Lecture 10. The Levi-Civita connection

In this lecture we will show that a Riemannian metric on a smooth manifold induces a unique connection.

10.1 Compatibility of a connection with the metric

Let M be a smooth Riemannian manifold with metric g. A connection ∇ on M is said to be *compatible with the metric* on M if for every pair of vector fields X and Y on M, and every vector $v \in T_xM$,

$$v\left(g(X,Y)\right) = g\left(\nabla_v X, Y\right) + g\left(X, \nabla_v Y\right).$$

Here on the left hand side we are applying the vector v (as a derivation) to the smooth function $x \mapsto g_x(X_x, Y_x)$. This is something which is well-defined without reference to any connection. The right-hand side does depend on the connection.

Example 10.1.1 Differentiation of vector fields on a vector space is compatible with any inner product on the vector space: We have

$$D_v\left(X \cdot Y\right) = D_v X \cdot Y + X \cdot D_v Y.$$

Compatibility of the connection with the metric can be expressed in terms of parallel transport: Suppose γ is a smooth curve in M, and E_1 and E_2 are smooth vector fields along γ. Then

$$\frac{d}{dt} g(E_1, E_2) = g(\nabla_t E_1, E_2) + g(E_1, \nabla_t E_2).$$

In the special case where E_1 and E_2 are parallel along γ, this implies that $\frac{d}{dt} g(E_1, E_2) = 0$. Therefore vector fields which are parallel with respect to a compatible connection have constant length (take $E_1 = E_2$) and make a constant angle to each other. A particular consequence is that geodesics have tangent vectors of constant length.

Exercise 10.1.2 Show that any connection for which the lengths and angles between parallel vector fields are constant must be compatible with the metric.

In particular it is very easy to parallel transport a vector along a geodesic for a compatible connection on a two-dimensional manifold, since there is a unique vector at each point which makes the same angle with the tangent vector of the geodesic.

10.2 Submanifolds

Let M be a submanifold of Euclidean space $\mathbb{R}^N$, and induce on M the submanifold connection (given by projecting the derivatives of vector fields onto the tangent space of M) and the submanifold metric (where the lengths of tangent vectors to M are given by the lengths of their image in $\mathbb{R}^N$ under the inclusion map).

Proposition 10.2.1 *The submanifold connection ∇ is compatible with the induced metric g.*

Proof. We compute directly:

$$\begin{aligned} vg(X,Y) &= v\langle X^\alpha e_\alpha, Y^\beta e_\beta\rangle \\ &= \langle (D_v X^\alpha)\, e_\alpha, Y^\beta e_\beta\rangle + \langle X^\alpha e_\alpha, \left(D_v Y^\beta\right) e_\beta\rangle \\ &= \langle \pi\,(D_v X^\alpha)\, e_\alpha, Y^\beta e_\beta\rangle + \langle X^\alpha e_\alpha, \pi\left(D_v Y^\beta\right) e_\beta\rangle \\ &= g\left(\nabla_v X, Y\right) + g\left(X, \nabla_v Y\right). \end{aligned}$$

□

10.3 The Levi-Civita Theorem

Proposition 10.3.1 *Let M be a smooth Riemannian manifold with metric g. Then there exists a unique connection ∇ on M whicj is symmetric and compatible with g.*

The connection given by this proposition is called the Levi-Civita connection, or sometimes the Riemannian connection. Note that the Levi-Civita connection on a submanifold of Euclidean space (with the metric induced by the standard inner product) is just the submanifold connection.

Proof. First we show uniqueness: Let X, Y, and Z be three smooth vector fields on M. Then we must have the symmetry conditions

$$\nabla_X Y - \nabla_Y X = [X, Y];$$
$$\nabla_Y Z - \nabla_Z Y = [Y, Z];$$
$$\nabla_Z X - \nabla_X Z = [Z, X],$$

and the compatibility conditions

$$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = Xg(Y, Z);$$
$$g(\nabla_Y Z, X) + g(Z, \nabla_Y X) = Yg(Z, X);$$
$$g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = Zg(X, Y).$$

Take the sum of the first two of the latter equations, and subtract the third. Then apply the symmetry conditions, yielding:

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & + g(Z, [X, Y]) + g(Y, [Z, X]) + g(X, [Z, Y]). \end{aligned}$$

This determines the inner product of $\nabla_x Y$ with any vector field Z purely in terms of the metric, and so implicitly determines $\nabla_x Y$. This completes the proof of uniqueness. To prove existence, it is only necessary to check that the formula above does indeed define a connection with the desired properties.

Denote the right-hand side of the formula by $C(X, Y, Z)$. C is C^∞ in X and in Z:

$$\begin{aligned} C(fX, Y, hZ) &= fX\,(hg(Y, Z)) + Y\,(fhg(Z, X)) - hZ\,(fg(X, Y)) \\ &\quad + hg(Z, [fX, Y]) + g(Y, [hZ, fX]) + fg(X, [hZ, Y]) \\ &= fhXg(Y, Z) + fhYg(Z, X) - fhZg(X, Y) \\ &\quad + f(Xh)g(Y, Z) + fY(h)g(Z, X) \\ &\quad + hY(f)g(Z, X) - hZ(f)g(X, Y) \\ &\quad + hfg(Z, [X, Y]) + hfg(Y, [Z, X]) + fhg(X, [Z, Y]) \\ &\quad - hY(f)g(Z, X) - fX(h)g(Y, Z) \\ &\quad + hZ(f)g(Y, X) - fY(h)g(X, Z) \\ &= fhC(X, Y, Z) \end{aligned}$$

It follows that the map $x \mapsto C(X, Y, Z)_x$ depends only on Y and the values X_x and Z_x of the vector fields X and Z at x In the second slot we have instead:

$$\begin{aligned} C(X, fY, Z) &= fC(X, Y, Z) + X(f)g(Y, Z) - Z(f)g(X, Y) \\ &\quad + X(f)g(Z, Y) + Z(f)g(X, Y) \\ &= fC(X, Y, Z) + 2X(f)g(Y, Z). \end{aligned}$$

Therefore if we define $\nabla_X Y$ by requiring that $g(\nabla_X Y, Z) = \frac{1}{2}C(X, Y, Z)$, then $(\nabla_X Y)_x$ depends only on X_x and Y, and

$$g(\nabla_X (fY), Z) = fg(\nabla_X Y, Z) + X(f)g(Y, Z) = g(f\nabla_X Y + X(f)Y, Z),$$

so ∇ satisfies the Leibniz rule and defines a connection. □

In working with the Levi-Civita connection, it is often convenient to look at the formula which defines it in terms of a local coordinate tangent basis: Choose a local chart $\varphi : U \to V$ for M. Then the formula above, applied with X, Y, and Z given by coordinate tangent vector fields ∂_i, ∂_j and ∂_k, gives the following expression for the connection coefficients:

$$\Gamma_{ij}{}^k = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right),$$

where we denote by g^{ij} the *inverse* of the metric g_{ij}.

Exercise 10.3.1 Suppose M is a smooth manifold, and N is a smooth Riemannian manifold with metric h. Let $F : M \to N$ be an immersion. Then h induces on M a metric g. For each $x \in M$ let $\pi_x : T_F(x)N \to T_xM$ be the map given by orthogonal projection onto $D_xF(T_xM)$, followed by application of $(D_xF)^{-1}$. Define

$$\nabla_v X = \pi_x \left(\nabla^{(h)}_{DF(v)} DF(X) \right),$$

where $\nabla^{(h)}$ is the Levi-Civita connection of the metric h on N. Show that this is well-defined (i.e. makes sense even though $DF(X)$ is only defined on the image $F(M) \subset N$) and defines a connection on M, which is the Levi-Civita connection on M.

As we have already observed, the geodesics of the Levi-Civita connection have tangent vectors of constant speed, and parallel transport along any curve preserves inner products between parallel vector fields.

10.4 Left-invariant metrics

Let G be a Lie group. Then we have a connection on G defined by taking the left-invariant vector fields to be parallel. We have seen that this connection does not give a symmetric connection, and so does not give the Levi-Civita connection for a left-invariant metric. However, this connection is compatible with any left-invariant metric.

Applying the formula above in the case of a left-invariant metric, we have the following expression for the Levi-Civita connection: Let $\{E_1, \ldots, E_n\}$ be left-invariant vector fields which are orthonormal for the metric. Then by assumption the inner products $g(E_i, E_j)$ are constant, and the first three terms become zero. This gives

$$\nabla_{E_i} E_j = \frac{1}{2} \sum_{k=1}^{n} \left(c_{ijk} + c_{kij} + c_{kji} \right) E_k,$$

where the *structure constants* of G are defined by

$$[E_i, E_j] = \sum_{k=1}^{n} c_{ijk} E_k.$$

In contrast the left-invariant connection would have $\nabla_{E_i} E_j = 0$.

10.5 Exponential normal coordinates

It is often convenient to use the exponential map of the Levi-Civita connection to produce a chart for a Riemannian manifold.

Proposition 10.5.1 *Let (M, g) be a smooth Riemannian manifold, and ∇ the Levi-Civita connection. Fix $x \in M$. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for $T_x M$, and define a chart φ on a neighbourhood U of x by*

$$\varphi^{-1}\left(x^1, \ldots, x^n\right) = \exp_x\left(x^j e_j\right).$$

In these coordinates (called exponential normal coordinates) we have

$$g_{ij}(0) = \delta_{ij};$$

and

$$\Gamma_{ij}{}^{k}(0) = 0.$$

Proof. Recall that the derivative of the exponential map at the origin is just the identity map. Therefore

$$g_{ij}(0) = g(\partial_i, \partial_j) = g(D_0 \exp_x(e_i), D_0 \exp_x(e_j)) = \delta_{ij}$$

since the $\{e_i\}$ were chosen to be orthonormal.

We also have

$$\nabla_{a^i \partial_i}\left(a^j \partial_j\right) = 0$$

at the origin for any constants $a^1, \ldots, a^n$, since $a^i \partial_i$ is the tangent vector to the geodesic through 0 in that direction. By symmetry we have at the origin

$$\begin{aligned} 0 &= \nabla_{\partial_i + \partial_j}(\partial_i + \partial_j) \\ &= \nabla_{\partial_i}\partial_i + \nabla_{\partial_j}\partial_j + \nabla_{\partial_i}\partial_j + \nabla_{\partial_j}\partial_i \\ &= 2\nabla_{\partial_i}\partial_j. \end{aligned}$$

for every i and j. Therefore all the connection coefficients vanish at the origin. □

Remark. By the same proof, the connection coefficients at the origin vanish with respect to exponential coordinates for a connection (not necessarily a Levi-Civita connection) if and only if the connection is symmetric. This gives another interpretation of the torsion of a connection.