Variants of Equivariant Seiberg-Witten Floer Homology

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Abstract

For a rational homology 3-sphere Y with a Spin^c structure $\mathfrak{s}$, we show that simple algebraic manipulations of our construction of equivariant Seiberg-Witten Floer homology in [5] lead to a collection of variants $HF^{SW,-}_{*,U(1)}(Y,\mathfrak{s})$, $HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$ $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$, $\widehat{HF}^{SW}_{*}(Y,\mathfrak{s})$ and $HF^{SW}_{red,*}(Y,\mathfrak{s})$ which are topological invariants. We establish a long exact sequence relating $HF^{SW,\pm}_{*,U(1)}(Y,\mathfrak{s})$ and $HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$. We show they satisfy a duality under orientation reversal, and we explain their relation to the equivariant Seiberg-Witten Floer (co)homologies introduced in [5]. We conjecture the equivalence of these versions of equivariant Seiberg-Witten Floer homology with the Heegaard Floer invariants introduced by Ozsváth and Szabó.

Key words: rational homology 3-spheres, equivariant Seiberg-Witten Floer homology, Spin^c structures, topological invariants.

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1 Introduction

For any rational homology 3-sphere Y with a Spin^c structure $\mathfrak{s}$, we constructed in [5] an equivariant Seiberg-Witten Floer homology $HF^{SW}_{*,U(1)}(Y,\mathfrak{s})$, which is a topological invariant. In this paper, we will generalize this construction to provide a collection of equivariant Seiberg-Witten Floer homologies $HF^{SW,-}_{*,U(1)}(Y,\mathfrak{s}), HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s}), HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}), \widehat{HF}^{SW}_{*}(Y,\mathfrak{s})$ and $HF^{SW}_{red,*}(Y,\mathfrak{s})$, all of which are topological invariants, such that $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$ is isomorphic to the equivariant Seiberg-Witten Floer homology $HF^{SW}_{*,U(1)}(Y,\mathfrak{s})$ constructed in [5]. The construction utilizes the $U(1)$-invariant forms on $U(1)$-manifolds twisted with coefficients in the Laurent polynomial algebra over integers.

In analogy to Austin and Braam's construction of equivariant instanton Floer homology in [1], the equivariant Seiberg-Witten Floer homology $HF^{SW}_{*,U(1)}(Y,\mathfrak{s})$ is the homology of the complex

$(CF^{SW}_{*,U(1)}(Y,\mathfrak{s}), D)$, where $CF^{SW}_{*,U(1)}(Y,\mathfrak{s})$ is generated by equivariant de Rham forms over all $U(1)$-orbits of the solutions of 3-dimensional Seiberg-Witten equations on $(Y,\mathfrak{s})$ modulo based gauge transformations (Cf.[5]). More specifically,

$$CF^{SW}_{*,U(1)}(Y,\mathfrak{s}) = \bigoplus_{a\in\mathcal{M}^*_Y(\mathfrak{s})} \mathbb{Z}[\Omega]\otimes(\mathbb{Z}\eta_a\oplus\mathbb{Z}1_a)\oplus\mathbb{Z}[\Omega]\otimes\mathbb{Z}1_\theta, \tag{1}$$

where $\mathcal{M}_Y(\mathfrak{s}) = \mathcal{M}^*_Y(\mathfrak{s})\cup\{\theta\}$ is the equivalence classes of solutions to the Seiberg-Witten equations for a good pair of metric and perturbations, consists of the irreducible monopoles $\mathcal{M}^*_Y(\mathfrak{s})$ and the unique reducible monopole θ. We used the notation η_a to denote a 1-form on $O_a\cong S^1$, such that the cohomology class $[\eta_a]$ is an integral generator of $H^1(O_a)$. Similarly, we denote by 1_a the 0-form given by the constant function.

Each generator is endowed with a grading such that, for any $k\geq 0$,

$$gr(\Omega^k\otimes\eta_a) = 2k+gr(a),\ gr(\Omega^k\otimes 1_a) = 2k+gr(a)+1,\ and\ \ gr(\Omega^k\otimes 1_\theta) = 2k, \tag{2}$$

where $gr : \mathcal{M}^*_Y(\mathfrak{s})\to\mathbb{Z}$ is the relative grading with respect to the reducible monopole θ. This corresponds to grading equivariant de Rham forms on each orbit O_a by codimension (Cf.[5] §5 for details).

The differential operator D can be expressed explicitly in components as the form:

$$\begin{aligned}
D(\Omega^k\otimes\eta_a) &= \sum_{\substack{b\in\mathcal{M}^*_Y(\mathfrak{s})\\ gr(a)-gr(b)=1}} n_{ab}\Omega^k\otimes\eta_b + \sum_{\substack{c\in\mathcal{M}^*_Y(\mathfrak{s})\\ gr(a)-gr(c)=2}} m_{ac}\Omega^k\otimes 1_c - \Omega^{k-1}\otimes 1_a \\
&\qquad + n_{a\theta}\Omega^k\otimes 1_\theta (\text{if gr (a) =1});\\
D(\Omega^k\otimes 1_a) &= -\sum_{\substack{b\in\mathcal{M}^*_Y(\mathfrak{s})\\ gr(a)-gr(b)=1}} n_{ab}\Omega^k\otimes 1_b;\\
D(\Omega^k\otimes 1_\theta) &= \sum_{\substack{d\in\mathcal{M}^*_Y(\mathfrak{s})\\ gr(d)=-2}} n_{\theta d}\Omega^k\otimes 1_d.
\end{aligned} \tag{3}$$

where $n_{ab}, n_{a\theta}$ and $n_{\theta d}$ is the counting of flowlines from a to b (if $gr(a)-gr(b)=1$), from a to θ (if $gr(a)=1$) and from θ to d (if $gr(d)=-2$), and m_{ac} (if $gr(a)-gr(c)=2$) is described as a relative Euler number associated to the 2-dimensional moduli space of flowlines from a to c (Cf. Lemma 5.7 of [5]). In the next section, we shall briefly review the construction and various relations among the coefficients, as established in [5]. These identities ensure that $D^2=0$. Notice that, in the complex $CF^{SW}_{*,U(1)}(Y,\mathfrak{s})$ and in the expression of the differential operator, only terms with non-negative powers of Ω are considered. We modify the construction as follows.

Definition 1.1. *Let* $CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$ *be the graded complex generated by*

$$\{\Omega^k \otimes \eta_a, \Omega^k \otimes 1_a, \Omega^k \otimes 1_\theta : a \in \mathcal{M}^*_Y(\mathfrak{s}), k \in \mathbb{Z}.\}$$

with the grading gr and the differential operator D given by (2) and (3) respectively. Let $CF^{SW,-}_{*,U(1)}(Y,\mathfrak{s})$ *be the subcomplex of* $CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$*, generated by those generators with negative power of* Ω*. The quotient complex is denoted by* $CF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$*. Their homologies are denoted by* $HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$*,* $HF^{SW,-}_{*,U(1)}(Y,\mathfrak{s})$ *and* $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$ *respectively.*

The main results in this paper relate these homologies to the equivariant Seiberg-Witten-Floer homology $HF^{SW}_{*,U(1)}(Y,\mathfrak{s})$ and cohomology $HF^{SW,*}_{U(1)}(Y,\mathfrak{s})$ constructed in [5] and establish some of their main properties.

Theorem 1.2. *For any rational homology 3-sphere Y with a* Spin^c *structure* $\mathfrak{s} \in \mathrm{Spin}^c(Y)$*, these homologies satisfy the following properties:*

1. $HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s}) \cong \mathbb{Z}[\Omega, \Omega^{-1}]$.

2. $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}) \cong HF^{SW}_{*,U(1)}(Y,\mathfrak{s})$ *where* $HF^{SW}_{*,U(1)}(Y,\mathfrak{s})$ *is the equivariant Seiberg-Witten Floer homology for* $(Y,\mathfrak{s})$ *constructed in [5].*

3. $HF^{SW,-}_{*,U(1)}(Y,\mathfrak{s}) \cong HF^{SW,*}_{U(1)}(-Y,\mathfrak{s})$ *where* $HF^{SW,*}_{U(1)}(-Y,\mathfrak{s})$ *is the equivariant Seiberg-Witten Floer cohomology for* $(-Y,\mathfrak{s})$ *constructed in [5].*

4. *There exists a long exact sequence*

$$\cdots \to HF^{SW,-}_{*,U(1)}(Y,\mathfrak{s}) \xrightarrow{l_*} HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s}) \xrightarrow{\pi_*} HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}) \xrightarrow{\delta_*} HF^{SW,-}_{*-1,U(1)}(Y,\mathfrak{s}) \to \cdots \quad (4)$$

   relating these homologies. Moreover, $HF^{SW,-}_{*,U(1)}(Y,\mathfrak{s})$*,* $HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$*,* $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$ *and* $HF^{SW}_{red,*}(Y,\mathfrak{s}) = Coker(\pi_*) \cong Ker(l_{*-1})$ *are all topological invariants of* $(Y,\mathfrak{s})$*.*

5. *There is a u-action on* $HF^{SW,-}_{*,U(1)}(Y,\mathfrak{s})$*,* $HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$ *and* $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$ *respectively which decreases the degree by two, and is related to the cutting down moduli spaces of flowlines by a geometric representative of a degree 2 characteristic form. The long exact sequence (4) is a long exact sequence of* $\mathbb{Z}[u]$*-modules.*

6. *There is a homology group* $\widehat{HF}_*^{SW}(Y,\mathfrak{s})$*, which is also a topological invariant of* $(Y,\mathfrak{s})$*, such that the following sequence is exact:*

$$\cdots \to \widehat{HF}_*^{SW}(Y,\mathfrak{s}) \longrightarrow HF_{*,U(1)}^{SW,+}(Y,\mathfrak{s}) \xrightarrow{u} HF_{*-2,U(1)}^{SW,+}(Y,\mathfrak{s}) \longrightarrow \widehat{HF}_{*-1}^{SW}(Y,\mathfrak{s}) \to \cdots \tag{5}$$

and that $\widehat{HF}^{SW}(Y,\mathfrak{s})$ *is non-trivial if and only if* $HF_{*,U(1)}^{SW,+}(Y,\mathfrak{s})$ *is non-trivial.*

The u-action in the main theorem is induced from a u-action on the chain complex

$$u: \qquad CF_{*,U(1)}^{SW,\infty} \to CF_{*,U(1)}^{SW,\infty},$$

which decreases the degree by 2. We will show that this u-action is homotopic to the obvious Ω^{-1} action on the chain complex $CF_{*,U(1)}^{SW,\infty}$. Thus, the induced u-action on $HF_{*,U(1)}^{SW,\pm}(Y,\mathfrak{s})$ endows them with $\mathbb{Z}[u]$-module structures.

Let $\widehat{CF}_*^{SW}(Y,\mathfrak{s})$ be the subcomplex of $CF_{*,U(1)}^{SW,+}(Y,\mathfrak{s})$ such that the following sequence is a short exact sequence of chain complexes:

$$0 \to \widehat{CF}_*^{SW}(Y,\mathfrak{s}) \longrightarrow CF_{*,U(1)}^{SW,+}(Y,\mathfrak{s}) \xrightarrow{\Omega^{-1}} CF_{*,U(1)}^{SW,+}(Y,\mathfrak{s}) \to 0$$

We can define $\widehat{HF}_*^{SW}(Y,\mathfrak{s})$ to be the homology of $\widehat{CF}_*^{SW}(Y,\mathfrak{s})$.

In recent work [7] [8], Ozsváth and Szabó introduced Heegaard Floer invariants $HF_*^{\pm}(Y,\mathfrak{s})$, $HF_*^{\infty}(Y,\mathfrak{s})$, $\widehat{HF}_*(Y,\mathfrak{s})$, and $HF_{red,*}(Y,\mathfrak{s})$, with exact sequences relating them. In view of their construction, the result of Theorem 1.2, together with the identification of our $HF_{*,U(1)}^{SW,\infty}(Y,\mathfrak{s})$ and the $HF_*^{\infty}(Y,\mathfrak{s})$ of Ozsváth and Szabó, suggest the following conjecture.

Conjecture 1.3. *For any rational homology 3-sphere* Y *with a* Spin^c *structure* $\mathfrak{s} \in \mathrm{Spin}^c(Y)$*, there are isomorphisms*

$$HF_{*,U(1)}^{SW,+}(Y,\mathfrak{s}) \cong HF_*^{+}(Y,\mathfrak{s}), \qquad HF_{*,U(1)}^{SW,-}(Y,\mathfrak{s}) \cong HF_*^{-}(Y,\mathfrak{s});$$

$$\widehat{HF}_*^{SW}(Y,\mathfrak{s}) \cong \widehat{HF}_*(Y,\mathfrak{s}), \qquad HF_{red,*}^{SW}(Y,\mathfrak{s}) \cong HF_{red,*}(Y,\mathfrak{s}).$$

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2 Review of equivariant Seiberg-Witten Floer homology

In this section, we recall some of basic ingredients in the definition of the equivariant Seiebrg-Witten Floer homology from [5] (See [5] for all the details).

Let $(Y, \mathfrak{s})$ be a rational homology 3-sphere Y with a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. For a good pair of metric and perturbation (a co-closed imaginary-valued 1-form ν) on Y, the 3-dimensional Seiebrg-Witten equations on $(Y, \mathfrak{s})$ (Cf. [2] [3] [4] [5]):

$$\begin{cases} *F_A = \sigma(\psi, \psi) + \nu \\ \partial\!\!\!/_A \psi = 0, \end{cases} \tag{6}$$

for a pair of Spin^c connection A and a spinor ψ, have only finitely many irreducible solutions (modulo the gauge transformations), denoted by $\mathcal{M}^*_Y(\mathfrak{s})$ the set of equivalence classes of irreducible solutions to (6), and θ is the unique reducible solution (modulo the gauge transformations). Write $\mathcal{M}_Y(\mathfrak{s}) = \mathcal{M}^*_Y(\mathfrak{s}) \cup \{\theta\}$.

Gauge classes of finite energy solutions to the 4-dimensional Seiebrg-Witten equations, perturbed as in [2] [3] [5], can be regarded as moduli spaces of flowlines of the Chern-Simons-Dirac functional on the gauge equivalence classes of Spin^c connections and spinors for $(Y, \mathfrak{s})$. These can be partitioned into moduli spaces of flowlines between pairs of critical points from $\mathcal{M}_Y(\mathfrak{s})$. Each is a smooth oriented manifold which can be compactfied to a smooth manifold with corners by adding broken flowlines that split through intermediate critical points.

The spectral flow of the Hessian operator of the Chern-Simons-Dirac functional defines a relative grading on $\mathcal{M}_Y(\mathfrak{s})$:

$$gr(\cdot, \cdot): \qquad \mathcal{M}_Y(\mathfrak{s}) \times \mathcal{M}_Y(\mathfrak{s}) \to \mathbb{Z}.$$

In particular, using the unique reducible point θ in $\mathcal{M}_Y(\mathfrak{s})$, there is a $\mathbb{Z}$-lifting of the relative grading given by $gr(a) = gr(a, \theta)$.

Let a be an irreducible monopole in $\mathcal{M}_Y(\mathfrak{s})$, then for any $b \neq a$ in $\mathcal{M}_Y(\mathfrak{s})$, the moduli space of flowlines from a to b, denoted by $\mathcal{M}(a, b)$ has dimension $gr(a) - gr(b) > 0$ (if non-empty). The moduli space of flowlines from θ to $d \in \mathcal{M}^*_Y(\mathfrak{s})$, denoted by $\mathcal{M}(\theta, d)$ has dimension $-gr(d) - 1 > 0$ (if non-empty). Note that all these moduli spaces of flowlines admit an $\mathbb{R}$-action given by the $\mathbb{R}$-translation of flowlines: the corresponding quotient spaces are denoted by $\widehat{\mathcal{M}}(a, b)$ and $\widehat{\mathcal{M}}(\theta, d)$, respectively.

For any irreducible critical points a and c in $\mathcal{M}_Y(\mathfrak{s})$ with $gr(a) - gr(c) = 2$, we can construct a canonical complex line bundle over $\mathcal{M}(a, c)$ and a canonical section as follows (see section 5.3 in [5]). Choose a base point (y_0, t_0) on $Y \times \mathbb{R}$, and a complex line ℓ_{y_0} in the fiber W_{y_0} of the spinor bundle W over $y_0 \in Y$. We choose ℓ_{y_0} so that it does not contain the spinor part ψ of any irreducible critical point. Since there are only finitely many critical points we can guarantee such choice exists. Denote the based moduli space of $\mathcal{M}(a, c)$ by $\mathcal{M}(O_a, O_c)$ as in [5], where O_a and O_c are the $U(1)$-orbits of monopoles on the based configuration space. We consider the line bundle

$$\mathcal{L}_{ac} = \mathcal{M}(O_a, O_c) \times_{U(1)} (W_{y_0}/\ell_{y_0}) \to \mathcal{M}(a, c) \tag{7}$$

with a section given by

$$s([A, \Psi]) = ([A, \Psi], \Psi(y_0, t_0)). \tag{8}$$

For a generic choice of (y_0, t_0) and ℓ_{y_0}, the section s of (8) has no zeroes on the boundary strata of the compactification of $\mathcal{M}(a, c)$. This determines a trivialization of $\mathcal{L}_{ac}$ away from a compact set in $\mathcal{M}(a, c)$. The line bundle $\mathcal{L}_{ac}$ over $\mathcal{M}(a, c)$, with the trivialization φ specified above, has a well-defined relative Euler class (Cf. Lemma 5.7 in [5]).

Definition 2.1. *1. For any two irreducible critical points a and b in $\mathcal{M}_Y(\mathfrak{s})$ with $gr(a) - gr(b) = 1$, we define $n_{ab} := \#(\hat{\mathcal{M}}(a, b))$, the number of flowlines in $\mathcal{M}(a, b)$ counting with orientations. Similarly, for any $a \in \mathcal{M}_Y(\mathfrak{s})$ with $gr(a) = 1$ and any $d \in \mathcal{M}_Y(\mathfrak{s})$ with $gr(d) = -2$, we define $n_{a\theta} := \#(\hat{\mathcal{M}}(a, \theta))$ and $n_{\theta d} := \#(\hat{\mathcal{M}}(\theta, b))$, respectively.*

2. For any two irreducible critical points a and c in $\mathcal{M}_Y(\mathfrak{s})$ with $gr(a) - gr(c) = 1$, we define m_{ac} to be the relative Euler number of the canonical line bundle $\mathcal{L}_{ac}$ (7) with the canonical trivialization given by (8).

The following proposition states various relations satisfied by the integers defined in Definition 2.1, whose proof can be found in Remark 5.8 of [5].

Proposition 2.2. *1. For any irreducible critical point a in $\mathcal{M}^*_Y(\mathfrak{s})$ and any critical point c in $\mathcal{M}_Y(\mathfrak{s})$ with $gr(a) - gr(c) = 2$, we have the following identity:*

$$\sum_{\substack{b \in \mathcal{M}^*_Y(\mathfrak{s}) \\ gr(a) - gr(b) = 1}} n_{ab} n_{bc} = 0.$$

2. Let a and d be two irreducible critical points with $gr(a) - gr(d) = 3$. Assume that all the critical points c with $gr(a) > gr(c) > gr(d)$ are irreducible. Then we have the identity

$$\sum_{c_1 : gr(a) - gr(c_1) = 1} n_{a,c_1} m_{c_1,d} - \sum_{c_2 : gr(c_2) - gr(d) = 1} m_{a,c_2} n_{c_2,d} = 0.$$

When $gr(a) = 1$ and $gr(d) = -2$, we have the identity

$$\sum_{c_1 : gr(c_1) = 0} n_{a,c_1} m_{c_1,d} + n_{a\theta} n_{\theta d} - \sum_{c_2 : gr(c_2) = -1} m_{a,c_2} n_{c_2,d} = 0.$$

With the help of this Proposition, we can check that the equivariant Seiberg-Witten-Floer complex $CF^{SW}_{*,U(1)}(Y, \mathfrak{s})$ as given in (1) with the grading and the differential operator given by (2) and (3) is well-defined, and we denote its homology by $HF^{SW}_{*,U(1)}(Y, \mathfrak{s})$. The equivariant Seiberg-Witten-Floer cohomology, denoted by $HF^{SW,*}_{U(1)}(Y, \mathfrak{s})$, is the homology of the dual complex $Hom(CF^{SW}_{*,U(1)}(Y, \mathfrak{s}), \mathbb{Z})$. The main result in [5] shows that the equivariant Seiberg-Witten Floer homology $HF^{SW}_{*,U(1)}(Y, \mathfrak{s})$ and cohomology $HF^{SW,*}_{U(1)}(Y, \mathfrak{s})$ are topological invariants of $(Y, \mathfrak{s})$.

3 Variants of equivariant Seiberg-Witten Floer homology

As mentioned in the introduction, we will generalize the construction of the equivariant Seiberg-Witten Floer homology in several ways.

First, we denote by $CF^{SW,\infty}_{*,U(1)}(Y, \mathfrak{s})$ the graded complex generated by

$$\{\Omega^k \otimes \eta_a, \Omega^k \otimes 1_a, \Omega^k \otimes 1_\theta : a \in \mathcal{M}^*_Y(\mathfrak{s}), k \in \mathbb{Z}.\}$$

More precisely, for any irreducible critical orbits O_a, we set

$$\begin{aligned} C^\infty_{*,U(1)}(O_a) &= \mathbb{Z}[\Omega, \Omega^{-1}] \otimes \Omega^*_0(O_a) \\ &:= \textstyle\bigoplus_{k \in \mathbb{Z}} \big(\mathbb{Z}\Omega^k \otimes \eta_a + \mathbb{Z}\Omega^k \otimes 1_a\big) \end{aligned}$$

with the grading $gr(\Omega^k \otimes \eta_a) = 2k + gr(a)$ and $gr(\Omega^k \otimes 1_a) = 2k + gr(a) + 1$, and we set

$$C^\infty_{*,U(1)}(\theta) = \bigoplus_{k \in \mathbb{Z}} \mathbb{Z}.\Omega^k \otimes 1_\theta$$

with the grading $gr(\Omega^k \otimes 1_\theta) = 2k$.

We then consider

$$CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s}) = \bigoplus_{a\in\mathcal{M}_Y(\mathfrak{s})} \mathbb{Z}[\Omega,\Omega^{-1}]\otimes\Omega_0^{*-\dim(O_a)}(O_a), \tag{9}$$

with the grading and the differential operator given by (2) and (3) respectively. That is, $CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$ is given by

$$\begin{aligned} & \bigoplus_{a\in\mathcal{M}_Y(\mathfrak{s})} C^{\infty}_{*,U(1)}(O_a) \\ = & \bigoplus_{a\in\mathcal{M}^*_Y(\mathfrak{s})} C^{\infty}_{*,U(1)}(O_a)\oplus C^{\infty}_{*,U(1)}(\theta). \end{aligned}$$

Theorem 3.1. *Define $HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$ to be the homology of $(CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s}),D)$. Then we have*

$$HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})\cong\mathbb{Z}[\Omega,\Omega^{-1}].$$

Proof. Consider the filtration of $CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$ according to the grading of the critical points

$$\mathcal{F}_n C^{\infty}_{*,U(1)} := \bigoplus_{gr(a)\leq n} C^{\infty}_{*,U(1)}(O_a)$$

the corresponding spectral sequence E^r_{kl}. The filtration is exhaustive, that is,

$$CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s}) = \bigcup_n \mathcal{F}_n C^{\infty}_{*,U(1)},$$

and

$$\cdots\subset\mathcal{F}_{n-1}C^{\infty}_{*,U(1)}\subset\mathcal{F}_n C^{\infty}_{*,U(1)}\subset\mathcal{F}_{n+1}C^{\infty}_{*,U(1)}\subset\cdots\subset CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s}).$$

Moreover, by the compactness of the moduli space of critical orbits, the set of indices $gr(a)$ is bounded from above and below, hence the filtration is bounded. Thus, the spectral sequence converges to $HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$.

We compute the E^0-term:

$$\begin{aligned} E^0_{kl} &= \mathcal{F}_k C^{\infty}_{k+l,U(1)}/\mathcal{F}_{k-1}C^{\infty}_{k+l,U(1)} \\ &= \bigoplus_{a\in\mathcal{M}_Y(\mathfrak{s}):gr(a)=i\leq k} C^{\infty}_{k+l-i,U(1)}(O_a)/\bigoplus_{a\in\mathcal{M}_Y(\mathfrak{s}):gr(a)=i\leq k-1} C^{\infty}_{k+l-i,U(1)}(O_a) \\ &= \bigoplus_{a\in\mathcal{M}_Y(\mathfrak{s}):gr(a)=k} C^{\infty}_{l,U(1)}(O_a). \end{aligned}$$

For $k \neq 0$ this complex is just the direct sum of the separate complexes $(C^{\infty}_{*,U(1)}(O_a), \partial_{U(1)})$ on each orbit O_a with $gr(a) = k$:

$$\cdots \to \mathbb{Z}.\Omega \otimes 1_a \xrightarrow{0} \mathbb{Z}.\Omega \otimes \eta_a \xrightarrow{-1} \mathbb{Z}.1 \otimes 1_a \xrightarrow{0} \mathbb{Z}.1 \otimes \eta_a \xrightarrow{-1} \mathbb{Z}.\Omega^{-1} \otimes 1_a \to \cdots \tag{10}$$

In the case $k = 0$ we have

$$E^0_{0,l} = C^{\infty}_{l,U(1)}(\theta) \oplus \bigoplus_{a \in \mathcal{M}^*_Y(\mathfrak{s}): gr(a)=0} C^{\infty}_{l,U(1)}(O_a),$$

which again is a direct sum of the complexes $(C^{\infty}_{*,U(1)}(O_a), \partial_{U(1)})$, here $\partial_{U(1)}$ is the equivariant de Rham differential, and of the complex with generators $\Omega^r \otimes 1_\theta\rangle$ in degree $l = 2r$ and trivial differentials.

We then compute the E^1_{pq} term directly: we have

$$E^1_{kl} = H_{k+l}(E^0_{k,*}) = \begin{cases} \mathbb{Z}.\Omega^r \otimes 1_\theta & k = 0, l = 2r \\ 0 & k \neq 0, \end{cases}$$

since each complex (10) is acyclic. Thus, the only non-trivial E^1-terms are of the form $E^1_{0l} = \mathbb{Z}.\Omega^r \otimes 1_\theta$, $l = 2r$, with trivial differentials, so that the spectral sequence collapses and we obtain the result. □

3.1 Long exact sequence

Definition 3.2. *Let $CF^{SW,-}_{*,U(1)}(Y, \mathfrak{s})$ be the subcomplex of $CF^{SW,\infty}_{*,U(1)}(Y, \mathfrak{s})$, generated by*

$$\{\Omega^k \otimes \eta_a, \Omega^k \otimes 1_a, \Omega^k \otimes 1_\theta : a \in \mathcal{M}^*_Y(\mathfrak{s}), k \in \mathbb{Z} \text{ and } k < 0\},$$

whose homology groups are denoted by $HF^{SW,-}_{*,U(1)}(Y, \mathfrak{s})$. The quotient complex is denoted by $CF^{SW,+}_{*,U(1)}(Y, \mathfrak{s})$, with the homology groups denoted by $HF^{SW,+}_{*,U(1)}(Y, \mathfrak{s})$.

Theorem 3.3. *1. $HF^{SW,+}_{*,U(1)}(Y, \mathfrak{s}) \cong HF^{SW}_{*,U(1)}(Y, \mathfrak{s})$, where $HF^{SW}_{*,U(1)}(Y, \mathfrak{s})$ is the equivariant Seiberg-Witten-Floer homology defined in [5].*

2. There is an exact sequence of $\mathbb{Z}$-modules which relates these variants of equivariant Seiberg-Witten-Floer homologies:

$$\cdots \to HF^{SW,-}_{*,U(1)}(Y, \mathfrak{s}) \xrightarrow{i_*} HF^{SW,\infty}_{*,U(1)}(Y, \mathfrak{s}) \xrightarrow{\pi_*} HF^{SW,+}_{*,U(1)}(Y, \mathfrak{s}) \xrightarrow{\delta_*} HF^{SW,-}_{*-1,U(1)}(Y, \mathfrak{s}) \to \cdots$$

Proof. It is easy to see that $CF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}) = CF^{SW}_{*,U(1)}(Y,\mathfrak{s})$, with the same grading and differentials, hence $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}) \cong HF^{SW}_{*,U(1)}(Y,\mathfrak{s})$. The long exact sequence in homology is induced by the short exact sequence of chain complexes:

$$0 \to CF^{SW,-}_{*,U(1)}(Y,\mathfrak{s}) \to CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s}) \to CF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}) \to 0.$$

$\square$

From the above long exact sequence, we can define

$$\begin{aligned} HF^{SW}_{red,*}(Y,\mathfrak{s}) &= Coker(\pi_*) \cong HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})/Ker(\delta_*) \\ &\cong Im(\delta_*) \cong Ker(l_{*-1}). \end{aligned} \tag{11}$$

3.2 The spectral sequence for $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$

We consider again the filtration by index of critical orbits,

$$\mathcal{F}_n C^+_{*,U(1)} := \bigoplus_{\mathrm{gr}(a)\leq n} C^+_{*,U(1)}(O_a),$$

for

$$C^+_{*,U(1)}(O_a) = \mathbb{Z}[\Omega] \otimes \Omega_0^{*-\dim(O_a)}(O_a).$$

We have

$$\begin{aligned} E^0_{kl} = &\ \mathcal{F}_k C^+_{k+l,U(1)}/\mathcal{F}_{k-1} C^+_{k+l,U(1)} \\ = &\ \textstyle\bigoplus_{\mathrm{gr}(a)=k} C^+_{l,U(1)}(O_a). \end{aligned}$$

This is a direct sum of the complexes

$$\cdots \stackrel{-1}{\to} \mathbb{Z}.\Omega \otimes 1_a \stackrel{0}{\to} \mathbb{Z}.\Omega \otimes \eta_a \stackrel{-1}{\to} \mathbb{Z}.1 \otimes 1_a \stackrel{0}{\to} \mathbb{Z}.1 \otimes \eta_a \to 0, \tag{12}$$

over each orbit $O_a \cong S^1$ and, in the case $k = 0$, the complex with generators $\Omega^r \otimes 1_\theta$ in degree $l = 2r$, and trivial differentials.

Thus, we obtain that $E^1_{pq} = H_{p+q}(E^0_{p*})$ is of the form

$$E^1_{pq} = \begin{cases} 0 & q > 0 \\ \mathbb{Z}.1 \otimes \eta_a & q = 0, \mathrm{gr}(a) = k \end{cases}$$

for $k \neq 0$, and

$$E^1_{0q} = \begin{cases} \mathbb{Z}.\Omega^r \otimes 1_\theta & q = 2r > 0 \\ \mathbb{Z}.1 \otimes \eta_a \oplus \mathbb{Z}.1 \otimes 1_\theta & q = 0, \mathrm{gr}(a) = 0. \end{cases}$$

The differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ is of the form

$$\begin{aligned} d^1(1 \otimes \eta_a) = \quad & n_{ab} 1 \otimes \eta_b \\ & + n_{a\theta} 1 \otimes 1_\theta \quad (\text{if } \mathrm{gr}(a) = 1) \end{aligned}$$

Thus, we obtain

$$E^2_{pq} = \begin{cases} HF^{SW}_p(Y, \mathfrak{s}) & p \neq 0, q = 0 \\ Ker(\Delta_1) & p = 1, q = 0 \\ HF^{SW}_0(Y, \mathfrak{s}) \oplus T_0 & p = 0, q = 0 \\ \mathbb{Z}.\Omega^r \otimes 1_\theta & p = 0, q = 2r > 0. \end{cases}$$

Here $HF^{SW}_*(Y, \mathfrak{s})$ denotes the non-equivariant (metric and perturbation dependent) Seiberg–Witten Floer homology. This is the homology of the complex with generators $1 \otimes \eta_a$ in degree $\mathrm{gr}(a)$ and boundary coefficients n_{ab} for $\mathrm{gr}(a) - \mathrm{gr}(b) = 1$. We also denoted by Δ_1 the map

$$\Delta_1 : HF^{SW}_1(Y, \mathfrak{s}) \to \mathbb{Z}.1 \otimes 1_\theta,$$

$$\Delta_1(\sum x_a 1 \otimes \eta_a) = \sum x_a n_{a\theta} 1 \otimes 1_\theta,$$

where the coefficients x_a satisfy $\sum x_a n_{ab} = 0$. Finally, the term T_0 denotes the term

$$T_0 = \mathbb{Z}.1 \otimes 1_\theta / Im(\Delta_1).$$

Notice then that the boundary $d^2 : E^2_{p,q} \to E^2_{p-2,q+1}$ is trivial, hence the $E^3_{p,q}$ terms are disposed as in the diagram:

$$\begin{tikzcd}
\cdots & 0 & 0 & 0 & 0 & \mathbb{Z}.\Omega^2 \otimes 1_\theta & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \mathbb{Z}.\Omega \otimes 1_\theta & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & HF^{SW}_4 & HF^{SW}_3 \arrow[uurrr, "d^3"] & HF^{SW}_2 & Ker(\Delta_1) & HF^{SW}_0 \oplus T_0 & HF^{SW}_{-1} & HF^{SW}_{-2} & \cdots
\end{tikzcd}$$

The differential $d^3 : E^3_{p,q} \to E^3_{p-3,q+2}$ is given by the expression

$$d^3([\sum x_a 1 \otimes \eta_a]) = \sum x_a m_{ac} n_{c\theta} \Omega \otimes 1_\theta, \tag{13}$$

for $\mathrm{gr}(a) - \mathrm{gr}(c) = 2$. The expression is obtained by considering the unique choice of a representative of the class $[\sum x_a 1 \otimes \eta_a]$ in $E^3_{p,q}$ whose boundary (3) defines a class in $E^3_{p-3,q+2}$.

The differential $d^4 : E^4_{p,q} \to E^4_{p-4,q+3}$ is again trivial, and we obtain the E^5_{pq} of the form

$$\begin{tikzcd}
\cdots & 0 & 0 & 0 & 0 & \mathbb{Z}.\Omega^2 \otimes 1_\theta & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & T_1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
HF^{SW}_5 \arrow[uuuurrrrr, "d^5"] & HF^{SW}_4 & Ker(\Delta_3) & HF^{SW}_2 & Ker(\Delta_1) & HF^{SW}_0 \oplus T_0 & HF^{SW}_{-1} & HF^{SW}_{-2} & \cdots
\end{tikzcd}$$

where again we denote by T_1 the term

$$T_1 = \mathbb{Z}.\Omega \otimes 1_\theta / Im(\Delta_3).$$

Thus, by iterating the process, we observe that all the differentials $d^{2k} : E^{2k}_{p,q} \to E^{2k}_{p-2k,q+2k+1}$ are trivial and the differentials $d^{2k+1} : E^{2k+1}_{p,q} \to E^{2k+1}_{p-2k-1,q+2k}$ consists of one map for $p = 2k+1$, $q = 0$:

$$\Delta_{2k+1} : HF^{SW}_{2k+1} \to \mathbb{Z}.\Omega^k \otimes 1_\theta,$$

induced by

$$\Delta_{2k+1}(\sum x_a 1 \otimes \eta_a) = \sum x_a m_{aa_{2k-1}} m_{a_{2k-1}a_{2k-3}} \cdots m_{a_3 a_1} n_{a_1\theta} \Omega^k \otimes 1_\theta.$$

Here we have $\mathrm{gr}(a) = 2k+1$ and $\mathrm{gr}(a_r) = r$. Notice that these maps agree with the morphism Δ_*, which is obtained in [5] as the connecting homomorphism in the long exact sequence relating equivariant and non-equivariant Seiberg–Witten Floer homologies.

We thus obtain the following structure theorem for equivariant Seiberg–Witten Floer homology.

Theorem 3.4. *The equivariant Seiberg–Witten Floer homology $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$ has the form*

$$HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}) = \begin{cases} Ker(\Delta_{2k+1}) & * = 2k+1 > 0 \\ HF^{SW}_{2k}(Y,\mathfrak{s}) \oplus T_k & * = 2k \geq 0 \\ HF^{SW}_{*}(Y,\mathfrak{s}) & * < 0 \end{cases}$$

where T_k is the term

$$T_k = \mathbb{Z}.\Omega^k \otimes 1_\theta / Im(\Delta_{2k+1}).$$

This result refines the long exact sequence obtained in [5]:

$$\begin{tikzcd} HF^{SW}_{*,U(1)}(Y,\mathfrak{s}) \arrow[r, "i_*"] & HF^{SW}_{*}(Y,\mathfrak{s},g,\nu) \arrow[dl, "\Delta_*"] \\ \mathbb{Z}[\Omega] \arrow[u, "j_*"] & \end{tikzcd}$$

Similar results can be obtained for $HF^{SW,-}_{*,U(1)}(Y,\mathfrak{s})$.

3.3 Topological invariance

Note that the definitions of these homologies depend on the Seiberg-Witten equations, which use the metric and perturbation on $(Y,\mathfrak{s})$. By the result of [5], we know that $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}) \cong HF^{SW}_{*,U(1)}(Y,\mathfrak{s})$ is a topological invariant of $(Y,\mathfrak{s})$, we first recall this topological invariance as stated in Theorem 6.1 [5].

Theorem 3.5. *(Theorem 6.1 [5]) Let $(Y,\mathfrak{s})$ be a rational homology sphere with a* Spin^c *structure. Suppose given two metrics g_0 and g_1 on Y and perturbations ν_0 and ν_1 such that $Ker(\partial\!\!\!/^{g_0}_{\nu_0}) = Ker(\partial\!\!\!/^{g_1}_{\nu_1}) = 0$, so that the corresponding monopole moduli spaces $\mathcal{M}_Y(\mathfrak{s},g_0,\nu_0)$ and $\mathcal{M}_Y(\mathfrak{s},g_1,\nu_1)$ consist of finitely many isolated points. Then there exists an isomorphism between the equivariant Seiberg-Witten Floer homologies $HF^{SW}_{*,U(1)}(Y,\mathfrak{s},g_0,\nu_0)$ and $HF^{SW}_{*,U(1)}(Y,\mathfrak{s},g_1,\nu_1)$, with a degree shift given by the spectral flow of the Dirac operator $\partial\!\!\!/^{g_t}_{\nu_t}$ along a path of metrics and perturbations connecting (g_0,ν_0) and (g_1,ν_1). That is, if the complex spectral flow along the path (g_t,ν_t) is denoted by $SF_{\mathbb{C}}(\partial\!\!\!/^{g_t}_{\nu_t})$, then for any $k \in \mathbb{Z}$,*

$$HF^{SW}_{k,U(1)}(Y,\mathfrak{s},g_0,\nu_0) \cong HF^{SW}_{k+2SF_{\mathbb{C}}(\partial\!\!\!/^{g_t}_{\nu_t}),U(1)}(Y,\mathfrak{s},g_1,\nu_1).$$

From Theorem 3.1, we know that

$$HF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s}) \cong \mathbb{Z}[\Omega,\Omega^{-1}]$$

is independent of $(Y, \mathfrak{s})$, up to a degree shift as given in Theorem 3.5. Thus, applying the five lemma to the long exact sequence in Theorem 3.3, we obtain that $HF^{SW,-}_{*,U(1)}(Y, \mathfrak{s})$ and $HF^{SW}_{red,*}(Y, \mathfrak{s})$ are also topological invariants of $(Y, \mathfrak{s})$.

Theorem 3.6. *$HF^{SW,-}_{*,U(1)}(Y, \mathfrak{s})$ and $HF^{SW}_{red,*}(Y, \mathfrak{s})$ are topological invariants of $(Y, \mathfrak{s})$, in the sense that, given any two metrics g_0 and g_1 on Y and perturbations ν_0 and ν_1, with $Ker(\not{\partial}^{g_0}_{\nu_0}) = Ker(\not{\partial}^{g_1}_{\nu_1}) = 0$, there exist isomorphisms*

$$\begin{aligned} HF^{SW,-}_{k,U(1)}(Y, \mathfrak{s}, g_0, \nu_0) &\cong HF^{SW,-}_{k+2SF_{\mathbb{C}}(\not{\partial}^{g_t}_{\nu_t}),U(1)}(Y, \mathfrak{s}, g_1, \nu_1) \\ HF^{SW}_{red,k}(Y, \mathfrak{s}, g_0, \nu_0) &\cong HF^{SW}_{red,k+2SF_{\mathbb{C}}(\not{\partial}^{g_t}_{\nu_t})}(Y, \mathfrak{s}, g_1, \nu_1). \end{aligned}$$

Here $SF_{\mathbb{C}}(\not{\partial}^{g_t}_{\nu_t})$ denotes the complex spectral flow of the Dirac operator $\not{\partial}^{g_t}_{\nu_t}$ along the path (g_t, ν_t).

4 Properties of equivariant Seiberg-Witten Floer homologies

In this section, we briefly discuss some of the algebraic structures and properties of the equivariant Seiberg-Witten Floer homologies defined in the previous section.

Note that for any irreducible critical points a and b in $\mathcal{M}^*_Y(\mathfrak{s})$, the associated integer m_{ac} is the counting of points in the geometric representative of the relative first Chern class of the canonical line bundle (7) over $\mathcal{M}(a, c)$, we can apply this fact to define a u-action on the chain complex $CF^{SW,\infty}_{*,U(1)}(Y, \mathfrak{s})$

$$u: \qquad CF^{SW,\infty}_{*,U(1)}(Y, \mathfrak{s}) \longrightarrow CF^{SW,\infty}_{*,U(1)}(Y, \mathfrak{s})$$

which decreases the grading by two. The action is given in terms of its actions on generators as follows:

$$\begin{aligned} u(\Omega^n \otimes \eta_a) &= \sum_{\substack{c \in \mathcal{M}^*(Y,\mathfrak{s}) \\ gr(a)-gr(c)=2}} m_{ac} \Omega^n \otimes \eta_c. \\ u(\Omega^n \otimes 1_a) &= \begin{cases} \displaystyle\sum_{\substack{c \in \mathcal{M}^*(Y,\mathfrak{s}) \\ gr(a)-gr(c)=2}} m_{ac} \Omega^n \otimes 1_c & \text{if } gr(a) \neq 1 \\ \displaystyle\sum_{\substack{c \in \mathcal{M}^*(Y,\mathfrak{s}) \\ gr(c)=-1}} m_{ac} \Omega^n \otimes 1_c + n_{a\theta} \Omega^n \otimes 1_\theta & \text{if } gr(a) = 1 \end{cases} \\ u(\Omega^n \otimes 1_\theta) &= \sum_{\substack{d \in \mathcal{M}^*_Y(\mathfrak{s}) \\ gr(d)=-2}} n_{\theta d} \Omega^n \times \eta_d + \Omega^{n-1} \otimes 1_\theta. \end{aligned} \tag{14}$$

Proposition 4.1. *The u-action defined (14) on the chain complex $CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$ is homotopic to the Ω^{-1}-action acting on $CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$. The induced actions on $CF^{SW,\pm}_{*,U(1)}(Y,\mathfrak{s})$ define $\mathbb{Z}[u]$-module structures on $HF^{SW,\pm}_{*,U(1)}(Y,\mathfrak{s})$.*

Proof. Define $H : CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s}) \longrightarrow CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$ by its actions on the generators as follows:

$$\begin{aligned} H(\Omega^n \otimes \eta_a) &= 0, \\ H(\Omega^n \otimes 1_a) &= \Omega^n \otimes \eta_a, \\ H(\Omega^n \otimes 1_\theta) &= 0. \end{aligned}$$

Then it is a direct calculation to show that we have:

$$(u - \Omega^{-1})(\Omega^k \otimes \eta_a) = m_{ac}\Omega^k \otimes \eta_c - \Omega^{k-1} \otimes \eta_a = (DH + HD)(\Omega^k \otimes \eta_a)$$

$$(u - \Omega^{-1})(\Omega^k \otimes 1_a) = m_{ac}\Omega^k \otimes 1_c - \Omega^{n-1} \otimes 1_a \; (+n_{a\theta}\Omega^n \otimes 1_\theta \text{ if } \mathrm{gr}(a) = 1) = (DH + HD)(\Omega^k \otimes 1_a),$$

$$(u - \Omega^{-1})(\Omega^k \otimes 1_\theta) = n_{\theta d}\Omega^n \otimes \eta_d = (DH + HD)(\Omega^k \otimes 1_\theta).$$

Thus the claim follows using the chain homotopy $u - \Omega^{-1} = D \circ H + H \circ D$.

□

Thus, on the homological level, we can identify the u-action with the induced Ω^{-1} action on various homologies. In particular, we see that there is a subcomplex $\widehat{CF}^{SW}_*(Y,\mathfrak{s})$ of $CF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$ such that the following short exact sequence of chain complexes holds:

$$0 \to \widehat{CF}^{SW}_*(Y,\mathfrak{s}) \longrightarrow CF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}) \xrightarrow{\Omega^{-1}} CF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}) \to 0. \tag{15}$$

Proposition 4.2. *Let $\widehat{HF}^{SW}_*(Y,\mathfrak{s})$ be the homology of $\widehat{CF}^{SW}_*(Y,\mathfrak{s})$, then $\widehat{HF}^{SW}_*(Y,\mathfrak{s})$ is also a topological invariant of $(Y,\mathfrak{s})$, and it is determined by the following long exact sequence*

$$\cdots \to \widehat{HF}^{SW}_*(Y,\mathfrak{s}) \longrightarrow HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}) \xrightarrow{u} HF^{SW,+}_{*-2,U(1)}(Y,\mathfrak{s}) \longrightarrow \widehat{HF}^{SW}_{*-1}(Y,\mathfrak{s}) \to \cdots.$$

Moreover, $\widehat{HF}^{SW}(Y,\mathfrak{s})$ is non-trivial if and only if $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$ is non-trivial.

Proof. The long exact sequence follows from the short exact sequence of chain complexes (15) and Proposition 4.1. This long exact sequence implies that $\widehat{HF}^{SW}_*(Y,\mathfrak{s})$ is also a topological invariant of $(Y,\mathfrak{s})$.

Note that, from the compactness of $\mathcal{M}_Y(\mathfrak{s})$, we see that each element in $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$ can be annihilated by a sufficiently large power of Ω^{-1}. Hence, u is an isomorphism on $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$ if and only if $HF^{SW,+}_{*,U(1)}(Y,\mathfrak{s})$ is trivial. Then the last claim follows from this observation and the long exact sequence. $\square$

If we think of the set of Spin^c structures on Y as the set of equivalence classes of nowhere vanishing vector fields on Y (Cf.[9]), then there is a natural bijection between $\mathrm{Spin}^c(Y)$ and $\mathrm{Spin}^c(-Y)$ where $-Y$ is the same Y with the opposite orientation.

Theorem 4.3. *Let $(Y,\mathfrak{s})$ be a rational homology 3-sphere with a Spin^c structure $\mathfrak{s}$, and $(-Y,\mathfrak{s})$ denote Y with the opposite orientation and the corresponding Spin^c structure. Then there is a natural isomorphism*

$$HF^{SW,*}_{U(1)}(Y,\mathfrak{s}) \cong HF^{SW,-}_{*,U(1)}(-Y,\mathfrak{s})$$

where $HF^{SW,*}_{U(1)}(Y,\mathfrak{s})$ is the equivariant Seiberg-Witten-Floer cohomology defined in [5].

Proof. Note that $HF^{SW,*}_{U(1)}(Y,\mathfrak{s})$ is the homology of the dual complex $Hom(CF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}),\mathbb{Z})$. We start to construct a natural pairing

$$\langle\cdot,\cdot\rangle : \qquad CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s}) \times CF^{SW,\infty}_{*,U(1)}(-Y,\mathfrak{s}) \longrightarrow \mathbb{Z} \tag{16}$$

which satisfies

$$\langle D_Y(\xi_1),\xi_2\rangle = \langle \xi_1, D_{-Y}(\xi_2)\rangle, \qquad \langle\Omega^{-1}(\xi_1),\xi_2\rangle = \langle\xi_1,\Omega^{-1}(\xi_2)\rangle. \tag{17}$$

for any element $\xi_1 \in CF^{SW,\infty}_{*,U(1)}(Y,\mathfrak{s})$ and any element $\xi_2 \in CF^{SW,\infty}_{*,U(1)}(-Y,\mathfrak{s})$.

Then we will show that the above pairing is non-degenerate when restricted to $CF^{SW,+}_{*,U(1)}(Y,\mathfrak{s}) \times CF^{SW,-}_{*,U(1)}(-Y,\mathfrak{s})$.

From the nature of the Seiberg-Witten equations, we see that there is an identification

$$\mathcal{M}_Y(\mathfrak{s}) \to \mathcal{M}_{-Y}(\mathfrak{s})$$

for a good pair of metric and perturbation on $(Y,\mathfrak{s})$ and the corresponding metric and perturbation on $(-Y,\mathfrak{s})$. Then the relative gradings with respect to the unique reducible monopole in $\mathcal{M}_Y(\mathfrak{s})$ and

$\mathcal{M}_{-Y}(\mathfrak{s})$ respectively, satisfies

$$gr_{-Y}(a^-) = -gr_Y(a) - 1,$$

where a^- is the element in $\mathcal{M}^*_{-Y}(\mathfrak{s})$ corresponding to $a \in \mathcal{M}^*_Y(\mathfrak{s})$, we assume that $gr_Y(\theta) = gr_{-Y}(\theta^-)$. Moreover, there is an natural identification between the moduli spaces of flowlines for $(Y, \mathfrak{s})$ and $(-Y, \mathfrak{s})$, that is,

$$\mathcal{M}_{Y\times\mathbb{R}}(a, b) \cong \mathcal{M}_{-Y\times\mathbb{R}}(b^-, a^-).$$

Now we define the pairing on $CF^{SW,\infty}_{*,U(1)}(Y, \mathfrak{s}) \times CF^{SW,\infty}_{*,U(1)}(-Y, \mathfrak{s})$ such that the following pairings are the only non-trivial pairings:

$$\langle \Omega^n \otimes \eta_a, \Omega^{-n-1} \otimes 1_{a^-} \rangle = 1$$

$$\langle \Omega^n \otimes 1_a, \Omega^{-n-1} \otimes \eta_{a^-} \rangle = 1$$

$$\langle \Omega^n \otimes 1_\theta, \Omega^{-n-1} \otimes 1_{\theta^-} \rangle = 1.$$

It is a direct calculation to show that this pairing satisfies the relation (17) and the restriction of this pairing to $CF^{SW,+}_{*,U(1)}(Y, \mathfrak{s}) \times CF^{SW,-}_{*,U(1)}(-Y, \mathfrak{s})$ is non-degenerate. Then the claim follows from the definition. $\square$

Let $\widehat{HF}^{SW,*}(Y, \mathfrak{s})$ and $HF^{SW,*}_{\pm,U(1)}(Y, \mathfrak{s})$ denote the homology groups of the dual complexes $Hom(\widehat{CF}^{SW}_*(Y, \mathfrak{s}), \mathbb{Z})$ and $Hom(CF^{SW,\pm}_{*,U(1)}(Y, \mathfrak{s}), \mathbb{Z})$ of $\widehat{CF}^{SW}_*(Y, \mathfrak{s})$ and $CF^{SW,\pm}_{*,U(1)}(Y, \mathfrak{s})$ respectively. From the proof the above Theorem 4.3, we actually establish the following duality between these homologies.

Theorem 4.4. *For any rational homology 3-sphere Y with a spinc structure $\mathfrak{s}$, there exist natural isomorphisms*

$$\widehat{HF}^{SW,*}(Y, \mathfrak{s}) \cong \widehat{HF}^{SW}_*(-Y, \mathfrak{s}), \qquad HF^{SW,*}_{\pm,U(1)}(Y, \mathfrak{s}) \cong HF^{SW,\mp}_{*,U(1)}(-Y, \mathfrak{s}). \tag{18}$$

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