Lecture 4. Tangent vectors

4.1 The tangent space to a point

Let $M^n$ be a smooth manifold, and $x$ a point in $M$. In the special case where $M$ is a submanifold of Euclidean space $\mathbb{R}^N$, there is no difficulty in defining a space of tangent vectors to $M$ at $x$: Locally $M$ is given as the zero level-set of a submersion $G : U \to \mathbb{R}^{N-n}$ from an open set $U$ of $\mathbb{R}^N$ containing $x$, and we can define the tangent space to be $\ker(D_xG)$, the subspace of vectors which map to 0 under the derivative of $G$. Alternatively, if we describe $M$ locally as the image of an embedding $\varphi : U \to \mathbb{R}^N$ from an open set $U$ of $\mathbb{R}^n$, then we can take the tangent space to $M$ at $x$ to be the subspace $\operatorname{rng}(D_{\varphi^{-1}(x)}\varphi) = \{D_{\varphi^{-1}(x)}\varphi(u) : u \in \mathbb{R}^n\}$, the image subspace of the derivative map.

If $M$ is an abstract manifold, however, then we do not have any such convenient notion of a tangent vector.

From calculus on $\mathbb{R}^n$ we have several complementary ways of thinking about tangent vectors: As $k$-tuples of real numbers; as ‘directions’ in space, such as the tangent vector of a curve; or as directional derivatives.

I will give three alternative candidates for the tangent space to a smooth manifold $M$, and then show that they are equivalent:

First, for $x \in M$ we define $T_xM$ to be the set of pairs $(\varphi, u)$ where $\varphi : U \to V$ is a chart in the atlas for $M$ with $x \in U$, and $u$ is an element of $\mathbb{R}^n$, modulo the equivalence relation which identifies a pair $(\varphi, u)$ with a pair $(\eta, w)$ if and only if $u$ maps to $w$ under the derivative of the transition map between the two charts:

$$D_{\varphi(x)}(\eta \circ \varphi^{-1})(u) = w. \quad (4.1)$$

Remark. The basic idea is this: We think of a vector as being an ‘arrow’ telling us which way to move inside the manifold. This information on which way to move is encoded by viewing the motion through a chart $\varphi$, and seeing which way we move ‘downstairs’ in the chart (this corresponds to a vector in $n$-dimensional space according to the usual notion of a velocity vector). The equivalence relation just removes the ambiguity of a choice of chart through which to follow the motion.

Another way to think about it is the following: We have a local description for $M$ using charts, and we know what a vector is ‘downstairs’ in each chart.
We want to define a space of vectors $T_x M$ ‘upstairs’ in such a way that the derivative map $D_x \varphi$ of the chart map $\varphi$ makes sense as a linear operator between the vector spaces $T_x M$ and $\mathbb{R}^n$, and so that the chain rule continues to hold. Then we would have for any vector $v \in T_p M$ vectors $u = D_x \varphi(v) \in \mathbb{R}^n$, and $w = D_x \eta(v) \in \mathbb{R}^n$. Writing this another way (implicitly assuming the chain rule holds) we have
\[
D_{\varphi(x)} \varphi^{-1}(u) = v = D_{\eta(x)} \eta^{-1}(w).
\]
The chain rule would then imply $D_{\varphi(x)} \left( \eta \circ \varphi^{-1} \right)(u) = w$.

![Fig.1: A tangent vector to $M$ at $x$ is implicitly defined by a curve through $x$](image)

The second definition expresses even more explicitly the idea of a ‘velocity vector’ in the manifold: We define $M_x$ to be the space of smooth paths in $M$ through $x$ (i.e. smooth maps $\gamma : I \to M$ with $\gamma(0) = x$) modulo the equivalence relation which identifies any two curves if they agree to first order (as measured in some chart): $\gamma \sim \sigma \iff (\varphi \circ \gamma)'(0) = (\varphi \circ \sigma)'(0)$ for some chart $\varphi : U \to V$ with $x \in U$. The equivalence does not depend on the choice of chart: If we change to a chart $\eta$, then we have
\[
(\eta \circ \gamma)'(0) = ((\eta \circ \varphi^{-1}) \circ (\varphi \circ \gamma))' (0) = D_{\varphi(x)} (\eta \circ \varphi^{-1}) (\varphi \circ \gamma)'(0)
\]
and similarly for sigma, so $(\eta \circ \gamma)'(0) = (\eta \circ \sigma)'(0)$.

Finally, we define $D_x M$ to be the space of derivations at $x$. Here a derivation is a map $v$ from the space of smooth functions $C^\infty(M)$ to $\mathbb{R}$, such that for any real numbers $c_1$ and $c_2$ and any smooth functions $f$ and $g$ on $M$,
\[
v(c_1 f + c_2 g) = c_1 v(f) + c_2 v(g) \quad \text{and} \quad v(fg) = f(x)v(g) + g(x)v(f) \quad (4.2)
\]
The archetypal example of a derivation is of course the directional derivative of a function along a curve: Given a smooth path $\gamma : I \to M$ with $\gamma(0) = x$, we can define
\[
v(f) = \frac{d}{dt} (f \circ \gamma) \bigg|_{t=0},
\]
and this defines a derivation at $x$. We will see below that all derivations are of this form.

**Proposition 4.1.1** There are natural isomorphisms between the three spaces $M_x, T_x M, \text{ and } D_x M$.

\[
\begin{array}{ccc}
TM_x & \to & M_x \\
\downarrow & & \downarrow \alpha \\
D_x M & \to & \end{array}
\]

*Proof.* First, we write down the isomorphisms: Given an equivalence class $[(\varphi, u)]$ in $T_x M$, we take $\alpha([(\varphi, u)])$ to be the equivalence class of the smooth path $\gamma$ defined by
\[
\gamma(t) = \varphi^{-1}(\varphi(x) + tu).
\]
(4.4)
The map $\alpha$ is well-defined, since if $[(\eta, w)]$ is another representative of the same equivalence class in $T_x M$, then $\alpha$ gives the equivalence class of the curve $\sigma(t) = \eta^{-1}(\eta(x) + tw)$, and
\[
(\varphi \circ \sigma)'(0) = ((\varphi \circ \eta^{-1}) \circ (\eta \circ \sigma))'(0)
\]
\[
= D_{\eta(x)} (\varphi \circ \eta^{-1}) (\eta \circ \sigma)'(0)
\]
\[
= D_{\eta(x)} (\varphi \circ \eta^{-1})(w)
\]
\[
= u
\]
\[
= (\varphi \circ \gamma)'(0),
\]
so $[\sigma] = [\gamma]$.

Given an element $[\gamma] \in M_x$, we take $\beta([\gamma])$ to be the natural derivation $v$ defined by Eq. (4.3). Again, we need to check that this is well-defined: Suppose $[\sigma] = [\gamma]$. Then
\[
\frac{d}{dt} (f \circ \gamma) \bigg|_{t=0} = D_{\varphi(x)} (f \circ \varphi^{-1}) (\varphi \circ \gamma)'(0)
\]
\[
= D_{\varphi(x)} (f \circ \varphi^{-1}) (\varphi \circ \sigma)'(0)
\]
\[
= \frac{d}{dt} (f \circ \sigma) \bigg|_{t=0}
\]
for any smooth function \( f \).

Finally, given a derivation \( v \), we choose a chart \( \varphi \) containing \( x \), and take \( \chi(v) \) to be the element of \( T_xM \) given by taking the equivalence class \([([\varphi, u])\]
where \( u = (v(\varphi^1), \ldots, v(\varphi^n))\). Here \( \varphi^i \) is the \( i \)th component function of the chart \( \varphi \).

There is a technicality involved here: In the definition, derivations were assumed to act on smooth functions defined on all of \( M \). However, \( \varphi^i \) is defined only on an open set \( U \) of \( M \). In order to overcome this difficulty, we will extend \( \varphi_i \) (somewhat arbitrarily) to give a smooth map on all of \( M \). For concreteness, we can proceed as follows (the functions we construct here will prove useful later on as well):

\[
\xi(z) = \begin{cases} 
\exp \left( \frac{1}{z^2-1} \right) & \text{for } -1 < z < 1; \\
0 & \text{for } |z| \geq 1.
\end{cases} \quad (4.5)
\]

and

\[
\rho(z) = \frac{\int_{-1}^{z} \xi(z')dz'}{\int_{-1}^{1} \xi(z')dz'} \quad (4.6)
\]

Then

- \( \xi \) is a \( C^\infty \) function on all of \( \mathbb{R} \), with \( \xi(z) \) equal to zero whenever \( |z| \geq 1 \), and \( \xi(z) > 0 \) for \( |z| < 1 \);
- \( \rho \) is a \( C^\infty \) function on all of \( \mathbb{R} \), which is zero whenever \( z \leq -1 \), identically 1 for \( z > 1 \), and strictly increasing for \( z \in (-1, 1) \).

Fig. 2: The ‘bump function’ or ‘cut-off function’ \( \xi \).

Fig. 3: A \( C^\infty \) ‘ramp function’ \( \rho \).

Now, given the chart \( \varphi : U \rightarrow V \) with \( x \in U \), choose a number \( r \) sufficiently small to ensure that the closed ball of radius \( 4r \) about \( \varphi(x) \) is contained in \( V \). Then define a function \( \tilde{\rho} \) on \( M \) by

\[
\tilde{\rho}(y) = \begin{cases} 
\rho \left( 3 - \left| \frac{\varphi(y) - \varphi(x)}{r} \right| \right) & \text{for } y \in U; \\
0 & \text{for all other } y \in M.
\end{cases} \quad (4.7)
\]

**Exercise 4.1** Prove that \( \tilde{\rho} \) is a smooth function on \( M \).
Note that this construction gives a function $\tilde{\rho}$ which is identically equal to 1 on a neighbourhood of $x$, and identically zero in the complement of a larger neighbourhood.

Now we can make sense of our definition above:

**Definition 4.1.2** If $f : U \to \mathbb{R}$ is a smooth function on an open set $U$ of $M$ containing $x$, we define $v(f) = v(\tilde{f})$, where $\tilde{f}$ is any smooth function on $M$ which agrees with $f$ on a neighbourhood of $x$.

For this to make sense we need to check that there is some smooth function $\tilde{f}$ on $M$ which agrees with $f$ on a neighbourhood of $x$, and that the definition does not depend on which such function we choose.

Without loss of generality we suppose $f$ is defined on $U$ as above, and define $\tilde{\rho}$ as above. Then we define $\tilde{f}(y) = \begin{cases} f(y)\tilde{\rho}(y) & \text{for } y \in U; \\ 0 & \text{for } y \in M \setminus U. \end{cases} \tag{4.8}$

$\tilde{f}$ is smooth, and agrees with $f$ on the set $\varphi^{-1}(B_{2r}(\varphi(x)))$.

Next we need to check that $v(\tilde{f})$ does not change if we choose a different function agreeing with $f$ on a neighbourhood of $x$.

**Lemma 4.1.3** Suppose $f$ and $g$ are two smooth functions on $M$ which agree on a neighbourhood of $x$. Then $v(f) = v(g)$.

*Proof.* Without loss of generality, assume that $f$ and $g$ agree on an open set $U$ containing $x$, and construct a ‘bump’ function $\tilde{\rho}$ in $U$ as above. Then we observe that $\tilde{\rho}(f - g)$ is identically zero on $M$, and that $v(0) = v(0) = 0$. Therefore

$$0 = v(\tilde{\rho}(f - g)) = \tilde{\rho}(x)v(f - g) + (f(x) - g(x))v(\tilde{\rho}) = v(f) - v(g)$$

since $f(x) - g(x) = 0$ and $\tilde{\rho}(x) = 1$. \hfill $\square$

This shows that the definition of $v(f)$ makes sense, and so our definition of $\chi(v)$ makes sense. However we still need to check that $\chi(v)$ does not depend on the choice of a chart $\varphi$. Suppose we instead use another chart $\eta$. Then we have in a small region about $x$,

$$\eta^i(y) = \eta^i(x) + \sum_{j=1}^{n} G^i_j(y)(\varphi^j(y) - \varphi^i(x)) \tag{4.9}$$

for each $i = 1, \ldots, n$, where $G^i_j(x)$ is a smooth function on a neighbourhood of $x$ for which $G^i_j(x) = \frac{\partial}{\partial z} (\eta^i \circ \varphi^{-1}) \bigg|_{\varphi(x)}$. To prove this, consider the Taylor expansion for $\eta^i \circ \varphi^{-1}$ on the set $V$, where $\varphi$ is the chart from $U$ to $V$. 
Now apply $v$ to Eq. (4.9). Note that the first term is a constant.

**Lemma 4.1.4** $v(c) = 0$ for any constant $c$.

**Proof.**

$v(1) = v(1.1) = 1.v(1) = 2v(1) \implies v(1) = 0 \implies v(c) = cv(1) = 0$.

$v$ applied to $\eta^i$ gives

$$v(\eta^i) = \sum_{j=1}^{n} G_j^i(x)v(\varphi^j) + (\varphi^j(x) - \varphi^i(x))v(G_j^i)$$

$$= \sum_{j=1}^{n} \frac{\partial}{\partial z^j} (\eta^i \circ \varphi^{-1}) \bigg|_{\varphi(x)} v(\varphi^j).$$

Therefore we have

$$\sum_{i=1}^{n} v(\eta^i)e_i = \sum_{i,j=1}^{n} \frac{\partial}{\partial z^j} (\eta^i \circ \varphi^{-1}) \bigg|_{\varphi(x)} v(\varphi^j)e_i$$

$$= D_{\varphi(x)} (\eta \circ \varphi^{-1}) \left( \sum_{i=1}^{n} v(\varphi^i)e_i \right)$$

and so $[(\varphi, \sum_{i=1}^{n} v(\varphi^i)e_i)] = [(\eta, \sum_{i=1}^{n} v(\eta^i)e_i)]$ and $\chi$ is independent of the choice of chart.

In order to prove the proposition, it is enough to show that the three triple compositions $\chi \circ \beta \circ \alpha$, $\beta \circ \alpha \circ \chi$, and $\alpha \circ \chi \circ \beta$ are just the identity map on each of the three spaces.

We have

$$\chi \circ \beta \circ \alpha([(\varphi, u)]) = \chi \circ \beta ([t \mapsto \varphi^{-1} (\varphi(x) + tu)])$$

$$= \chi (f \mapsto D_{\varphi(x)} (f \circ \varphi^{-1})(u))$$

$$= \left[ \varphi, \sum_{i=1}^{n} D_{\varphi(x)} (\varphi^i \circ \varphi^{-1})(u)e_i \right]$$

$$= [(\varphi, u)].$$

Similarly we have

$$\alpha \circ \chi \circ \beta([\sigma]) = \alpha \circ \chi (f \mapsto \frac{d}{dt} (f \circ \gamma) \bigg|_{t=0})$$

$$= \alpha \left[ \left[ \varphi, \sum_{i=1}^{n} (\varphi \circ \gamma)^i(0) \right] \right]$$

$$= [t \mapsto \varphi^{-1} (\varphi(x) + t (\varphi \circ \gamma)^i(0))].$$
Finally, we have
\[ \beta \circ \alpha \circ \chi (v) = \beta \circ \alpha \left( \left[ \varphi, \sum_{i=1}^{n} v(\varphi_i) e_i \right] \right) \]
\[ = \beta \left[ \left. t \mapsto \varphi^{-1} \left( \varphi(x) + t \sum_{i=1}^{n} v(\varphi_i) e_i \right) \right|_{\phi(x)} \right] \]
\[ = \left( \left. f \mapsto \sum_{i=1}^{n} \frac{\partial}{\partial z^i} (f \circ \varphi^{-1}) \right|_{\phi(x)} v(\varphi_i) \right). \]

We need to show that this is the same as \( v \). To show this, we note (using the Taylor expansion for \( f \circ \varphi^{-1} \)) that
\[ f(y) = f(x) + \sum_{i=1}^{n} G_i(y) (\varphi^i(y) - \varphi^i(x)) \]
for \( y \) in a sufficiently small neighbourhood of \( x \), where \( G_i \) is a smooth function with \( G_i(x) = \frac{\partial}{\partial z^i} (f \circ \varphi^{-1}) \big|_{\phi(x)} \). Applying \( v \) to this expression, we find
\[ v(f) = \sum_{i=1}^{n} G_i(x) v(\varphi^i) = \sum_{i=1}^{n} \left. \frac{\partial}{\partial z^i} (f \circ \varphi^{-1}) \right|_{\phi(x)} v(\varphi^i) \]
and the right hand side is the same as \( \beta \circ \alpha \circ \gamma v(f) \).

Having established the equivalence of the three spaces \( M_x, T_x, \) and \( D_x M \), I will from now on keep only the notation \( T_x M \) (the tangent space to \( x \) at \( M \)) while continuing to use all three different notions of a tangent vector.

4.2 The differential of a map

**Definition 4.2.1** Let \( f : M \to \mathbb{R} \) be a smooth function. Then we define the differential \( df \) of \( f \) at the point \( x \in M \) to be the linear function on the tangent space \( T_x M \) given by \( (df)(v) = v(f) \) for each \( v \in T_x M \) (thinking of \( v \) as a derivation). Let \( F : M \to N \) be a smooth map between two manifolds. Then we define the differential \( DF \) of \( F \) at \( x \in M \) to be the linear map from \( T_x M \) to \( T_{F(x)} N \) given by \( ((DF)(v))(f) = v(f \circ F) \) for any \( v \in T_x M \) and any \( f \in C^\infty(N) \).

It is useful to describe the differential of a map in terms of the other representations of tangent vectors. If \( v \) is the vector corresponding to the equivalence class \( [\varphi, u] \), then we have \( v : f \mapsto D_{\varphi(x)}(f \circ \varphi^{-1})(u) \), and so by the definition above, \( DF(v) \) sends a smooth function \( f \) on \( N \) to \( v(f \circ F) \):
\( D_x F(v) : f \mapsto D_{\varphi(x)} (f \circ F \circ \varphi^{-1}) (u) \)
\[ = D_{\eta(F(x))} (f \circ \eta^{-1}) \circ D_{\varphi(x)} (\eta \circ F \circ \varphi^{-1}) (u) \]

which is the vector corresponding to \([\eta, D_{\varphi(x)} (\eta \circ F \circ \varphi^{-1}) (u)]\).

Alternatively, if we think of a vector \( v \) as the tangent vector of a curve \( \gamma \), then we have \( v : f \mapsto (f \circ \gamma)'(0) \), and so \( D_x F(v) : f \mapsto (f \circ F \circ \gamma)'(0) \), which is the tangent vector of the curve \( F \circ \gamma \). In other words,
\[ D_x F([\gamma]) = [F \circ \gamma]. \tag{4.11} \]

In most situations we can use the differential of a map in exactly the same way as we use the derivative for maps between Euclidean spaces. In particular, we have the following results:

**Theorem 4.2.2 The Chain Rule** If \( F : M \to N \) and \( G : N \to P \) are smooth maps between manifolds, then so is \( G \circ F \), and
\[ D_x (G \circ F) = D_{F(x)} G \circ D_x F. \]

**Proof.** By Eq. (4.11),
\[ D_x (G \circ F)([\gamma]) = [G \circ F \circ \gamma] = D_{F(x)} G([F \circ \gamma]) = D_{F(x)} G(D_x F([\gamma])). \]

**Theorem 4.2.3 The inverse function theorem** Let \( F : M \to N \) be a smooth map, and suppose \( D_x F \) is an isomorphism for some \( x \in M \). Then there exists an open set \( U \subset M \) containing \( x \) and an open set \( V \subset N \) containing \( F(x) \) such that \( F \big|_U \) is a diffeomorphism from \( U \) to \( V \).

**Proof.** We have \( D_x F([(\varphi, u)]) = [\eta, D_{\varphi(x)} (\eta \circ F \circ \varphi^{-1}) (u)] \), so \( D_x F \) is an isomorphism if and only if \( D_{\varphi(x)} (\eta \circ F \circ \varphi^{-1}) \) is an isomorphism. The result follows by applying the usual inverse function theorem to \( \eta \circ F \circ \varphi^{-1} \).
4.4 The tangent bundle

**Theorem 4.2.4 The implicit function theorem (surjective form)** Let $F : M \rightarrow N$ be a smooth map, with $D_x F$ surjective for some $x \in M$. Then there exists a neighbourhood $U$ of $x$ such that $F^{-1}(F(x)) \cap U$ is a smooth submanifold of $M$.

**Theorem 4.2.5 The implicit function theorem (injective form)** Let $F : M \rightarrow N$ be a smooth map, with $D_x F$ injective for some $x \in M$. Then there exists a neighbourhood $U$ of $x$ such that $F|_U$ is an embedding.

These two theorems follow directly from the corresponding theorems for smooth maps between Euclidean spaces.

4.3 Coordinate tangent vectors

Given a chart $\varphi : U \rightarrow V$ with $x \in U$, we can construct a convenient basis for $T_x M$: We simply take the vectors corresponding to the equivalence classes $[(\varphi, e_i)]$, where $e_1, \ldots, e_n$ are the standard basis vectors for $\mathbb{R}^n$. We use the notation $\partial_i = [(\varphi, e_i)]$, suppressing explicit mention of the chart $\varphi$. As a derivation, this means that $\partial_i f = \frac{\partial}{\partial z} (f \circ \varphi^{-1}) \bigg|_{\varphi(x)}$. In other words, $\partial_i$ is just the derivation given by taking the $i$th partial derivative in the coordinates supplied by $\varphi$. It is immediate from Proposition 4.1.1 that $\{\partial_1, \ldots, \partial_n\}$ is a basis for $T_x M$.

4.4 The tangent bundle

We have just constructed a tangent space at each point of the manifold $M$. When we put all of these spaces together, we get the tangent bundle $TM$ of $M$:

$$TM = \{ (p, v) : p \in M, v \in T_p M \}.$$ 

If $M$ has dimension $n$, we can endow $TM$ with the structure of a $2n$-dimensional manifold, as follows: Define $\pi : TM \rightarrow M$ to be the projection which sends $(p, v)$ to $p$. Given a chart $\varphi : U \rightarrow V$ for $M$, we can define a chart $\tilde{\varphi}$ for $TM$ on the set $\pi^{-1}(U) = \{ (p, v) \in TM : p \in U \}$, by

$$\tilde{\varphi}(p, v) = (\varphi(p), v(\varphi^1), \ldots, v(\varphi^n)) \in \mathbb{R}^{2n}.$$ 

Thus the first $n$ coordinates describe the point $p$, and the last $n$ give the components of the vector $v$ with respect to the basis of coordinate tangent vectors $\{\partial_i\}_{i=1}^n$, since by Eq. (4.10),
\[ v(f) = \sum_{i=1}^{n} \frac{\partial}{\partial z^i} (f \circ \varphi^{-1}) \bigg|_{\varphi(x)} v(\varphi^i) = \sum_{i=1}^{n} v(\varphi^i) \partial_i(f) \]  

(4.12)

for any smooth \( f \), and hence \( v = \sum_{i=1}^{n} v(\varphi^i) \partial_i \). For convenience we will often write the coordinates on \( TM \) as \((x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n)\)

To check that these charts give \( TM \) a manifold structure, we need to compute the transition maps. Suppose we have two charts \( \varphi : U \to V \) and \( \eta : W \to Z \), overlapping non-trivially. Then \( \tilde{\eta} \circ \tilde{\varphi}^{-1} \) first takes a \( 2n \)-tuple \((x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n)\) to the element \((\varphi^{-1}(x^1, \ldots, x^n), \sum_{i=1}^{n} \dot{x}^i \varphi^i)\) of \( TM \), then maps this to \( \mathbb{R}^{2n} \) by \( \tilde{\eta} \). Here we add the superscript \((\varphi)\) to distinguish the coordinate tangent vectors coming from the chart \( \varphi \) from those given by the chart \( \eta \). The first \( n \) coordinates of the result are then just \( \eta \circ \varphi^{-1}(x^1, \ldots, x^n) \). To compute the last \( n \) coordinates, we need to \( \partial_i(\varphi) \) in terms of the coordinate tangent vectors \( \partial_j(\eta) \): We have

\[
\partial_i^{(\varphi)} f = D_{\varphi(x)} (f \circ \varphi^{-1}) (e_i)
= D_{\varphi(x)} \left((f \circ \eta^{-1}) \circ (\eta \circ \varphi^{-1})\right) (e_i)
= D_{\eta(x)} \left(f \circ \eta^{-1}\right) \circ D_{\varphi(x)} \left(\eta \circ \varphi^{-1}\right) (e_i)
= D_{\eta(x)} \left(f \circ \eta^{-1}\right) \left(\sum_{j=1}^{n} D_{\varphi(x)} (\eta \circ \varphi^{-1})_j^i e_j\right)
= \sum_{j=1}^{n} D_{\varphi(x)} (\eta \circ \varphi^{-1})_j^i D_{\eta(x)} \left(f \circ \eta^{-1}\right) (e_j)
= \sum_{j=1}^{n} D_{\varphi(x)} (\eta \circ \varphi^{-1})_j^i \partial_j^{(\eta)} f
\]

for every smooth function \( f \). Therefore

\[
\sum_{i,j=1}^{n} \dot{x}^i \partial_i^{(\varphi)} = \sum_{i,j=1}^{n} D_{\varphi(x)} (\eta \circ \varphi^{-1})_j^i \partial_j^{(\eta)}
\]

and

\[
\tilde{\eta} \circ \tilde{\varphi}^{-1}(x, \dot{x}) = (\eta \circ \varphi^{-1}(x), D_{\varphi(x)} (\eta \circ \varphi^{-1}) (\dot{x})).
\]

Since \( \eta \circ \varphi^{-1} \) is smooth by assumption, so is its matrix of derivatives \( D(\eta \circ \varphi^{-1}) \), so \( \tilde{\eta} \circ \tilde{\varphi}^{-1} \) is a smooth map on \( \mathbb{R}^{2n} \).
4.5 Vector fields

A vector field on $M$ is given by choosing a vector in each of the tangent spaces $T_x M$. We require also that the choice of vector varies smoothly, in the following sense: A choice of $V_x$ in each tangent space $T_x M$ gives us a map from $M$ to $TM$, $x \mapsto V_x$. A smooth vector field is defined to be one for which the map $x \mapsto V_x$ is a smooth map from the manifold $M$ to the manifold $TM$.

In order to check whether a vector field is smooth, we can work locally. In a chart $\phi$, the vector field can be written as $V_x = \sum_{i=1}^{n} V^i_x \partial x_i$, and this gives us $n$ functions $V^1_x, \ldots, V^n_x$. Then it is easy to show that $V$ is a smooth vector field if and only if these component functions are smooth as functions on $M$. That is, when viewed through a chart the vector field is smooth, in the usual sense of an $n$-tuple of smooth functions.

Our notion of a tangent vector as a derivation allows us to think of a vector field in another way:

**Proposition 4.5.1** Smooth vector fields are in one-to-one correspondence with derivations $V : C^\infty(M) \to C^\infty(M)$ satisfying the two conditions

$$V(c_1 f_1 + c_2 f_2) = c_1 V(f_1) + c_2 V(f_2)$$

$$V(f_1 f_2) = f_1 V(f_2) + f_2 V(f_1)$$

for any constants $c_1$ and $c_2$ and any smooth functions $f_1$ and $f_2$.

**Proof**: Given such a derivation $V$, and $p \in M$, the map $V_p : C^\infty(M) \to \mathbb{R}$ given by $V_p(f) = (V(f))(p)$ is a derivation at $p$, and hence defines an element of $T_p M$. The map $p \mapsto V_p$ from $M$ to $TM$ is therefore a vector field, and it remains to check smoothness. The components of $V_p$ with respect to the coordinate tangent basis $\partial_1, \ldots, \partial_n$ for a chart $\phi : U \to V$ is given by
\[ V^i_p = V_p(\varphi^i) \]

which is by assumption a smooth function of \( p \) for each \( i \) (since \( \varphi^i \) is a smooth function and \( V \) maps smooth functions to smooth functions – here one should really multiply \( \varphi^i \) by a smooth cut-off function to convert it to a smooth function on the whole of \( M \)). Therefore the vector field is smooth.

Conversely, given a smooth vector field \( x \mapsto V_x \in T_x M \), the map

\[ (V(f))(x) = V_x(f) \]

satisfies the two conditions in the proposition and takes a smooth function to a smooth function. \( \square \)

A common notation is to refer to the space of smooth vector fields on \( M \) as \( \mathcal{X}(M) \). Over a small region of a manifold (such as a chart), the space of smooth vector fields is in 1 : 1 correspondence with \( n \)-tuples of smooth functions. However, when looked at over the whole manifold things are not so simple. For example, a theorem of algebraic topology says that there are no continuous vector fields on the sphere \( S^2 \) which are everywhere non-zero ("the sphere has no hair"). On the other hand there are certainly nonzero functions on \( S^2 \) (constants, for example).