Lecture 1. Introduction

In Differential Geometry we study spaces which are smooth enough to do calculus. These include such familiar objects as curves and surfaces in space.

In its earliest manifestation, differential geometry consisted of the study of curves and surfaces in the plane and in space — this goes back at least as far as Newton and Leibniz who applied calculus to the study of curves in the plane, and to Monge and Euler who gave analytic treatments of surfaces. This concentration on geometry in Euclidean space seems quite natural, since curves and surfaces arise very naturally as trajectories or as level sets of functions — for example, in mechanics it is common to consider phase spaces, and within them surfaces on which some preserved quantity (such as energy) is constant. A very substantial body of results was developed in what is now often called 'classical differential geometry', covering such topics as evolutes and involutes, developable surfaces and ruled surfaces, envelopes, minimal surfaces, parallel surfaces, and so on. None of these will feature very much in this course, though there are many fascinating aspects to all of this.

The material covered in this course is almost all of much more recent vintage, but serves as an excellent basis for the treatment of all the subjects mentioned above. The notion of differentiable manifold unifies and simplifies most of the computations involved in these more classical subjects, so that they are now more sensibly treated in a second course, applying the fundamental ideas developed here.

Hermann Weyl

Hassler Whitney

Carl Friedrich Gauss

The notion of a differentiable manifold was not clearly formulated until relatively recently: Hermann Weyl first gave a concise definition in 1912 (in his work making rigorous the theory of Riemann surfaces), but this did not really come into common use until much later, after a series of papers by Hassler Whitney around 1936. However the idea has its roots much earlier. Gauss had a major interest in differential geometry, and published many papers on the subject. His most renowned work in the area was *Disquisitiones generales circa superficies curva* (1828). This paper contained extensive discussion on geodesics and what are now called 'Gaussian coordinates' and 'Gauss curvature', which he called the 'measure of curvature'. The paper also includes the famous theorema egregium:

> *"If a curved surface can be developed (i.e. mapped isometrically) upon another surface, the measure of curvature at every point remains unchanged after the development."*

This result led Gauss to a fundamental insight:

> *"These theorems lead to the consideration of curved surfaces from a new point of view ... where the very nature of the curved surface is given by means of the expression of any linear element in the form* $\sqrt{E\mathrm{d}p^2 + 2F\mathrm{d}p\mathrm{d}q + G\mathrm{d}q^2}$*."*

In other words, he saw that it is posible to consider the geometry of a surface as defined by a metric (here he is locally parametrizing the surface with coordinates p and q), without reference to the way the surface lies in space.

In 1854 Riemann worked with locally defined metrics (now of course known as Riemannian metrics) in any number of dimensions, and defined the object we now call the Riemann curvature. These computations were local, and Riemann only gave a rather imprecise notion of how a curved space can be defined globally.


Bernhard Riemann

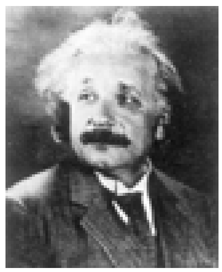

G. Ricci-Curbastro


T. Levi-Civita

Despite the lack of a precise notion of how a manifold should be defined, significant advances were made in differential geometry, particularly in developing the local machinery for computing curvatures — Christoffel and Levi-Civita introduced connections (not precisely in the sense we will define them in this course), and Ricci and Schouten developed the use of tensor calculus in geometric computations. Henri Poincaré worked with manifolds, but never precisely defined them.

Riemann's work surfaced again in spectacular fashion in Einstein's formulation of the theory of general relativity – Einstein had been trying for several years to find a way of expressing mathematically his principle of equivalence, and had attempted to use Gauss's ideas to encapsulate this. In 1912 he learned from his friend Marcel Grossman about Riemann's work and its mathematical development using tensors by Christoffel, Ricci and Levi-Civita, and succeeded in adapting it to his requirements after a further three years of struggle. This development certainly gave great impetus to the further development of the field, but Einstein still worked entirely on the local problem of interpreting the equivalence principle, and never worked in any systematic way on the global spacetime manifold.

E. Christoffel



Albert Einstein

The definition of a manifold encapsulates the idea that there are no preferred coordinates, and therefore that geometric computations must be invariant under coordinate change. Since this is built automatically into our framework, we never have to spend much time checking that things are geometrically well-defined, or invariant under changes of coordinates. In contrast, in the works of Riemann, Ricci and Levi-Civita these computations take some considerable effort.

The abstract notion of a manifold, without reference to any 'background' Euclidean spaces, also arises naturally from several directions. One of these is of course general relativity, where we do not want the unnecessary baggage associated with postulating a larger space in which the physical spacetime should lie. Another comes from the work of Riemann in complex analysis, in what is now called the theory of Riemann surfaces. Consider an analytic function f on a region of the complex plane. This can be defined in a neighbourhood of a point z_0 by a convergent power series. This power series converges in some disk of radius r_0 about z_0. If we now move to some other point z_1 in this disk, we can look at the power series for f about z_1, and this in general converges on a different region, and can be used to extend f beyond the original disk. Using analytic extensions in this way, we can move around the complex plane as long as we avoid poles and singularities of f. However, it may happen that the value of the function that we obtain depends on the path we took to get there, as in the case $f(z) = z^{1/2}$. In this case we can think of the function as defined not on the plane, but instead on an abstract surface which projects onto the plane (if $f(z) = z^{1/2}$, this surface covers $\mathbb{C}\backslash\{0\}$ twice). Another place where it is natural to work with abstract manifolds is in the theory of Lie groups, which are groups with a manifold structure.

1.1 Differentiable Manifolds

Definition 1.1.1 (Manifolds and atlases)

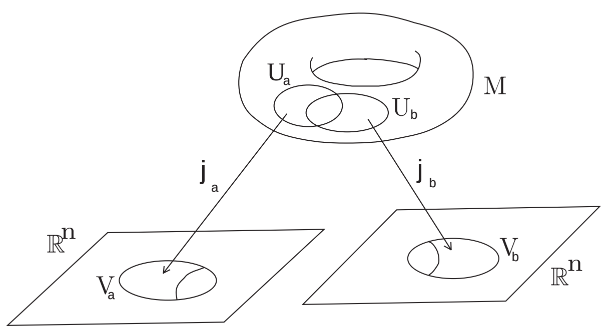
A *manifold* M of dimension n is a (Hausdorff, paracompact) topological space[1] M, such that every point $x \in M$ has a neighbourhood which is homeomorphic to an open set in Euclidean space $\mathbb{R}^n$.

A *chart* for M is a homeomorphism $\varphi : U \to V$ where U is open in M and V is open in $\mathbb{R}^n$.

A collection of charts $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha |\ \alpha \in \mathcal{I}\}$ is called an *atlas* for M if $\cup_{\alpha\in\mathcal{I}} = M$.

Next we want to impose some 'smooth' structure:

Definition 1.1.2 (Differentiable structures) Let M be an n-dimensional manifold. An atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha |\ \alpha \in \mathcal{I}\}$ for M is *differentiable* if for every α and β in $\mathcal{I}$, the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is differentiable[2] (as a map between open subsets of $\mathbb{R}^n$).



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[1] If you are not familiar with these topological notions, it is sufficient to consider metric spaces, or even subsets of Euclidean spaces. If you want to know more, see Appendix A.

[2] I will use the terms 'differentiable' and 'smooth' interchangeably, and both will mean 'infinitely many times differentiable'.

Two differentiable atlases $\mathcal{A}$ and $\mathcal{B}$ are *compatible* if their union is also a differentiable atlas — equivalently, for every chart ϕ in $\mathcal{A}$ and η in $\mathcal{B}$, $\phi \circ \eta^{-1}$ and $\eta \circ \phi^{-1}$ are smooth.

A *differentiable structure* on a manifold M is an equivalence class of differentiable atlases, where two atlases are deemed equivalent if they are compatible.

A *differentiable manifold* is a manifold M together with a differentiable structure on M.

We will usually abuse notation by simply referring to a 'differentiable manifold M' without referring to a differentiable atlas. This is slightly dangerous, because the same manifold can carry many inequivalent differentiable atlases, and each of these defines a different differentiable manifold.

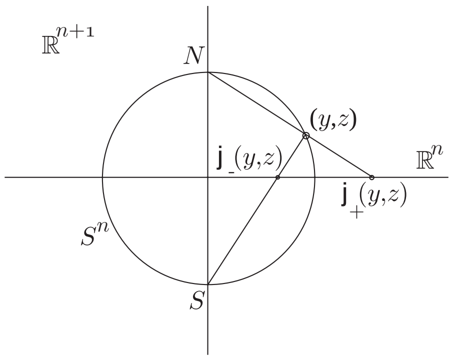
Example 1.1.1 Consider the set $M = \mathbb{R}$ (with the usual topology). This can be made a manifold is many different ways: The obvious way is to take the atlas $\mathcal{A} = \{\mathrm{Id} : \mathbb{R} \to \mathbb{R}\}$. Another way is to take the atlas $\mathcal{B} = \{x \mapsto x^3 : \mathbb{R} \to \mathbb{R}\}$. More generally, any homeomorphism of $\mathbb{R}$ to itself (or to an open subset of itself) can be used to define an atlas. The atlases $\mathcal{A}$ and $\mathcal{B}$ are incompatible because the union is $\{x \mapsto x,\ x \mapsto x^3\}$, which is not an atlas because the first map composed with the inverse of the second is the map $x \mapsto x^{1/3}$, which is not smooth. This example can be extended to show there are infinitely many different differentiable structures on the real line. This seems ridiculously complicated, but it will turn out that these differentiable structures are all equivalent in a sense to be defined later.

Remark. Given a differentiable structure on a manifold M, we can in principle choose a canonical atlas on M, namely the maximal atlas consisting of all those charts which are compatible with some differentiable atlas representing the differentiable structure. This is occasionally useful as a theoretical device but is completely unworkable in practice, as such a maximal atlas necessarily contains uncountable many charts.

Example 1.1.2: Euclidean space. A trivial example of a differentiable manifold is the Euclidean space $\mathbb{R}^n$, equipped with the atlas consisting only of the identity map.

Example 1.1.3: An atlas for S^n. The sphere S^n is the set $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$. Since this is a closed subset of Euclidean space, the topological requirements are satisfied. Define maps $\varphi_+ : S^n \backslash \{N\} \to \mathbb{R}^n$ and $\varphi_- : S^n \backslash \{S\} \to \mathbb{R}^n$ as follows, where N is the "north pole" $(0, \ldots, 0, 1)$ and S the "south pole" $(0, \ldots, 0, -1)$: Writing $x \in \mathbb{R}^{n+1}$ as (y, z) where $y \in \mathbb{R}^n$ and $z \in \mathbb{R}$, we take $\varphi_+(y, z) = \frac{y}{1-z}$ and $\varphi_-(y, z) = \frac{y}{1+z}$. Then we have $\varphi_+^{-1}(w) = \frac{(2w, |w|^2-1)}{|w|^2+1}$ and $\varphi_-^{-1}(w) = \frac{(2w, 1-|w|^2)}{|w|^2+1}$. It follows that $\varphi_\pm$ is a homeomorphism, since clearly

$\varphi_\pm^{-1}$ is continuous, and $\varphi_\pm$ is the restriction to S^n of a continuous map defined on all of $\mathbb{R}^{n+1}$. Also $\varphi_+ \circ \varphi_-^{-1}(w) = \frac{w}{|w|^2}$ and $\varphi_- \circ \varphi_+^{-1}(w) = \frac{w}{|w|^2}$ for all $w \in \mathbb{R}^n \backslash \{0\}$. Since these maps are differentiable, the two charts $\varphi_\pm$ form a differentiable atlas, and so define a differentiable manifold.



The maps $\varphi_\pm$ in the last example are the *stereographic projections* from the north and south poles. Next we consider an example which does not come from a subset of $\mathbb{R}^{n+1}$:

Example 1.1.4: The real projective spaces. The n-dimensional real projective space $\mathbb{R}P^n$ is the set of lines through the origin in $\mathbb{R}^{n+1}$ (an observer at the origin sees anything along such a line as being at the same position in the field of view – thus real projective space captures the geometry of perspective drawing). Equivalently, $\mathbb{R}P^n = (\mathbb{R}^{n+1}\backslash\{0\})/\sim$ where $x \sim y \iff x = \lambda y$ for some $\lambda \in \mathbb{R}\backslash\{0\}$. A point in $\mathbb{R}P^n$ is denoted $[x_1, x_2, \ldots, x_{n+1}]$, meaning the equivalence class under $\sim$ of the point $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$. We place a topology on $\mathbb{R}P^n$ by taking the open sets to be images of open sets in $\mathbb{R}^{n+1}$ under the projection onto $\mathbb{R}P^n$. This topology is Hausdorff: Take any two non-zero points x and y in $\mathbb{R}^{n+1}$ not lying on the same line. Choose an open set U about x which is disjoint from the line through y and the origin. Then choose an open set V about y which is disjoint from the set $\{\lambda z : \ z \in U, \ \lambda \in \mathbb{R}\}$. Then $U/\sim$ and $V/\sim$ are disjoint open sets in $\mathbb{R}P^n$ with $[x] \in U/\sim$ and $[y] \in V/\sim$. Define subsets V_i of $\mathbb{R}P^n$ for $i = 1, \ldots, n+1$ by $V_i = \{[x_1, \ldots, x_{n+1}] : x_i \neq 0\}$. Note that V_i is well-defined. Define maps $\varphi_i : V_i \to \mathbb{R}^n$ for $i = 1, \ldots, n+1$ by $\varphi_i([x_1, \ldots, x_{n+1}]) = \left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right)$. This has the inverse $\varphi_i^{-1}(x_1, \ldots, x_n) = [x_1, x_2, \ldots, x_{i-1}, 1, x_i, \ldots, x_n]$. The maps φ and their inverses are continuous; the open sets V_i cover $\mathbb{R}P^n$; and (for $i < j$)

$$\varphi_i \circ \varphi_j^{-1}(x_1, \ldots, x_n) = \left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{j-1}}{x_i}, \frac{1}{x_i}, \frac{x_j}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

which is smooth on the set $\varphi_j(V_i \cap V_j) = \{(x_1, \ldots, x_n) : \; x_i \neq 0\}$. The case $i > j$ is similar. Therefore the maps φ_i form an atlas for $\mathbb{R}P^n$.

Exercise 1.1.1 Define $\mathbb{C}P^n$ to be $\left(\mathbb{C}^{n+1}\backslash\{0\}\right) / \sim$, where $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{C}\backslash\{0\}$. Find a differentiable atlas which makes $\mathbb{C}P^n$ a $2n$-dimensional smooth manifold.

Example 1.1.5: Open subsets. Let M be a differentiable manifold, with atlas $\mathcal{A}$. Let U be any open subset of M. Then U is a differentiable manifold with the atlas $\mathcal{A}_{\mathcal{U}} = \{\varphi\big|_{\mathcal{U}} : \; \varphi \in \mathcal{A}\}$. Any open set in a Euclidean space is trivially a manifold. Other examples of manifolds obtained in this way are:

i). The general linear group $GL(n, \mathbb{R})$ (the set of non-singular $n \times n$ matrices) – this is an open set of the set of $n \times n$ matrices, which is naturally identified with $\mathbb{R}^{n^2}$;

ii). The multiplicative group $\mathbb{C}\backslash\{0\}$, clearly open in $\mathbb{C} \simeq \mathbb{R}^2$;

iii). The complement of a Cantor set (i.e. The set of real numbers which do not have a base 3 expansion consisting only of the digits 0 and 2).

Usually we will find ways to avoid things like the last example — by considering manifolds which are connected, or which satisfy some topological or geometric condition (such as compactness).

Example 1.1.6: Product manifolds. Let $(M^n, \mathcal{A})$ and $(N^k, \mathcal{B})$ be two manifolds. Then the topological product $M \times N$ can be made a manifold with the atlas $\mathcal{A}\#\mathcal{B} = \{(\varphi, \eta) : \; \varphi \in \mathcal{A}, \; \eta \in \mathcal{B}\}$. Here $(\varphi, \eta)(x, y) = (\varphi(x), \eta(y)) \in \mathbb{R}^{n+k}$ for each $(x, y) \in U \times W$, where $\varphi : U \to V$ is in $\mathcal{A}$ and $\eta : W \to Z$ is in $\mathcal{B}$.