mathbb{R}$. Since we know that $\mathcal{M}_{Y_1(r)\times\mathbb{R}}(\tilde{a}_1, a_1) = \emptyset$ if $\deg_{Y_1}(a_1) = \deg_{Y_1}(\tilde{a}_1)$, we have shown that, for $\epsilon \to 0$, we also have

$$\mathcal{M}^{W_1}_\ell(\tilde{a}_1, j(a_1)) = \emptyset,$$

hence the relation

$$\langle w^1_*(\tilde{a}_1), j(a_1)\rangle = 0$$

holds for small enough ϵ, if $a_1 \neq \tilde{a}_1$ in $\mathcal{M}^*_{Y_1}$.
$\diamond$

We derive from Lemma 5.3 the second fundamental relation.

Corollary 5.4 *We have*

$$N_\ell^{W_1}(a_i^{(1)}, a_q^{(1)}) = 0$$

whenever $i \neq q$ and

$$\deg_{Y_1}(a_i^{(1)}) = \deg_{Y_1}(a_q^{(1)}).$$

Similarly, for W_0 we have

$$N_k^{W_0}(a_j^{(0)}, a_p^{(0)}) = 0$$

whenever $j \neq p$ and

$$\deg_{Y_0}(a_j^{(0)}) = \deg_{Y_0}(a_p^{(0)}).$$

Notice how the previous results do not give any information about the components of the maps w_*^1 and w_*^0 that interchange the critical points $a_i^{(1)}$ and $a_j^{(0)}$. In fact, in this case there are in general non-degenerate triangles, as $\epsilon \to 0$ in the surgery perturbation, cf. the result of Lemma 3.4. In other words, the presence of these non-degenerate holomorphic triangles measures the difference between the chain maps w_*^1 and w_*^0 and the group homomorphisms j and π defined by the inclusion and projection, in the identification of the generators on Y with generators on Y_1 and Y_0.

6 Exactness in the middle term

Recall that the results of [6] imply that the ranks of the Floer complexes in the sequence

$$0 \to C_*(Y_1) \to C_*(Y, \mu) \to \oplus_k C_{(*)}(Y_0, \mathfrak{s}_k) \to 0$$

are as prescribed for the existence of an exact sequence. Moreover, in the previous section we have proved injectivity of the first map and surjectivity of the last. Now we analyze the middle term.

The result of Lemma 4.4, together with Corollary 5.2 and Corollary 5.4, yields the following.

Lemma 6.1 *Suppose given $a \in \mathcal{M}_{Y,\mu} \backslash j(\mathcal{M}_{Y_1})$ and $a_1 \in \mathcal{M}_{Y_1}$ and let π be the identification $\pi : \mathcal{M}_{Y,\mu} \backslash j(\mathcal{M}_{Y_1}) \to \cup_k \mathcal{M}_{Y_0}(\mathfrak{s}_k)$. The coefficients of the composite map $w_*^0 \circ w_*^1$ satisfy the relation*

$$N_{\ell,k}^W(a_1, \pi(a)) = N_\ell^{W_1}(a_1, a) + N_k^{W_0}(j(a_1), \pi(a)). \tag{30}$$

The main purpose of this section is to show that the counting in (30) is zero. We then verify that this is sufficient to prove exactness in the middle term. Establishing the relation $w^0_* \circ w^1_* = 0$ depends again essentially on the analysis of the geometric limits of solutions on the cobordisms, following the technique of [17]. We need a preliminary discussion of the geometric limits on $V \times \mathbb{R}$ which completes the results of Part II, [17].

6.1 The moduli space on $V \times \mathbb{R}$

Consider elements $a_i^{(1)} \in \mathcal{M}^*_{Y_1}$ and $a_j^{(0)}(\epsilon) \in \mathcal{M}^*_{Y,\mu}$, which we can write as

$$a_i^{(1)} = [(A_i^-, \psi_i^-)\#(a^-_{\infty,i}, 0)]$$

$$a_j^{(0)}(\epsilon) = [(A_j^+(\epsilon), \psi_j^+(\epsilon))\#(a^+_{\infty,j}, 0)].$$

Assume that we have solutions $[\mathcal{A}_1(r), \Psi_1(r)]$ in $\mathcal{M}_\ell^{W_1(r)}(a_i^{(1)}, a_j^{(0)}(\epsilon))$, for all sufficiently large $r \geq r_0$. Then these solutions define geometric limits as in Proposition 3.1 (a) - (h).

In particular, we list here separately the limits on $V \times \mathbb{R}$. Our purpose now is to simplify and group together in a more efficient way the information on the geometric limits given in Proposition 3.1.

Remark 6.2 *A family of solutions* $[\mathcal{A}_1(r), \Psi_1(r)]$ *in* $\mathcal{M}_\ell^{W_1(r)}(a_i^{(1)}, a_j^{(0)}(\epsilon))$ *defines the following limits on* $V \times \mathbb{R}$*:*
(a). A finite energy solution $[\mathcal{A}', \Psi']^\epsilon$ *of the perturbed equations (11) on* $V \times \mathbb{R}$*, with a radial limit* $a_\infty(\epsilon)$ *in* $\partial_\infty(\mathcal{M}^*_V) \subset \chi_0(T^2, V)$*, and with temporal limits* $[A, \psi]^\epsilon_1$ *and* $[\tilde{A}, \tilde{\psi}]^\epsilon_1$ *in*

$$\partial_\infty^{-1}(a_\infty(\epsilon)) \subset \mathcal{M}^*_V.$$

(b). Two paths $[A(t), \psi(t)]^\epsilon_1$ *in* $\mathcal{M}^*_V$*, for* $t \in [-1, 0)$ *and* $t \in (0, 1]$*, with*

$$[A(-1), \psi(-1)]^\epsilon_1 = [A_i^-, \psi_i^-]$$

$$\lim_{t \to 0-} [A(t), \psi(t)]^\epsilon_1 = [A, \psi]^\epsilon_1$$

$$\lim_{t \to 0+} [A(t), \psi(t)]^\epsilon_1 = [\tilde{A}, \tilde{\psi}]^\epsilon_1$$

$$[A(1), \psi(1)]^\epsilon_1 = [A_j^+(\epsilon), \psi_j^+(\epsilon)].$$

These paths induce a continuous, piecewise smooth path

$$a_1^\epsilon(t) \subset \partial_\infty(\mathcal{M}_V^*) \subset \chi_0(T^2, V)$$

satisfying

$$a_1^\epsilon(t) = \partial_\infty[A(t), \psi(t)]_1^\epsilon,$$

with

$$a_1^\epsilon(-1) = a_{\infty,i}^- \quad a_1^\epsilon(0) = a_\infty(\epsilon) \quad a_1^\epsilon(1) = a_{\infty,j}^+(\epsilon).$$

As $\epsilon \to 0$, these geometric limits define paths $[A(t), \psi(t)]$ and $a(t)$ with similar properties, and with

$$a(-1) = a_{\infty,i}^- \quad a(0) = a_\infty \quad a(1) = a_{\infty,j}^+,$$

in $\partial_\infty(\mathcal{M}_V^*) \subset \chi_0(T^2, V)$, with $a_\infty = \lim_\epsilon a_\infty(\epsilon)$.

(c) Moreover, we have a holomorphic triangle in $H^1(T^2, \mathbb{R})$ with vertices

$$\{a_{\infty,i}^-, \vartheta_1, a_{\infty,j}^+(\epsilon)\}$$

and sides given by parameterized arcs along the lines $\ell_1 = \{v - u = 1\}$, $\ell_\mu = \{v = f'(u)\}$ and by

$$\{a_1^\epsilon(t)\} \subset \ell = \partial_\infty(\mathcal{M}_V^*).$$

This is obtained from the non-uniform limits of Proposition 3.1.

We now describe how to assemble together these geometric limits in a suitable moduli space. This will be useful in the following subsection, in the proof of Theorem 6.9 that establishes the exactness in the middle term.

Let a_∞ be an element in $\chi_0(T^2, V)$. We define the configuration space

$$\mathcal{A}_{k,\delta}(V \times \mathbb{R}, a_\infty)$$

as follows.

We can write a pair $(\mathcal{A}, \Psi)$ of a $U(1)$-connection and a spinor on $V \times \mathbb{R}$ in the form

$$\mathcal{A} = a(w, s, t) + f(w, s, t)ds + h(w, s, t)dt$$

in the region $T^2 \times [0, \infty) \times \mathbb{R}$ inside $V \times \mathbb{R}$. A pair $(\mathcal{A}, \Psi)$ as above is in the configuration space $\mathcal{A}_{k,\delta}(V \times \mathbb{R}, a_\infty)$ if $(\mathcal{A}, \Psi)$ is in L_k^2 on $V \times \mathbb{R}$. Moreover, we also require that, after the change of coordinates $s + it = e^{\rho+i\theta}$, and the corresponding change of variables

$$\begin{aligned} a(w,\rho,\theta) &= a(w,e^{\rho+i\theta}) \\ f(w,\rho,\theta) &= e^{-\rho}\cos\theta\ h(w,e^{\rho+i\theta}) - e^{-\rho}\sin\theta\ f(w,e^{\rho+i\theta}) \\ h(w,\rho,\theta) &= e^{-\rho}\cos\theta\ f(w,e^{\rho+i\theta}) + e^{-\rho}\sin\theta\ h(w,e^{\rho+i\theta}), \end{aligned} \tag{31}$$

we have

$$(a - a_\infty, h, f - f_0, \alpha, \beta) \in L^2_{k,\delta}(T^2 \times [-\pi/2, \pi/2] \times [\rho_0, \infty)),$$

where f_0 is a constant. The $L^2_{k,\delta}$ norm we consider is defined by

$$\|F\|_{L^2_{k,\delta}} = \|e_\delta \cdot F\|_{L^2_k},$$

where e_δ is a smooth non-negative function satisfying

$$e_\delta(w,\rho,\theta) = \exp(\delta e^\rho),$$

for (w,ρ,θ) in the range

$$T^2 \times [\rho_0, \infty) \times [-\pi/2, \pi/2].$$

We have a group of gauge transformations acting on this configuration space, namely the group

$$\mathcal{G}_{V\times\mathbb{R},k+2,\delta}$$

given by maps $\lambda : V \times \mathbb{R} \to U(1)$ of the form $\lambda = e^{i\ell}$ with $\ell : V \times \mathbb{R} \to \mathbb{R}$, such that $\ell - \ell_\infty$ is in $L^2_{k+2,\delta}$ on the domain

$$T^2 \times [\rho_0, \infty) \times [-\pi/2, \pi/2] \subset V \times \mathbb{R}. \tag{32}$$

Here $\lambda_\infty = e^{i\ell_\infty}$ is a gauge transformation on T^2 that extends to V, that is, $\lambda_\infty \in \mathcal{G}_V$.

Recall that, by the analysis of [17], any finite energy solution $(\mathcal{A}, \Psi)$ on $V \times \mathbb{R}$ has the property that in radial gauge, $(\mathcal{A}, \Psi)$ has limits

$$a_\infty(\theta) = \lambda(\theta) a_\infty,$$

with $\theta \in [-\pi/2, \pi/2]$ and with $\lambda(\theta)$ a family of gauge transformations on T^2 that extend to gauge transformations on V, and $[a_\infty]$ a fixed gauge class of flat connections on T^2. With a slight abuse of notation, we write a_∞ for this gauge class. Thus, we can represent all finite energy solutions by elements in some configuration space $\mathcal{A}_{k,\delta}(V\times\mathbb{R},a_\infty)$.

Given a fixed element $\Gamma_{a_\infty}=(\mathcal{A},\Psi)$ in the configuration space $\mathcal{A}_{k,\delta}(V\times\mathbb{R},a_\infty)$, we define the slice at Γ_{a_∞} as

$$\mathcal{S}_{\Gamma_{a_\infty}}=\{(\tilde{a},\tilde{\Phi})\in\mathcal{A}_{k,\delta}(V\times\mathbb{R},a_\infty)|G^*_{\Gamma_{a_\infty}}(\tilde{a},\tilde{\Phi})=0\},$$

where the operator $G^*_{\Gamma_{a_\infty}}$ is the L^2_δ adjoint of the infinitesimal gauge action $G_{\Gamma_{a_\infty}}$. In the region (32) the elements $(\tilde{a},\tilde{\Phi})$ can be written in the form

$$(\tilde{a},\tilde{\Phi})=(a,h,f,\alpha,\beta),$$

with

$$\tilde{a}=a+f\,ds+h\,dt \quad\text{and}\quad \tilde{\Phi}=(\alpha,\beta),$$

as above.

We define the moduli space $\mathcal{M}_{V\times\mathbb{R}}(a_\infty)$ as the set of solutions of the Seiberg-Witten equations in $\mathcal{A}_{k,\delta}(V\times\mathbb{R},a_\infty)$, modulo the action of the gauge group $\mathcal{G}_{V\times\mathbb{R},k+2,\delta}$. We denote by $\hat{\mathcal{M}}_{V\times\mathbb{R}}(a_\infty)$ the *balanced energy* moduli space, namely the elements in $\mathcal{M}_{V\times\mathbb{R}}(a_\infty)$ satisfying

$$\begin{aligned}&\textstyle\int_{-\infty}^{0}(\|\partial_t A(t)\|^2_{L^2(V)}+\|\partial_t\psi(t)\|^2_{L^2(V)})dt=\\&\textstyle\int_{0}^{+\infty}(\|\partial_t A(t)\|^2_{L^2(V)}+\|\partial_t\psi(t)\|^2_{L^2(V)})dt,\end{aligned}$$

for a temporal gauge representative $(A(t),\psi(t))$.

The linearization $\mathcal{L}_{(\mathcal{A},\Psi)}$ at a solution $(\mathcal{A},\Psi)$ in the slice $\mathcal{S}_{\Gamma_{a_\infty}}$ is given by the operator

$$\mathcal{L}_{(\mathcal{A},\Psi)}(\tilde{a},\tilde{\Phi})=\begin{cases} d^+\tilde{a}-\frac{1}{2}Im(\Psi\cdot\tilde{\Phi})\\ D_{\mathcal{A}}\tilde{\Phi}+\tilde{a}.\Psi\\ G^*_{\Gamma_{a_\infty}}(\tilde{a},\tilde{\Phi})\end{cases} \tag{33}$$

The virtual dimension of the moduli space $\hat{\mathcal{M}}_{V\times\mathbb{R}}(a_\infty)$ at a solution $(\mathcal{A},\Psi)$ in the slice $\mathcal{S}_{\Gamma_{a_\infty}}$ is given by

$$Index(\mathcal{L}_{(\mathcal{A},\Psi)}).$$

Assuming that the element $a_\infty\neq\vartheta$ is away from the bad point ϑ in the character variety of T^2, we know, by the result of Section 3.2 of [17], that all finite energy solutions $(\mathcal{A}, \Psi)$ have a uniform exponential decay in radial gauge, hence they can be regarded as elements of the moduli spaces $\hat{\mathcal{M}}_{V\times\mathbb{R}}(a_\infty)$ introduced here, for some δ which depends only on a_∞.

Recall also that in Proposition 3.4 of Part II we proved that, given a finite energy solution $(\mathcal{A}, \Psi)$ in $\mathcal{A}_{k,\delta}(V \times \mathbb{R}, a_\infty)$ of the Seiberg-Witten equations on $V \times \mathbb{R}$, with asymptotic value in the gauge class of a_∞, by applying a gauge transformation $\lambda(\theta)$ in $\mathcal{G}_{V\times\mathbb{R},k+2,\delta}$, we obtain a solution $\lambda(\mathcal{A}, \Psi)$ with $h \equiv 0$, and $f - f_0 \equiv 0$. In particular, when inverting the change of variables (31), the condition that $\lambda(\mathcal{A}, \Psi)$ is in L^2_k gives $f_0 \equiv 0$. In particular the resulting solution is in a temporal gauge, hence we can write $\lambda(\mathcal{A}, \Psi)$ as $(A(t), \psi(t))$. Thus, given a solution $(\mathcal{A}, \Psi)$ in $\mathcal{A}_{k,\delta}(V \times \mathbb{R}, a_\infty)$, up to gauge transformations, we obtain two classes $[A, \psi]$ and $[\tilde{A}, \tilde{\psi}]$ in

$$\partial_\infty^{-1}(a_\infty) \subset \mathcal{M}^*_V$$

defined by the asymptotic values as $t \to \pm\infty$ of the temporal gauge representative $(A(t), \psi(t)) = \lambda(\mathcal{A}, \Psi)$, cf. the result of Lemma 3.9 of [17].

Thus, we can break the moduli space $\hat{\mathcal{M}}_{V\times\mathbb{R}}(a_\infty)$ into a union of components

$$\hat{\mathcal{M}}_{V\times\mathbb{R}}(a_\infty) = \bigcup_{[A,\psi],[\tilde{A},\tilde{\psi}]} \hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty),$$

with

$$[A,\psi],[\tilde{A},\tilde{\psi}] \in \partial_\infty^{-1}(a_\infty) \subset \mathcal{M}^*_V.$$

We can rephrase the calculation of the virtual dimension as follows.

Proposition 6.3 *Let $[\mathcal{A}, \Psi]$ be a gauge class in*

$$\hat{\mathcal{M}}_{V\times\mathbb{R}}(a_\infty).$$

Let $(A(t), \psi(t))$ be a temporal gauge representative of $[\mathcal{A}, \Psi]$, which satisfies

$$\lim_{t\to-\infty}[A(t),\psi(t)] = [A,\psi],$$

$$\lim_{t\to\infty}[A(t),\psi(t)] = [\tilde{A},\tilde{\psi}].$$

Let $H_{A(t),\psi(t)}$ be the operator

$$H_{A(t),\psi(t)} = \begin{cases} L_{A(t),\psi(t)}(\alpha,\phi) + G_{A(t),\psi(t)}(f) \\ G^*_{A(t),\psi(t)}(\alpha,\phi), \end{cases}$$

with

$$L_{A(t),\psi(t)}(\alpha,\phi) = (*_3 d\alpha - 2iIm(\psi(t),\phi), \partial_{A(t)}\phi + \alpha\cdot\psi(t)),$$

$$G_{A(t),\psi(t)}(f) = (-df, e^{if}\psi(t)),$$

and $G^*_{A(t),\psi(t)}$ *is the* L^2_δ *adjoint of* $G_{A(t),\psi(t)}$. *Let* Q_{a_∞} *be the asymptotic operator of* $H_{A(t),\psi(t)}$, *for each* t, *given by*

$$Q_{a_\infty}(a,\alpha,\beta) = (0, -i\bar\partial^*_{a_\infty}\beta, i\bar\partial_{a_\infty}\alpha),$$

as in [6].

Each moduli space

$$\mathcal{M}_{V\times\mathbb{R}}([A,\psi],[\tilde A,\tilde\psi],a_\infty),$$

for a fixed choice of $[A,\psi]$ *and* $[\tilde A,\tilde\psi]$ *in* $\partial_\infty^{-1}(a_\infty)$ *in* $\mathcal{M}^*_V$, *is a smooth finite dimensional oriented manifold, of dimension given by the spectral flow*

$$SF(H_{A(t),\psi(t)}) = Index(\partial_t + H_{A(t),\psi(t)}).$$

Thus, for the balanced energy case, we have virtual dimension

$$virtdim\hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde A,\tilde\psi],a_\infty) = SF(H_{A(t),\psi(t)}) - 1.$$

An orientation of $\hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde A,\tilde\psi],a_\infty)$ *is obtained by considering the determinant line bundle of the operator* $\mathcal{L}_{\mathcal{A},\Psi} = \partial_t + H_{A(t),\psi(t)}$.

6.2 Admissible elements in $\hat{\mathcal{M}}_{V\times\mathbb{R}}(a_\infty)$

Now consider $a(t)$ a regular parameterization of an arc in $\partial_\infty(\mathcal{M}^*_V)$ inside $\chi_0(T^2,V)$, which satisfies

$$a(-1) = a^- \quad a(0) = a_\infty \quad a(1) = a^+,$$

with given

$$a^- \in \partial_\infty(\mathcal{M}^*_V)\cap\{u-v=1\}$$
$$a^+ \in \partial_\infty(\mathcal{M}^*_V)\cap\{v=f'(u)\}.$$

Let us assume that the path $a(t)$ also satisfies the condition $a(t)\neq\vartheta$, for all $t\in[-1,1]$, and that the path $a(t)$ avoids all the boundary points of

$\partial_\infty(\mathcal{M}^*_V)$ on $\chi(V)\backslash\vartheta$, cf. [6]. By the analysis of [6], we know then that the fiber

$$\partial_\infty^{-1}(a(t)) \subset \mathcal{M}^*_V$$

is a finite set of points, for each fixed t. Moreover, under the current hypotheses, the set

$$\cup_{t\in[-1,1]}\partial_\infty^{-1}(a(t))$$

describes a cobordism between

$$\partial_\infty^{-1}(a(-1)) \quad \text{and} \quad \partial_\infty^{-1}(a(0))$$

and between

$$\partial_\infty^{-1}(a(0)) \quad \text{and} \quad \partial_\infty^{-1}(a(1)).$$

Now suppose given two assigned elements

$$[A^-,\psi^-] \in \partial_\infty^{-1}(a^-)$$

$$[A^+,\psi^+] \in \partial_\infty^{-1}(a^+),$$

and elements

$$[A,\psi], [\tilde{A},\tilde{\psi}] \in \partial_\infty^{-1}(a_\infty).$$

In the cobordism

$$\cup_{t\in[-1,1]}\partial_\infty^{-1}(a(t)),$$

there is at most one path $[A(t),\psi(t)]$ in $\mathcal{M}^*_V$, for $t \in [-1,0]$ that satisfies $\partial_\infty[A(t),\psi(t)] = a(t)$ and $[A(-1),\psi(-1)] = [A^-,\psi^-]$. Similarly, there is at most one path $[A(t),\psi(t)]$ in $\mathcal{M}^*_V$, for $t \in [0,1]$ that satsifies $\partial_\infty[A(t),\psi(t)] = a(t)$ and $[A(1),\psi(1)] = [A^+,\psi^+]$, cf. the Figure 3. The case (d) of Figure 3 illustrates an example of a parameterization $a(t)$ and a choice of $[A^-,\psi^-]$ and $[A^+,\psi^+]$ for which a path with the desired properties does not exist. Such case does not arise as a geometric limit.

Thus, we can introduce the following notation, to distinguish which choices of elements $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ can arise as part of the geometric limits of solutions in $\mathcal{M}^{W_1(r)}_\ell(a_1,a)$ (or $\mathcal{M}^{W_0(r)}_k(a_1,a_0)$, or $\mathcal{M}_{Y(r)\times\mathbb{R}}(a,b)$).

Definition 6.4 *We say that a triple* $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ *is* admissible*, with respect to the endpoints* (a_1,a)*, if the following conditions hold. The element* a_∞ *lies on a path component of* $\partial_\infty(\mathcal{M}^*_V)$ *connecting* a^- *and* a^+*. Moreover, there exists a smooth regular parameterization* $a(t)$*, for* $t \in [-1,1]$ *of the path in* $\partial_\infty(\mathcal{M}^*_V)$ *connecting* a^- *and* a^+*, such that* $a(0) = a_\infty$*, and corresponding* *smooth paths* $[A(t),\psi(t)]$ *in* $\mathcal{M}^*_V$*, for* $t\in[-1,0)$ *and* $t\in(0,1]$*, satisfying* $\partial_\infty[A(t),\psi(t)]=a(t)$*, and with*

$$[A(-1),\psi(-1)]=[A^-,\psi^-] \quad \textit{and} \quad \lim_{t\to 0_-}[A(t),\psi(t)]=[A,\psi]$$

$$\lim_{t\to 0_+}[A(t),\psi(t)]=[\tilde{A},\tilde{\psi}] \quad \textit{and} \quad [A(1),\psi(1)]=[A^+,\psi^+],$$

with $[A,\psi]$ *and* $[\tilde{A},\tilde{\psi}]$ *in* $\partial_\infty^{-1}(a_\infty)$ *in* $\mathcal{M}^*_V$ *and*

$$a_1=[(A^-,\psi^-)\#(a^-,0)]$$

$$a=[(A^+,\psi^+)\#(a^+,0)].$$

In the examples of Figure 3, the cases (a)-(b) represent admissible elements, and case (c) is not an admissible element, because the parameterization $a(t)=\partial_\infty[A(t),\psi(t)]$ is not regular ($\partial_t a(t)=0$ for some t), and the case (d) is also non admissible because no path $[A(t),\psi(t)]$ with the desired properties exists. An element $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ can appear as part of the geometric limits of solutions on $W_1(r)$ (or $W_0(r)$, or $Y(r)\times\mathbb{R}$, etc.) only if it is admissible with respect to the endpoints (a_1,a) (or (a,a_0), or (a,b)).

Notice that, in general, if we consider different solutions in $\mathcal{M}^{W_1(r)}_\ell(a_1,a)$ (or $\mathcal{M}^{W_0(r)}_k(a_1,a_0)$, or $\mathcal{M}_{Y(r)\times\mathbb{R}}(a,b)$), these will give rise to geometric limits with different parameterizations $a(t)$, and in general different a_∞. In the next subsection we describe how to assemble the various geometric limits.

6.3 Assembling the geometric limits

Let us first return to the setting of [17] and consider moduli spaces of flow-lines. We shall then generalize our statements to the case of finite energy solutions on one of the surgery cobordisms.

Consider first the case of a zero-dimensional moduli space of flow lines, $\hat{\mathcal{M}}_{Y\times\mathbb{R},\mu}(a,b)$, with $a,b\in\mathcal{M}^*_{Y,\mu}$ satisfying $\deg_{Y,\mu}(a)-\deg_{Y,\mu}(b)=1$. We recall the results on the splitting of the spectral flow that we used in Part I [6] in order to compare the relative gradings, cf. [7].

We can describe the critical points a and b in $\mathcal{M}^*_{Y,\mu}$ as

$$a=[(A^-,\psi^-)\#_r(a^-,0)]$$

$$b=[(A^+,\psi^+)\#_r(a^+,0)].$$

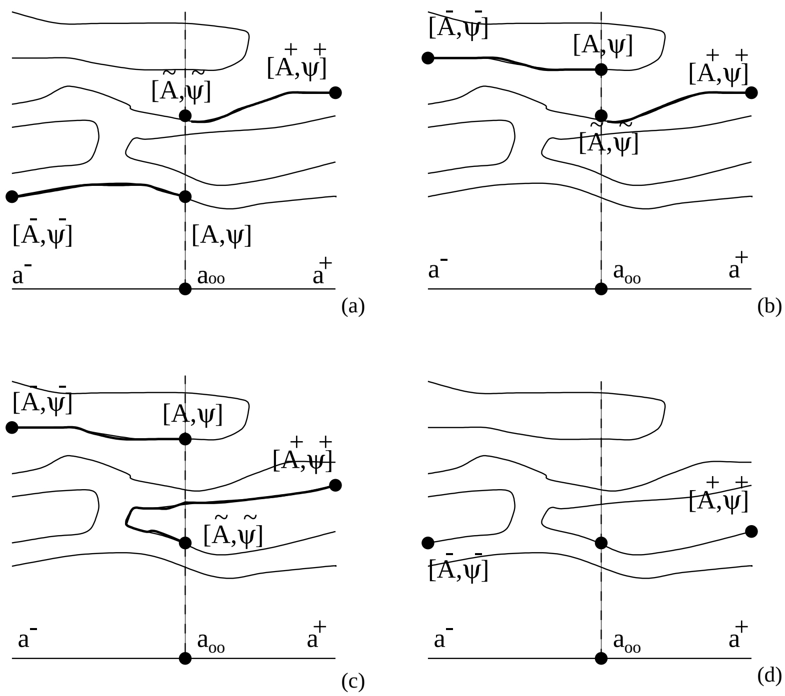

Figure 3: The paths $[A(t),\psi(t)]$ in $\mathcal{M}^*_V$

The relative grading is given by

$$1 = \deg_{Y,\mu}(a) - \deg_{Y,\mu}(b) = \frac{1}{r^2} SF_{Y(r)}(H_{(A_r(t),\psi_r(t))}),$$

with the notation as in [6], [3]. Then the spectral flow can be written as

$$\begin{aligned} &\tfrac{1}{r^2} SF_{Y(r)}(H_{(A_r(t),\psi_r(t))}) = \epsilon SF_{V(r)}(H_{(A(t),\psi(t)),\tilde{\ell}_V(t)})+ \\ &\epsilon SF_{\nu(K)(r)}(H_{(a''(t),0),\tilde{\ell}_\nu(t)}) + Maslov(\tilde{\ell}_V(t),\tilde{\ell}_\nu(t)) \end{aligned} \tag{34}$$

Here we follow the same convention of [3] regarding the definition of the ϵ-spectral flow. The boundary conditions are prescribed by assigning a choice of Lagrangian subspaces $(\tilde{\ell}_V(t),\tilde{\ell}_\nu(t))$ in $H^1(T^2,\mathbb{R})$.

We consider the following Lagrangians, which we already introduced in Part I, [6]. Let ℓ^*_μ be the piecewise smooth Lagrangian submanifold of $\chi_0(T^2,Y)$ described in Part I, [6], and let $\tilde{\ell}_\mu$ be the path of Lagrangian subspaces of $H^1(T^2,\mathbb{R})$ defined as in Part I, [6], which is given by the tangent spaces of ℓ^*_μ where the latter is smooth, completed with a specific choice of paths at the singular points, as discussed in [6]. Let ℓ be the union of the arcs in the Lagrangian submanifold with boundary $\partial_\infty(\mathcal{M}_V^*)$ in $\chi_0(T^2, Y)$ that connect the points a^- and a^+. By $\partial_\infty(\mathcal{M}_V^*) \subset \chi_0(T^2, Y)$ we mean the pullback of $\partial_\infty(\mathcal{M}_V^*) \subset \chi_0(T^2, V)$ under the covering map $\chi_0(T^2, Y) \to \chi_0(T^2, V)$. Under the assumption that these arcs avoid the boundary points $\partial_\infty(\partial\mathcal{M}_V^*)$, we can consider regular parameterizations $a'(t)$ and $a''(t)$ of the arcs of ℓ_μ^* and ℓ, respectively, connecting a^- to a^+. We consider the corresponding paths of Lagrangians $\tilde{\ell}(t) = T_{a'(t)}\ell$ and $\tilde{\ell}_\mu(t)$, for $t \in [0,1]$. We can also assume that ℓ_μ^* and ℓ intersect transversely.

Now recall that, in Part II, we identified the geometric limits of flowlines in $\hat{\mathcal{M}}_{Y(r)\times\mathbb{R}}(a,b)$ with finite energy solutions in $\hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$, together with paths $[A(t),\psi(t)] \in \mathcal{M}_V^*$, and a holomorphic disk

$$\Delta : (D^2, \partial D^2) \to (\chi_0(T^2, Y), \ell \cup \ell_\mu^*),$$

which determines the regular parameterization $a(t)$ of the arcs of ℓ and ℓ_μ^*, hence the admissible data $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$. Thus, we are interested in understanding the space of inequivalent holomorphic disks Δ, which can appear in the geometric limits. We have the following result, see for instance [12] pg.3. The set of equivalence classes under $Aut(D^2) \simeq PSL(2,\mathbb{R})$ of holomorphic disks

$$\Delta : (D^2, \partial D^2) \to (\chi_0(T^2, Y), \ell \cup \ell_\mu^*),$$

in a given homotopy class $\beta = [\Delta]$ in $\pi_2(\chi_0(T^2, Y), \ell \cup \ell_\mu^*)$ has virtual dimension

$$\mu(\beta) - 1,$$

with $\mu(\beta)$ the Maslov class, defined as in [12] pg. 3. The original formula given in [12] for this virtual dimension is $n + \mu(\beta) - 2$, where $2n$ is the real dimension of the ambient symplectic manifold. In our case, we have $n = 1$. We can describe the Maslov class $\mu(\beta)$ in terms of the Maslov index $Maslov(\tilde{\ell}(t), \tilde{\ell}_\mu(t))$ as follows.

Lemma 6.5 *Consider the piecewise smooth Lagrangian submanifolds ℓ and ℓ_μ^* in $\chi_0(T^2, Y)$, defined as above. Consider all possible holomorphic disks*

$$\Delta : (D^2, \partial D^2) \to (\chi_0(T^2, Y), \ell \cup \ell_\mu^*),$$

up to automorphisms $Aut(D^2) \simeq PSL(2,\mathbb{R})$, which map the boundary to arcs of the Lagrangians connecting two points a^- and a^+ in $\ell \cap \ell_\mu^*$, in the *homotopy class β specified by the regular parameterizations $a'(t)$ and $a''(t)$ of these arcs of Lagrangians. Then the Maslov class satisfies*

$$\mu(\beta) = Maslov(\tilde{\ell}(t), \tilde{\ell}_\mu(t)) + 1.$$

Proof. Here we follow the notation of Section 13 of [4]. First let us define

$$\nu_{\pm}^{a^{\pm}}(T_{a^{\pm}}\ell_1, T_{a^{\pm}}\ell_2)$$

as the path of Lagrangian subspaces that connects the two Lagrangian specified subspaces $(T_{a^+}\ell_1, T_{a^+}\ell_2)$, rotating in the positive or negative direction, according to sign. Recall that, given a pair of Lagrangian paths $f(t) = (\tilde{\ell}_1(t), \tilde{\ell}_2(t))$, we use the notation $\hat{f}(t) = f(1-t)$, and

$$f_{flip}(t) = (\tilde{\ell}_2(t), \tilde{\ell}_1(t)).$$

Thus, given $f(t) = (\tilde{\ell}_1(t), \tilde{\ell}_2(t))$ with $\ell_1(0) = \ell_2(0) = a^-$ and $\ell_1(1) = \ell_2(1) = a^+$, we obtain four possible loops $f_{\pm\pm}(t)$, given as the concatenation of paths

$$f_{ij}(t) = \tilde{\ell}_1(t) * \nu_i^{a^+}(t) * \widehat{\tilde{\ell}_2(t)} * (\nu_j^{a^-}(t))_{flip},$$

with $i, j = \pm$.

The definition of the Maslov class given in [12] pg. 3, for one smooth Lagrangian submanifold ℓ inside a symplectic manifold X and disks $\Delta : (D^2, \partial D^2) \to (X, \ell)$, coincides with the notion of Maslov index for closed loops *à la* Floer [10]. Using the results of Section 13 of [4], we can relate this to the Maslov index, as described in the Lemma. Since the arguments of Part II, [17] actually ensure the existence of a holomorphic disk on a slightly larger domain, which maps a subdomain homeomorphic to $(D^2, \partial D^2)$ to a disk filling a region bounded by arcs of the Lagrangians $\ell \cup \ell_\mu^*$ with endpoints a^- and a^+, we obtain that the Maslov class is computed by

$$\mu(\beta) = \mu(f_{++}(t)),$$

with the notation as above, and $\mu(f_{++}(t))$ the Maslov index of a loop of Lagrangians *à la* Floer, cf. Section 13 of [4], where we have

$$f(t) = (\tilde{\ell}(t), \tilde{\ell}_\mu(t)).$$

Now the result of Section 13 of [4] gives

$$\mu(f_{++}(t)) = Maslov(\tilde{\ell}(t), \tilde{\ell}_\mu(t)) + 1.$$

$\diamond$

Notice that we can think of these holomorphic disks equivalently as equivalence classes of

$$\Delta : (D^2, \partial D^2) \to (\chi_0(T^2, Y), \ell \cup \ell^*_\mu)$$

that map $\Delta(-1) = a^-$ and $\Delta(+1) = a^+$, modulo the subgroup of $Aut(D^2)$ that fixes the points ± 1, or then again, equivalently, as holomorphic maps Δ of an infinite strip

$$\mathbb{R} \times [0,1] \to \chi_0(T^2, Y)$$

that map the boundary $\mathbb{R} \times \{0,1\}$ to arcs of the Lagrangians ℓ and ℓ^*_μ, and with asymptotic values along $t \in \mathbb{R}$, as $t \to \mp\infty$, equal to a^- and a^+. In this case, we consider classes modulo the action of $\mathbb{R}$ by reparameterizations.

Thus, the solutions in the zero-dimensional moduli space $\hat{\mathcal{M}}_{Y\times\mathbb{R},\mu}(a,b)$ are obtained by gluing solutions in a zero-dimensional moduli space

$$\hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$$

with a holomorphic disk

$$\Delta : (D^2, \partial D^2) \to (\chi_0(T^2, Y), \ell \cup \ell^*_\mu),$$

in a fixed homotopy class $\beta = [\Delta]$ in $\pi_2(\chi_0(T^2, Y), \ell \cup \ell^*_\mu)$. The admissible data $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ is determined by the holomorphic disk Δ, as we discuss in the following Lemma.

Lemma 6.6 *Suppose given a zero-dimensional moduli space $\hat{\mathcal{M}}_{Y\times\mathbb{R},\mu}(a,b)$. The space of pre-glued solutions, obtained by pasting together the geometric limits of solutions in $\hat{\mathcal{M}}_{Y\times\mathbb{R},\mu}(a,b)$, as described in [17], is given by*

$$\bigcup_{\Delta\in\Xi} \hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty).$$

where Δ is a choice of one particular representative in each equivalence class in the set Ξ of classes of holomorphic disks

$$\Delta : (D^2, \partial D^2) \to (\chi_0(T^2, Y), \ell \cup \ell^*_\mu),$$

in the fixed homotopy class $[\Delta]$ in $\pi_2(\chi_0(T^2, Y), \ell \cup \ell^*_\mu)$, modulo the action of $Aut(D^2) \simeq PSL(2,\mathbb{R})$. Each such representative Δ determines the corresponding admissible element $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$.

Proof. The statement of the Lemma follows from the previous discussion. In order to see the dependence of the data $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ on the holomorphic disk Δ, recall that the choice of Δ in particular fixes the parameterization $a(t)$ of the arc of ℓ connecting a^- and a^+. This determines the point $a_\infty = a(0)$. However, the choice of the parameterization $a(t)$ also determines the choice of the points $[A,\psi]$ and $[\tilde{A},\tilde{\psi}]$ in $\partial_\infty^{-1}(a_\infty)$. Notice that each solution $(\mathcal{A}(r),\Psi(r))$, representing an element in $\hat{\mathcal{M}}_{Y(r)\times\mathbb{R},\mu}(a,b)$, determines a holomorphic disk

$$\Delta = \Delta(\mathcal{A}(r),\Psi(r))$$

in the geometric limits. Each such disk determines a possibly different parameterization $a(t)$, hence different admissible data

$$([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty).$$

Thus, the set of pre-glued solutions can be written as

$$\bigcup_\Delta \hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty),$$

where Δ varies in the set of holomorphic disks

$$\Delta : (D^2,\partial D^2) \to (\chi_0(T^2,Y),\ell\cup\ell^*_\mu),$$

in the fixed homotopy class $[\Delta]$ in $\pi_2(\chi_0(T^2,Y),\ell\cup\ell^*_\mu)$. However, we only count the disks Δ up to $Aut(D^2)\simeq PSL(2,\mathbb{R})$.

For geometric limits of a zero-dimensional moduli space, the set of such holomorphic disks, up to $Aut(D^2)$ is also a zero-dimensional moduli space. This follows by the result of Lemma 6.5, in the case where we have

$$Maslov(\tilde{\ell}(t),\tilde{\ell}_\mu(t)) = 0.$$

In fact, in our case, with disks in $\chi_0(T^2,Y)$, the homotopy class is fixed by the choice of the arcs of the Lagrangians ℓ and ℓ^*_μ connecting a^- and a^+, and the representatives in each class only differ by reparameterizations of the boundary. A choice of a representative Δ in each class identifies uniquely a corresponding moduli space

$$\hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty).$$

Thus, with the notation as above, we obtain

$$n_{Y,\mu}(a,b) = \sum_{\Delta} \# \hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty).$$

$\diamond$

Thus, we can rephrase the gluing theorem of [17] as the following statement. Let us introduce the following notation:

$$\hat{\mathcal{M}}_{V\times\mathbb{R}}([A^-,\psi^-],[A^+,\psi^+]) := \bigcup_{\Delta} \hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty),$$

with the admissible data $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ and the corresponding holomorphic disk Δ as in Lemma 6.6.

Proposition 6.7 *The gluing map gives an orientation preserving diffeomorphism*

$$\# : \hat{\mathcal{M}}_{V\times\mathbb{R}}([A^-,\psi^-],[A^+,\psi^+]) \cong \hat{\mathcal{M}}_{Y\times\mathbb{R}}(a,b),$$

with

$$a = [(A^-,\psi^-)\#_{a^-}(a^-,0)]$$
$$b = [(A^+,\psi^+)\#_{a^+}(a^+,0)].$$

Now we can rephrase the result in the case of moduli spaces of finite energy solutions of the Seiberg–Witten equations on one of the surgery cobordisms. In this case, we can assemble the geometric limits in a similar way, to obtain spaces

$$\mathcal{M}_{V\times\mathbb{R}}([A^-,\psi^-],[A^+(\epsilon),\psi^+(\epsilon)]) := \bigcup_{\Delta_1^\epsilon} \mathcal{M}_{V\times\mathbb{R}}([A,\psi]_1^\epsilon,[\tilde{A},\tilde{\psi}]_1^\epsilon,a^1_\infty(\epsilon)), \quad (35)$$

in the case of W_1, with the admissible data

$$([A,\psi]_1^\epsilon,[\tilde{A},\tilde{\psi}]_1^\epsilon,a^1_\infty(\epsilon))$$

determined by the holomorphic triangle Δ_1^ϵ, or

$$\mathcal{M}_{V\times\mathbb{R}}([A^-(\epsilon),\psi^-(\epsilon)],[A^+,\psi^+]) := \bigcup_{\Delta_0^\epsilon} \mathcal{M}_{V\times\mathbb{R}}([A,\psi]_0^\epsilon,[\tilde{A},\tilde{\psi}]_0^\epsilon,a^0_\infty(\epsilon)), \quad (36)$$

in the case of W_0, with the admissible data

$$([A,\psi]_0^\epsilon, [\tilde{A},\tilde{\psi}]_0^\epsilon, a_\infty^0(\epsilon))$$

determined by the holomorphic triangle Δ_0^ϵ. Here the Δ_i^ϵ vary in the set of inequivalent holomorphic triangles in $H^1(T^2,\mathbb{R})$, namely

$$\Delta_1^\epsilon : (D^2, \partial D^2) \to (H^1(T^2,\mathbb{R}), \ell \cup \ell_1 \cup \ell_\mu),$$

in the case of W_1, or

$$\Delta_0^\epsilon : (D^2, \partial D^2) \to (H^1(T^2,\mathbb{R}), \ell \cup \ell_\mu \cup_k \ell_k),$$

in the case of W_0, with $\ell_1 = \{u - v = 1\}$, $\ell_\mu = \{v = f'(u)\}$, and $\ell_k = \{u = 2k\}$ or $\{u = \eta\}$ in the case of ℓ_0, as in [6]. The image of each Δ_i^ϵ describes a triangle in $H^1(T^2,\mathbb{R})$ with vertices $\{a^-, \vartheta_i, a^+(\epsilon)\}$ or $\{a^-(\epsilon), \vartheta_i, a^+\}$ and sides along the Lagrangians, as specified. Here the points ϑ_i are the intersection of the lines ℓ_1 and ℓ_μ for W_1 and ℓ_k and ℓ_μ for W_0, that is, the restriction to $T^2 = \partial\nu = \partial\tilde{\nu}$ of the unique reducible point θ_{S^3} at the puncture in the cobordism.

Thus, we obtain the following result on the gluing theorem for the moduli spaces $\mathcal{M}_\ell^{W_1}(a_1, a)$, or for the minimal energy component of $\mathcal{M}_k^{W_0}(a, a_0)$. We state the result in the case of W_1.

Lemma 6.8 *Suppose given a pair $a_1 \in \mathcal{M}_{Y_1}$, and $a \in \mathcal{M}_{Y,\mu}$. Suppose that we have the decomposition*

$$a_1 = [(A^-, \psi^-)\#_{a^-}(a^-, 0)]$$
$$a = [(A^+(\epsilon), \psi^+(\epsilon))\#_{a^+(\epsilon)}(a^+(\epsilon), 0)].$$

Then the gluing map gives an orientation preserving diffeomorphism

$$\# : \mathcal{M}_{V\times\mathbb{R}}([A^-,\psi^-],[A^+(\epsilon),\psi^+(\epsilon)]) \to \mathcal{M}_\ell^{W_1}(a_1, a),$$

where the first moduli space is defined as in (35), with the union over inequivalent holomorphic triangles Δ with vertices $\{a^-, \vartheta_1, a^+(\epsilon)\}$ and sides along the union of Lagrangians $\ell \cup \ell_1 \cup \ell_\mu$, with ℓ defined by the asymptotic values $\partial_\infty(\mathcal{M}_V^*)$. Different choices of the Spin^c*-structure $\mathfrak{s}_\ell$ correspond to moduli spaces of different dimension. If $\mathcal{M}_\ell^{W_1}(a_1, a)$ is non-empty, then, under the gluing map we obtain*

$$\iota(W_1, \mathfrak{s}_\ell, a_1, a) = \mu([\Delta]) - 1 + virtdim\mathcal{M}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}], a_\infty).$$

Again we can observe that each holomorphic triangle fixes a parameterization $a(t)$, hence a choice of the admissible data $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$. The argument then proceeds as in the case of flow lines. In the case of the moduli spaces $\mathcal{M}_k^{W_0}(a,a_0)$, the minimal energy condition ensures that the path $a(t)$ is contractible in $\chi_0(T^2,V)$, so that the path along the union of Lagrangians can be filled by a holomorphic triangle of minimal energy.

We now return to our analysis of the sequence

$$0 \to C_*(Y_1) \stackrel{w_*^1}{\to} C_*(Y,\mu) \stackrel{w_*^0}{\to} \bigoplus_k C_{(*)}(Y_0,\mathfrak{s}_k) \to 0.$$

In the next subsection, we use the results obtained in this section on the moduli spaces $\mathcal{M}_{V\times\mathbb{R}}([A^-,\psi^-],[A^+,\psi^+])$ to prove exactness in the middle term.

6.4 The relation $w_*^0 \circ w_*^1 = 0$

By Lemma 6.1, we can show that the composite map $w_*^0 \circ w_*^1$ is trivial by proving the following result.

Theorem 6.9 *For small enough ϵ and large $r \geq r_0$, there is an orientation reversing diffeomorphism*

$$\mathcal{M}_\ell^{W_1(r)}(a_i^{(1)},a_j^{(0)}(\epsilon)) \cong \mathcal{M}_k^{W_0(r),\ (0)}(a_i^{(1)}(\epsilon),a_j^{(0)}),$$

where ℓ is the unique positive Spin^c *structure on $W_1(r)$ such that the moduli space $\mathcal{M}_\ell^{W_1(r)}(a_i^{(1)},a_j^{(0)}(\epsilon))$ is zero dimensional, and $\mathcal{M}_k^{W_0(r),\ (0)}(a_i^{(1)}(\epsilon),a_j^{(0)})$ is the zero-dimensional components of $\mathcal{M}_k^{W_0(r)}(a_i^{(1)}(\epsilon),a_j^{(0)})$.*

Proof. By our assumptions, the moduli spaces $\mathcal{M}_\ell^{W_1(r)}(a_i^{(1)},a_j^{(0)}(\epsilon))$ and $\mathcal{M}_k^{W_0(r),\ (0)}(a_i^{(1)}(\epsilon),a_j^{(0)})$ are smooth, compact, oriented 0-dimensional manifolds. We write the critical points $a_i^{(1)}$, $a_j^{(0)}(\epsilon)$ and $a_i^{(1)}(\epsilon),a_j^{(0)}$ according to the following decomposition:

$$a_i^{(1)} = [(A_i(r),\psi_i(r))] = [(A_i^-,\psi_i^-)\#_{a^-_{\infty,i}}(a^-_{\infty,i},0)]$$

on $Y_1(r)$,

$$\begin{aligned} a_i^{(1)}(\epsilon) &= j([(A_i(r),\psi_i(r))]) = [(A_i^-(\epsilon),\psi_i^-(\epsilon))\#_{a^-_{\infty,i}(\epsilon)}(a^-_{\infty,i}(\epsilon),0)] \\ a_j^{(0)}(\epsilon) &= [(A_j^+(\epsilon),\psi_j^+(\epsilon))\#_{a^+_{\infty,j}(\epsilon)}(a^+_{\infty,j}(\epsilon),0)] \end{aligned}$$

on $Y(r)$, and

$$a_j^{(0)} = [(A_j^+, \psi_j^+)\#^r_{a^+_{\infty,j}}(a^+_{\infty,j}, 0)]$$

on $Y_0(r)$. In the limit $\epsilon \to 0$, the asymptotic values satisfy

$$\lim_{\epsilon\to 0} a^-_{\infty,i}(\epsilon) = a^-_{\infty,i}$$

$$\lim_{\epsilon\to 0} a^+_{\infty,j}(\epsilon) = a^+_{\infty,j},$$

in $H^1(T^2, i\mathbb{R})$ and

$$\lim_{\epsilon\to 0}[A_i^-(\epsilon), \psi_i^-(\epsilon)] = [A_i^-, \psi_i^-]$$
$$\lim_{\epsilon\to 0}[A_j^+(\epsilon), \psi_j^+(\epsilon)] = [A_j^+, \psi_j^+]$$

in $\mathcal{M}_V^*$.

We now apply the geometric limits results of Proposition 3.1 to study the moduli spaces $\mathcal{M}_\ell^{W_1(r)}(a_i^{(1)}, a_j^{(0)}(\epsilon))$ and $\mathcal{M}_k^{W_0(r),\ (0)}(a_i^{(1)}(\epsilon), a_j^{(0)})$.

Let $[\mathcal{A}_1(r), \Psi_1(r)]$ be solutions in $\mathcal{M}_\ell^{W_1(r)}(a_i^{(1)}, a_j^{(0)}(\epsilon))$, for sufficiently large $r \geq r_0$. Then these solutions define geometric limits as recalled in Remark 6.2, cf. Proposition 3.1.

For all sufficiently small ϵ, the geometric limits listed in Proposition 3.1 can be grouped together to give a holomorphic triangle

$$\Delta_1^\epsilon : (D^2, \partial D^2) \to (H^1(T^2, \mathbb{R}), \ell \cup \ell_1 \cup \ell_\mu)$$

with vertices $\{a^-_{\infty,i}, \vartheta_1, a^+_{\infty,j}(\epsilon)\}$ and sides along the Lagrangians $\ell_1 = \{v - u = 1\}$, $\ell = \partial_\infty \mathcal{M}_V^*$, and $\ell_\mu = \{v = f'(u)\}$, and an element in a moduli space

$$\mathcal{M}_{V\times\mathbb{R}}([A,\psi]_1^\epsilon, [\tilde{A}, \tilde{\psi}]_1^\epsilon, a^1_\infty(\epsilon)),$$

for some admissible data

$$([A,\psi]_1^\epsilon, [\tilde{A}, \tilde{\psi}]_1^\epsilon, a^1_\infty(\epsilon))$$

determined by the holomorphic triangle Δ_1^ϵ.

We use the notation $\Delta_1^\epsilon = \Delta^\epsilon(\mathcal{A}_1(r), \Psi_1(r))$, for any such holomorphic triangle determined by a family of solutions $(\mathcal{A}_1(r), \Psi_1(r))$ representing an element in

$$\mathcal{M}_\ell^{W_1(r)}(a_i^{(1)}, a_j^{(0)}(\epsilon)).$$

The orientation of this triangle is the same as the orientation of the region of non-uniform convergence. The orientation of

$$\hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi]^\epsilon_1,[\tilde{A},\tilde{\psi}]^\epsilon_1,a^1_\infty(\epsilon))$$

at a solution $[\mathcal{A}',\Psi']^\epsilon$ is given by the determinant line bundle of the linearization $\mathcal{L}_{(\mathcal{A}',\Psi'^\epsilon)}$, as discussed previously.

Now we can proceed to compare the geometric limits of solutions in $\mathcal{M}^{W_1(r)}_\ell(a^{(1)}_i,a^{(0)}_j(\epsilon))$ and $\mathcal{M}^{W_0(r),\ (0)}_k(a^{(1)}_i(\epsilon),a^{(0)}_j)$.

Recall that, as discussed previously (cf. also [17]), we can pre-glue the geometric limits to form an approximate monopole on $W_1(r)$, for sufficiently large $r\geq r_0$. That is, as discussed in the previous subsection, a holomorphic triangle Δ determines the regular parameterizations of the arcs of Lagrangians, hence the admissible data $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$. Therefore, the space of pre-glued monopoles can be identified with the moduli space

$$\mathcal{M}_{V\times\mathbb{R}}([A^-_i,\psi^-_i],[A^+_j(\epsilon),\psi^+_j(\epsilon)])=\bigcup_{\Delta^\epsilon_1}\mathcal{M}_{V\times\mathbb{R}}([A,\psi]^\epsilon_1,[\tilde{A},\tilde{\psi}]^\epsilon_1,a^1_\infty(\epsilon)),$$

with the admissible data specified by

$$\Delta^\epsilon_1=\Delta^\epsilon(\mathcal{A}_1(r),\Psi_1(r)),$$

ranging over the set of inequivalent holomorphic triangles in $H^1(T^2,\mathbb{R})$, with vertices $\{a^-_{\infty,i},\vartheta_1,a^+_{\infty,j}(\epsilon)\}$ and sides along arcs of the Lagrangians ℓ, ℓ_1, and ℓ_μ connecting these points. The orientation on the moduli space $\hat{\mathcal{M}}_{V\times\mathbb{R}}([A^-_i,\psi^-_i],[A^+_j(\epsilon),\psi^+_j(\epsilon)])$ is the product orientation of the pairs

$$([\mathcal{A}',\Psi']^\epsilon,\Delta^\epsilon_1)$$

with $[\mathcal{A}',\Psi']^\epsilon$ an element of the moduli space

$$\hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi]^\epsilon_1,[\tilde{A},\tilde{\psi}]^\epsilon_1,a^1_\infty(\epsilon))$$

determined by the corresponding triangle Δ^ϵ_1. The gluing map gives an orientation preserving diffeomorphism

$$\mathcal{M}^{W_1(r)}_\ell(a^{(1)}_i,a^{(0)}_j(\epsilon))\cong\mathcal{M}_{V\times\mathbb{R}}([A^-_i,\psi^-_i],[A^+_j(\epsilon),\psi^+_j(\epsilon)]),\tag{37}$$

as in Lemma 6.8 in the previous subsection. Moreover, by an analogous argument, we have a similar result for the zero-dimensional component $\mathcal{M}^{W_0(r),\ (0)}_k(a^{(1)}_i(\epsilon),a^{(0)}_j)$.

Claim: For $\mathcal{M}_k^{W_0(r),\ (0)}(a_i^{(1)}(\epsilon), a_j^{(0)})$, the zero-dimensional component of $\mathcal{M}_k^{W_0(r)}(a_i^{(1)}(\epsilon), a_j^{(0)})$, there is no energy loss in the process of stretching along the r-direction. In this case, we have a similar orientation preserving diffeomorphism

$$\mathcal{M}_k^{W_0(r),\ (0)}(a_i^{(1)}(\epsilon), a_j^{(0)}) \cong \hat{\mathcal{M}}_{V(\infty)\times\mathbb{R}}([A_i^-(\epsilon), \psi_i^-(\epsilon)], [A_j^+, \psi_j^+]). \tag{38}$$

Here the moduli space of pre-glued solutions is given by

$$\mathcal{M}_{V\times\mathbb{R}}([A_i^-(\epsilon), \psi_i^-(\epsilon)], [A_j^+, \psi_j^+] = \bigcup_{\Delta_0^\epsilon} \mathcal{M}_{V\times\mathbb{R}}([A,\psi]_0^\epsilon, [\tilde{A}, \tilde{\psi}]_0^\epsilon, a_\infty^0(\epsilon)),$$

with the admissible data determined by

$$\Delta_0^\epsilon = \Delta^\epsilon(\mathcal{A}_0(r), \Psi_0(r)),$$

ranging over the set of inequivalent holomorphic triangles in $H^1(T^2,\mathbb{R})$, with vertices $\{a_{\infty,i}^-(\epsilon), \vartheta_0, a_{\infty,j}^+\}$ and sides along arcs of the Lagrangians ℓ, ℓ_μ, and $\cup_k \ell_k$ connecting these points.
Claim: As $\epsilon \to 0$, we can identify the moduli spaces

$$\mathcal{M}_{V\times\mathbb{R}}([A_i^-, \psi_i^-], [A_j^+(\epsilon), \psi_j^+(\epsilon)])$$

and

$$\mathcal{M}_{V\times\mathbb{R}}([A_i^-(\epsilon), \psi_i^-(\epsilon)], [A_j^+, \psi_j^+])$$

as sets of points.

In fact, if the triangles Δ_1^ϵ and Δ_0^ϵ are non-degenerate, for $\epsilon \to 0$, then these holomorphic triangles reduce to holomorphic triangles

$$\Delta : (D^2, \partial D^2) \to (H^1(T^2,\mathbb{R}), \ell \cup \ell_1 \cup_k \ell_k),$$

in the same homotopy class in $\pi_2(H^1(T^2,\mathbb{R}), \ell \cup \ell_1 \cup_k \ell_k)$, with sides along arcs of the Lagrangians $\ell \cup \ell_1 \cup_k \ell_k$ connecting the points $a_{\infty,i}^-$, $a_{\infty,j}^+$ and the intersection point between the lines ℓ_1 and ℓ_k. Notice that the set theoretic difference between the triangles Δ_1^ϵ and Δ_0^ϵ shrinks to zero size as $\epsilon \to 0$, hence they define the same triangle Δ as a limit. Again we can count these up to automorphisms of D^2. Thus, in the limit as $\epsilon \to 0$ we can identify both moduli spaces with

$$\mathcal{M}_{V\times\mathbb{R}}([A_i^-, \psi_i^-], [A_j^+, \psi_j^+]) = \bigcup_{\Delta} \mathcal{M}_{V\times\mathbb{R}}([A,\psi], [\tilde{A}, \tilde{\psi}], a_\infty),$$

with the admissible data determined by Δ. Notice that the parameterizations $a(t)$ determined by the holomorphic triangles Δ are sufficiently close in the $\mathcal{C}^\infty$ topology to the parameterizations $a^1(t)$ and $a^0(t)$ determined by the Δ_1^ϵ and Δ_0^ϵ, for sufficiently small ϵ, and the corresponding admissible data $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ are also sufficiently close to the admissible data

$$([A,\psi]_1^\epsilon,[\tilde{A},\tilde{\psi}]_1^\epsilon,a_\infty^1(\epsilon)) \quad \text{and} \quad ([A,\psi]_0^\epsilon,[\tilde{A},\tilde{\psi}]_0^\epsilon,a_\infty^0(\epsilon)).$$

Thus, for sufficiently close data, the arguments of the previous subsections show that the moduli spaces

$$\begin{aligned}&\mathcal{M}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)\cong\\&\mathcal{M}_{V\times\mathbb{R}}([A,\psi]_1^\epsilon,[\tilde{A},\tilde{\psi}]_1^\epsilon,a_\infty^1(\epsilon))\cong\\&\mathcal{M}_{V\times\mathbb{R}}([A,\psi]_0^\epsilon,[\tilde{A},\tilde{\psi}]_0^\epsilon,a_\infty^0(\epsilon))\end{aligned}$$

can also be identified.

Thus, we have obtained that the moduli spaces $\mathcal{M}_\ell^{W_1(r)}(a_i^{(1)},a_j^{(0)}(\epsilon))$ and $\mathcal{M}_k^{W_0(r),\ (0)}(a_i^{(1)}(\epsilon),a_j^{(0)})$ can be identified as sets of points. This implies that the composite map satisfies $w_*^0\circ w_*^1\equiv 0\,(mod\,2)$. In order to obtain the result over the integers, we need to compare the orientations of these moduli spaces as they appear in the decomposition of $\mathcal{M}_{\ell,k}^{W(r)}(a_i^{(1)},a_j^{(0)})$ in Lemma 4.4.

Claim: In the decomposition

$$\mathcal{M}_{\ell,k}^{W(r)}(a_i^{(1)},a_j^{(0)})\cong\mathcal{M}_\ell^{W_1(r)}(a_i^{(1)},a_j^{(0)}(\epsilon))\cup\mathcal{M}_k^{W_0(r),\ (0)}(a_i^{(1)}(\epsilon),a_j^{(0)}),$$

as in Lemma 4.4, the moduli spaces

$$\mathcal{M}_\ell^{W_1(r)}(a_i^{(1)},a_j^{(0)}(\epsilon)) \ \text{ and } \ \mathcal{M}_k^{W_0(r),\ (0)}(a_i^{(1)}(\epsilon),a_j^{(0)})$$

have opposite orientations.

In fact, it is sufficient to notice that, when we glue the punctured cobordisms $\hat{W}_1(r)$ and $\hat{W}_0(r)$ along the common boundary Y, to obtain

$$\hat{W}(r)=\hat{W}_1(r)\cup_{Y(r)}\hat{W}_0(r),$$

there is a long product region from a solid torus in Q_1 to a solid torus in Q_0. Thus, the orientations of the two rescaled regions of non-uniform convergence, that is, of the domains of the two holomorphic triangles

$$\Delta_1^\epsilon=\Delta^\epsilon(\mathcal{A}_1(r),\Psi_1(r)) \ \text{ and } \ \Delta_0^\epsilon=\Delta^\epsilon(\mathcal{A}_0(r),\Psi_0(r))$$

have opposite orientations. This implies that, when regarded as solutions in

$$\mathcal{M}^{W(r)}_{\ell,k}(a_i^{(1)}, a_j^{(0)}),$$

one of the triangles is antiholomorphic (holomorphic up to a change of orientation). In fact, the triangles in $H^1(T^2, \mathbb{R})$ have the same orientation, whereas the rescaled regions of non-uniform convergence for the triangles on $\hat{W}_1$ and $\hat{W}_0$ differ by a change of orientations when compared inside the composite cobordism W.

In other words, we obtain an orientation reversing diffeomorphism

$$\mathcal{M}^{W_1(r)}_{\ell}(a_i^{(1)}, a_j^{(0)}(\epsilon)) \cong \mathcal{M}^{W_0(r),\ (0)}_{k}(a_i^{(1)}(\epsilon), a_j^{(0)}),$$

when both moduli spaces are identified with solutions in

$$\mathcal{M}^{W(r)}_{\ell,k}(a_i^{(1)}, a_j^{(0)}).$$

$\diamond$

We now derive from Theorem 6.9 the third fundamental relation.

Corollary 6.10 *Let $a_i^{(1)}$ and $a_j^{(0)}$ be critical points on $Y_1(r)$ and $Y_0(r)$ respectively with*

$$deg_{Y_1}(a_i^{(1)}) = deg_{Y_0,\mathfrak{s}_k}(a_j^{(0)}),$$

and let $a_i^{(1)}(\epsilon)$ and $a_j^{(0)}(\epsilon)$ be the corresponding critical points on $Y(r)$, for sufficiently small ϵ and large $r \geq r_0$, then we have

$$N^{W_1(r)}_{\ell}(a_i^{(1)}, a_j^{(0)}(\epsilon)) = -N^{W_0(r)}_{k}(a_i^{(1)}(\epsilon), a_j^{(0)}).$$

6.5 The inclusion $Ker(w^0_*) \subset Im(w^1_*)$

Recall that, for a fixed pair of asymptotic values $(a_i^{(1)}, a_j^{(0)})$, there exists a unique non-negative Spin^c structure $\mathfrak{s}_\ell$ satisfying

$$\iota(\mathfrak{s}_\ell, W_1, a_i^{(1)}, a_j^{(0)}) = 0.$$

We denote the corresponding ℓ with ℓ_{ij}. The coefficient of the map w^1_* is then given by

$$N^{W_1}_{\ell_{ij}}(a_i^{(1)}, a_j^{(0)}).$$

Using the results of Section 6, we can write explicitly the condition that a Floer chain on Y is in the image of w^1_*.

Lemma 6.11 *Suppose given a Floer chain $a \in C_*(Y,\mu)$,*

$$a = \sum_{i=1}^{m} n_i a_i^{(1)} + \sum_{j=m+1}^{n} n_j a_j^{(0)},$$

with coefficients n_i and n_j in $\mathbb{Z}$. If we assume that the parameter $\epsilon > 0$ in the surgery perturbation is sufficiently small, then the chain $a \in C_*(Y,\mu)$ is in the image $Im(w_*^1)$ under the morphism

$$w_*^1 : C_*(Y_1) \to C_*(Y,\mu)$$

if and only if the coefficients satisfy the relation

$$n_j = \sum_{i=1}^{m} n_i N_{\ell_{ij}}^{W_1}(a_i^{(1)}, a_j^{(0)}),$$

for all $j = m+1, \ldots, n$.

Proof. Assume that the element

$$a = \sum_{i=1}^{m} n_i a_i^{(1)} + \sum_{j=m+1}^{n} n_j a_j^{(0)}$$

is in $Im(w_*^1)$. Then it satisfies

$$a = w_*^1(\sum_{i=1}^{m} p_i a_i^{(1)}),$$

for some coefficients $p_i \in \mathbb{Z}$. For sufficiently small $\epsilon > 0$, we know from the results of Section 6 that the relation

$$\langle a_i^{(1)}, w_*^1(a_k^{(1)})\rangle = \delta_{ik}$$

holds. This gives

$$a = \sum_{i=1}^{m} p_i a_i^{(1)} + \sum_{j=m+1}^{n} \sum_{i=1}^{m} p_i N_{\ell_{ij}}^{W_1}(a_i^{(1)}, a_j^{(0)}).$$

Thus, we obtain

$$n_i = p_i \ \text{ for all } \ i = 1, \ldots, m$$

and

$$n_j = \sum_{i=1}^m p_i N_{\ell_{ij}}^{W_1}(a_i^{(1)}, a_j^{(0)}).$$

$\diamond$

On the other hand, we can also write explicitly the condition that a Floer chain on Y is in the kernel of w_*^0, using the expression for the map w_*^0 for sufficiently small $\epsilon > 0$ in the surgery perturbation.

Lemma 6.12 *Suppose given a Floer chain $a \in C_*(Y, \mu)$,*

$$a = \sum_{i=1}^m n_i a_i^{(1)} + \sum_{j=m+1}^n n_j a_j^{(0)},$$

with coefficients n_i and n_j in $\mathbb{Z}$. For sufficiently small $\epsilon > 0$ in the surgery perturbation, we have that $a \in C_*(Y, \mu)$ is in the kernel $Ker(w_*^0)$ if and only if the coefficients satisfy

$$n_j = -\sum_{i=1}^m n_i N_{k_j}^{W_0}(a_i^{(1)}, a_j^{(0)}),$$

for all $j = m+1, \ldots, n$, with

$$a_j^{(0)} \in \mathcal{M}_{Y_0}(\mathfrak{s}_{k_j}).$$

Proof. Assume that the element

$$a = \sum_{i=1}^m n_i a_i^{(1)} + \sum_{j=m+1}^n n_j a_j^{(0)}$$

is in $Ker(w_*^0)$. Then it satisfies

$$\left(\sum_{i=1}^m \sum_{j=m+1}^n n_i N_{k_j}^{W_0}(a_i^{(1)}, a_j^{(0)}) + \sum_{j=m+1}^n n_j \right) a_j^{(0)} = 0.$$

Here we use the condition, proved in Section 6, that for sufficiently small $\epsilon > 0$ we have

$$\langle a_j^{(0)}, w_*^0(a_k^{(0)}) \rangle = \delta_{ik}.$$

The above condition can be rewritten as

$$n_j = -\sum_{i=1}^{m} n_i N_{k_j}^{W_0}(a_i^{(1)}, a_j^{(0)}) \text{ for all } j = m+1, \ldots, n.$$

Notice that this means that we can choose arbitrarily the first $\{n_i\}_{i=1,\ldots m}$ coefficients and the remaining $\{n_j\}_{j=m+1,\ldots,n}$ are determined by the above relation.

◇

We then derive the exactness result as follows.

Proposition 6.13 *The inclusion $Ker(w_*^0) \subset Im(w_*^1)$ is satisfied.*

Proof. Suppose given an element $a \in Ker(w_*^0)$. By Lemma 6.12, for sufficiently small $\epsilon > 0$ in the surgery perturbation, we can write

$$a = \sum_{i=1}^{m} n_i a_i^{(1)} + \sum_{j=m+1}^{n} n_j a_j^{(0)}$$

with n_i and n_j in $\mathbb{Z}$, satisfying

$$n_j = -\sum_{i=1}^{m} n_i N_{k_j}^{W_0}(a_i^{(1)}, a_j^{(0)}).$$

Consider the element

$$a_1 = \sum_{i=1}^{m} n_i a_i^{(1)},$$

with the coefficients n_i as above. We have

$$w_*^1(a_1) = \sum_{i=1}^{m} n_i a_i^{(1)} + \sum_{j=m+1}^{n} \sum_{i=1}^{m} n_i N_{\ell_{ij}}^{W_1}(a_i^{(1)}, a_j^{(0)}).$$

The relation

$$N_{\ell_{ij}}^{W_1}(a_i^{(1)}, a_j^{(0)}) = -N_{k_j}^{W_0}(a_i^{(1)}, a_j^{(0)})$$

implies that we have

$$a = w_*^1(a_1).$$

◇

The result of Proposition 6.13, together with the relation $w^0_* \circ w^1_* = 0$ proved in the previous section, implies that we have exactness in the middle term of the sequence

$$0 \to C_q(Y_1) \stackrel{w^1_q}{\to} C_q(Y,\mu) \stackrel{w^0_q}{\to} \bigoplus_{\mathfrak{s}_k} C_{(q)}(Y_0,\mathfrak{s}_k) \to 0.$$

7 The connecting homomorphism

We have established the exactness of the sequence

$$0 \to C_*(Y_1) \stackrel{w^1_*}{\to} C_*(Y,\mu) \stackrel{w^0_*}{\to} \oplus_k C_{(*)}(Y_0,\mathfrak{s}_k) \to 0.$$

Now we can give a more precise description of the connecting homomorphism in the induced long exact sequence of Floer homologies. As we are going to see, the connecting homomorphism is described in terms of the discrepancy between the boundary operator ∂_Y of the Floer complex $C_*(Y,\mu)$ and the operator $\partial_{Y_1} \oplus \partial_{Y_0}$, with $\partial_{Y_0} = \oplus_k \partial_{Y_0,k}$ on

$$C_*(Y_1) \oplus \oplus_k C_{(*)}(Y_0,\mathfrak{s}_k).$$

More explicitly, let us identify again the points of $\mathcal{M}^*_{Y,\mu}$ with

$$\mathcal{M}^*_{Y,\mu} = \{a^{(1)}_i\}_{i=1,\dots m} \cup \{a^{(0)}_j\}_{i=m+1,\dots n},$$

as in (28). Consider a cycle $\sum_j x_j a^{(0)}_j$ in $\oplus_k C_{(*)}(Y_0,\mathfrak{s}_k)$. We have

$$\partial_{Y_0}(\sum_j x_j a^{(0)}_j) = 0.$$

Now consider the corresponding element $\sum_j x_j a^{(0)}_j(\epsilon)$ in $C_*(Y,\mu)$. We have the following result.

Lemma 7.1 *Given a cycle $\sum_j x_j a^{(0)}_j$ in $C_{(*)}(Y_0,\mathfrak{s}_k)$, we have*

$$\begin{array}{l} \partial_{Y,\mu}(\sum_j x_j a^{(0)}_j(\epsilon)) = \\ \sum_{i,j} x_j n_{Y,\mu}(a^{(0)}_j(\epsilon), a^{(1)}_i(\epsilon)) a^{(1)}_i(\epsilon) \\ - \sum_{r,j,p} x_j n_{Y,\mu}(a^{(0)}_j(\epsilon), a^{(1)}_r(\epsilon)) N^{W_0}_k(a^{(1)}_r(\epsilon), a^{(0)}_p) a^{(0)}_p(\epsilon). \end{array} \tag{39}$$

Here the first sum is over $a_i^{(1)}$ in $\mathcal{M}^*_{Y_1}$ with

$$\deg_{Y,\mu}(a_j^{(0)}(\epsilon)) - \deg_{Y,\mu}(a_i^{(1)}(\epsilon)) = 1$$

and the second sum is over $a_p^{(0)}$ in $\mathcal{M}_{Y_0}(\mathfrak{s}_k)$ satisfying

$$\deg_{Y_0,\mathfrak{s}_k}(a_j^{(0)}) - \deg_{Y_0,\mathfrak{s}_k}(a_p^{(0)}) = 1$$

and $a_r^{(1)}$ in $\mathcal{M}^*_{Y_1}$ satisfying

$$\deg_{Y,\mu}(a_j^{(0)}(\epsilon)) - \deg_{Y,\mu}(a_r^{(1)}(\epsilon)) = 1$$

and

$$\iota(W_0, \mathfrak{s}_k, a_i^{(1)}(\epsilon), a_p^{(0)}) = 0.$$

Proof. It is sufficient to check that we have

$$\begin{aligned}
&\pi \circ \partial_{Y,\mu}(a_j^{(0)}(\epsilon)) = \partial_{Y_0}(a_j^{(0)}) \\
&- \textstyle\sum_{r,j,p} x_j n_{Y,\mu}(a_j^{(0)}(\epsilon), a_r^{(1)}(\epsilon)) N_k^{W_0}(a_r^{(1)}(\epsilon), a_p^{(0)}) a_p^{(0)}(\epsilon),
\end{aligned}$$

where $\pi : C_q(Y,\mu) \to \oplus_k C_{(q)}(Y_0, \mathfrak{s}_k)$ is the projection induced from the identification of the moduli spaces in [6]. In order to prove this relation, consider 1-dimensional moduli spaces $\mathcal{M}_k^{W_0}(a_j^{(0)}(\epsilon), a_p^{(0)})$ and their compactification. We have boundary strata

$$\begin{aligned}
&\partial \mathcal{M}_k^{W_0}(a_j^{(0)}(\epsilon), a_p^{(0)}) = \\
&\textstyle\bigcup_{a_s^{(0)}} \hat{\mathcal{M}}_{Y,\mu}(a_j^{(0)}(\epsilon), a_s^{(0)}(\epsilon)) \times \mathcal{M}_k^{W_0}(a_s^{(0)}(\epsilon), a_p^{(0)}) \cup \\
&\textstyle\bigcup_{a_i^{(1)}} \mathcal{M}_{Y,\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)) \times \mathcal{M}_k^{W_0}(a_i^{(1)}(\epsilon), a_p^{(0)}) \cup \\
&\textstyle\bigcup_{a_s^{(0)}} \mathcal{M}_k^{W_0}(a_j^{(0)}(\epsilon), a_s^{(0)}) \times \hat{\mathcal{M}}_{Y_0,\mathfrak{s}_k}(a_s^{(0)}, a_p^{(0)}).
\end{aligned}$$

Using the results of Corollary 5.2 and 5.4 we can identify these expressions with

$$\begin{aligned}
&\mathcal{M}_{Y,\mu}(a_j^{(0)}(\epsilon), a_p^{(0)}(\epsilon)) \cup \\
&\textstyle\bigcup_{a_i^{(1)}} \mathcal{M}_{Y,\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)) \times \mathcal{M}_k^{W_0}(a_i^{(1)}(\epsilon), a_p^{(0)}) \cup \\
&\hat{\mathcal{M}}_{Y_0,\mathfrak{s}_k}(a_j^{(0)}, a_p^{(0)}).
\end{aligned}$$

When keeping into account the orientations, this yields the desired formula. $\diamond$

Using this result we obtain the following.

Lemma 7.2 *Suppose given a cycle in* $\sum_j x_j a_j^{(0)}$ *in* $C_{(*)}(Y_0, \mathfrak{s}_k)$*. The image of* $\sum_j x_j a_j^{(0)}$ *under the connecting homomorphism* Δ *is given by*

$$\Delta(\sum_j x_j a_j^{(0)}) = \sum_j x_j n_{Y,\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)) a_i^{(1)},$$

that is, by the algebraic counting of flowlines in

$$\hat{\mathcal{M}}_{Y\times\mathbb{R},\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)),$$

for $\deg_{Y,\mu}(a_j^{(0)}(\epsilon)) - \deg_{Y,\mu}(a_i^{(1)}(\epsilon)) = 1$.

Proof. Using the result of Theorem 6.9 we can write (39) equivalently as

$$\begin{aligned}&\partial_{Y,\mu}(\textstyle\sum_j x_j a_j^{(0)}(\epsilon) = \\ &\textstyle\sum_{i,j} x_j n_{Y,\mu}(a_j^{(0)}, a_i^{(1)}) a_i^{(1)}(\epsilon) \\ &+ \textstyle\sum_{i,j,p} x_j n_{Y,\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)) N_\ell^{W_1}(a_i^{(1)}, a_p^{(0)}(\epsilon)) a_p^{(0)}(\epsilon).\end{aligned}$$

By comparing this expression with

$$\begin{aligned}&w_*^1(\textstyle\sum_j x_j n_{Y,\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)) a_i^{(1)}) = \\ &\textstyle\sum_j x_j n_{Y,\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)) a_i^{(1)}(\epsilon) + \\ &\textstyle\sum_j x_j n_{Y,\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)) N_\ell^{W_1}(a_i^{(1)}, a_p^{(0)}(\epsilon)) a_p^{(0)}(\epsilon),\end{aligned}$$

we obtain

$$\Delta(\sum_j x_j a_j^{(0)}) = \sum_j x_j n_{Y,\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)) a_i^{(1)}.$$

This completes the proof.
$\diamond$

7.1 The surgery triangle

In this section we give a different description of the connecting homomorphism. This will prove that, in the $\epsilon \to 0$ limit of the surgery perturbation, the exact triangle for Seiberg-Witten Floer homology is a surgery triangle, that is, the connecting homomorphism in the exact sequence can also be described as a map $\bar{w}^2_*$ induced by a surgery cobordism $\bar{W}_2$, and the resulting diagram

$$C_*(Y_1) \stackrel{w^1_*}{\to} C_*(Y,\mu) \stackrel{w^0_*}{\to} \oplus_k C_{(*)}(Y_0,\mathfrak{s}_k) \stackrel{\bar{w}^2_*}{\to} C_*(Y_1)[-1]$$

is a distinguished triangle, cf. [9], [2], [24].

Proposition 7.3 *The connecting homomorphism Δ in the exact triangle is given by the following expression,*

$$\Delta(\sum_j x_j a^{(0)}_j) = \bar{w}^2_*(\sum_j x_j a^{(0)}_j),$$

for any cycle $\sum_j x_j a^{(0)}_j$ in $C_{(*)}(Y_0,\mathfrak{s}_k)$, for some $\mathfrak{s}_k$, where

$$\bar{w}^2_* : \oplus_k C_{(*)}(Y_0,\mathfrak{s}_k) \to C_*(Y,\mu)$$

is the homomorphism defined by counting solutions in the zero-dimensional components of the moduli spaces

$$\mathcal{M}^{\bar{W}_2}_k(a^{(0)}_j, a^{(1)}_i),$$

over the cobordism $\bar{W}_2$.

Proof. We only need to prove that the algebraic counting of flow lines in

$$\hat{\mathcal{M}}_{Y(r)\times\mathbb{R},\mu}(a^{(0)}_j(\epsilon), a^{(1)}_i(\epsilon)),$$

agrees with the algebraic counting of monopoles in the zero-dimensional components of $\mathcal{M}^{\bar{W}_2}_k(a^{(0)}_j, a^{(1)}_i)$.

We begin with a few observations on the topology of the cobordisms. Recall that we have $\bar{W} = \bar{W}_0 \cup_Y \bar{W}_1$. Moreover, we can write $\bar{W} = \bar{W}_2 \# \mathbb{C}P^2$, where the generator of the homology of the $\mathbb{C}P^2$ summand is the surface $S^2 = D^2_0 \cup_K D^2_1$, with D^2_i the core handles of the cobordisms $\bar{W}_0$ and $\bar{W}_1$. In our analysis of the geometric limits of solutions, we have considered the punctured cobordisms $\hat{W}_1$ and $\hat{W}_0$. It is convenient here to consider the punctured $\overline{\hat{W}_1} = \bar{W}_1 \backslash \{x_1\}$ and $\overline{\hat{W}_0} = \bar{W}_0 \backslash \{x_0\}$, obtained by removing points $\{x_i\}$ inside the core disks D^2_i. Now consider the doubly punctured space

$$\bar{W} \backslash \{x_0, x_1\} \cong \overline{\hat{W}_1} \cup_Y \overline{\hat{W}_0},$$

with a metric that has cylindrical ends $S^3 \times [0, \infty)$ at the punctures and stretched regions $T^2 \times [-r, r]$ inside the standard Heegaard splittings of these S^3 and along the product regions connecting the solid tori in the Heegaard splittings of S^3 to the tubular neighborhoods of the knot in Y_1, Y, and Y_0. Then the sphere $S^2 = D^2_0 \cup_K D^2_1$ corresponds to a cylinder

$$S^2 \backslash \{x_0, x_1\} \cong S^1 \times \mathbb{R}$$

in $\bar{W} \backslash \{x_0, x_1\}$.

Thus, when removing the points x_0 and x_1, the $\mathbb{C}P^2$ summand in $\bar{W}$ becomes a disk bundle $\mathcal{S}$ over the cylinder $S^1 \times \mathbb{R}$, which, with respect to a fixed trivialization at the two ends, has Euler number $+1$. In turn, the doubly punctured cobordism $\bar{W}_2 \backslash \{x_0, x_1\}$ is obtained by replacing this disk bundle with the trivial disk bundle $\mathcal{S}_0 = D^2 \times S^1 \times \mathbb{R}$.

Now we proceed to compare the geometric limits of solutions in

$$\mathcal{M}_{Y \times \mathbb{R}, \mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)) \quad \text{and} \quad \mathcal{M}_k^{\bar{W}_2}(a_j^{(0)}, a_i^{(1)}).$$

There are two distinct cases: these correspond to (d) and (e) of Figure 4.
Case 1. Consider geometric limits of solutions on the doubly punctured cobordism $\bar{W} \backslash \{x_0, x_1\}$. By the previous analysis of the geometric limits of solutions on the punctured cobordisms W_1 and W_0, together with the gluing theorem for $\mathcal{M}_{\ell,k}^{\bar{W}}(a_j^{(0)}, a_i^{(1)})$, which in this case gives

$$\begin{aligned}
&\mathcal{M}_{\ell,k}^{\bar{W}}(a_j^{(0)}, a_i^{(1)}) \cong \\
&\mathcal{M}_k^{\bar{W}_0}(a_j^{(0)}, a_i^{(1)}(\epsilon)) \times \mathcal{M}_\ell^{\bar{W}_1}(a_i^{(1)}(\epsilon), a_i^{(1)}) \cup \\
&\mathcal{M}_k^{\bar{W}_0}(a_j^{(0)}, a_j^{(0)}(\epsilon)) \times \mathcal{M}_\ell^{\bar{W}_1}(a_j^{(0)}(\epsilon), a_i^{(1)})
\end{aligned}$$

for zero-dimensional moduli spaces, we obtain geometric limits as in Figure 4, (a), (b). These reduce to solutions on the once punctured $\bar{W}_2 \backslash \{x_2\}$ that have geometric limits as in Figure 4 (d). The counting

$$N_k^{\bar{W}_0}(a_j^{(0)}, a_i^{(1)}(\epsilon)) + N_\ell^{\bar{W}_1}(a_j^{(0)}(\epsilon), a_i^{(1)}) = 0$$

implies that the total counting of these solutions in the zero-dimensional moduli space $\mathcal{M}_k^{\bar{W}_2}(a_j^{(0)}, a_i^{(1)})$ is zero. Notice that, in this case we have

$$\mathcal{M}_{Y(r) \times \mathbb{R}, \mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)) = \emptyset,$$

therefore we also have $n_{Y,\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)) = 0$.

Case 2. In this case, the zero-dimensional moduli space $\mathcal{M}_k^{\bar{W}_2}(a_j^{(0)}, a_i^{(1)})$ is obtained from approximate solutions of the form

$$\hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)\#\Delta,$$

with Δ a triangle as in Figure 4 (e), and $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ is the admissible triple with respect to $(a_j^{(0)}, a_i^{(1)})$. (The corresponding holomorphic triangle for geometric limits on $\bar{W}-\{x_0,x_1\}$ is illustrated in Figure 4 (c).) This means that, in forming the preglued solutions, elements in the moduli spaces

$$\mathcal{M}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$$

that differ by a translation will give rise to the same solution on $\bar{W}_2$. Notice that geometric limits for $\mathcal{M}_{Y(r)\times\mathbb{R}}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon))$ are given by

$$\mathcal{M}_{V\times\mathbb{R}}([A,\psi]^\epsilon,[\tilde{A},\tilde{\psi}]^\epsilon,a_\infty)\#\Delta^\epsilon$$

with Δ^ϵ a holomorphic disk in $H^1(T^2,\mathbb{R})$ with boundary along arcs of ℓ_μ and ℓ connecting the asymptotic limits of $a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)$ respectively, the corresponding admissible triple is $([A,\psi]^\epsilon,[\tilde{A},\tilde{\psi}]^\epsilon,a_\infty)$ for $(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon))$.

As we let $\epsilon\to 0$ these geometric limits approximate geometric limits

$$\hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)\#\Delta$$

of $\mathcal{M}_k^{\bar{W}_2}(a_j^{(0)}, a_i^{(1)})$. Under the gluing map, we obtain that the algebraic counting of solutions in the zero-dimensional moduli space $\mathcal{M}_k^{\bar{W}_2}(a_j^{(0)}, a_i^{(1)})$ agrees with the algebraic counting of flow lines in $\hat{\mathcal{M}}_{Y\times\mathbb{R},\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon))$,

$$N_k^{\bar{W}_2}(a_j^{(0)}, a_i^{(1)}) = n_{Y,\mu}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon)).$$

This completes the proof.

$\diamond$

Let us recall briefly the following definitions of homological algebra, cf. [13].

A triangle (in a category of complexes) is a diagram of the form

$$K_\bullet \xrightarrow{u} L_\bullet \xrightarrow{v} M_\bullet \xrightarrow{w} K_\bullet[-1].$$

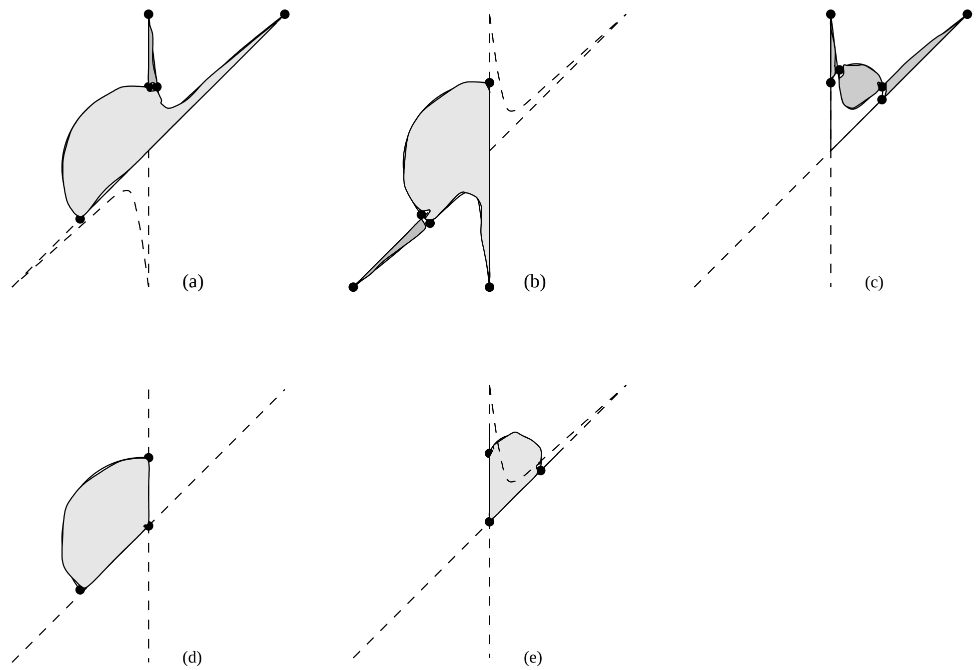

Figure 4: triangles on $\bar{W}_2$

Here we take complexes with differentials of degree (-1) for consistency with our notation. $K_\bullet[n]$ is the n-th shift with $K_i[n] = K_{i+n}$, and $d_{K[n]} = (-1)^n d_K$. A triangle is *distinguished* if it is *quasi–isomorphic* to a diagram of the form

$$K_\bullet \xrightarrow{\tilde{u}} \mathrm{Cyl}(u) \xrightarrow{\pi} \mathrm{C}(u) \xrightarrow{\delta} K_\bullet[-1].$$

Here $\mathrm{Cyl}(u)$ and $\mathrm{C}(u)$ are the cylinder and the cone on the morphism u, defined as in [13] pg.154 (up to adjusting notations to differentials of degree (-1)), and δ is the connecting homomorphism.

With this notation and the previous results, we can rephrase our main result as the following Proposition.

Proposition 7.4 *For sufficiently small $\epsilon \leq \epsilon_0$ in the surgery perturbation μ, the sequence*

$$C_*(Y_1) \xrightarrow{w^1_*} C_*(Y,\mu) \xrightarrow{w^0_*} \oplus_k C_{(*)}(Y_0,\mathfrak{s}_k) \xrightarrow{\bar{w}^2_*} C_*(Y_1)[-1]$$

is a distinguished triangle.

Proof. The result follows from the previous results, together with Proposition 5, pg.157 of [13]. In fact, we have proven that, as $\epsilon \to 0$, the sequence

$$0 \to C_*(Y_1) \stackrel{w^1_*}{\to} C_*(Y, \mu) \stackrel{w^0_*}{\to} \oplus_k C_{(*)}(Y_0, \mathfrak{s}_k) \to 0$$

is an exact triple of complexes ([13] pg. 42). Proposition 5, pg.157 of [13] then gives the result.
$\diamond$

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Matilde Marcolli, Max–Planck–Institut für Mathematik, D-53111 Bonn, Germany. marcolli@mpim-bonn.mpg.de

Bai-Ling Wang, Department of Pure Mathematics, University of Adelaide, Adelaide SA 5005. bwang@maths.adelaide.edu.au
Max–Planck–Institut für Mathematik, D-53111 Bonn, Germany.
bwang@mpim-bonn.mpg.de