Lecture 14. Stokes' Theorem

In this section we will define what is meant by integration of differential forms on manifolds, and prove Stokes' theorem, which relates this to the exterior differential operator.

14.1 Manifolds with boundary

In defining integration of differential forms, it will be convenient to introduce a slightly more general notion of manifold, allowing for the possibility of a boundary.

Definition 14.1.1 A *boundary chart* $\varphi : U \to V$ for a topological space M about a point $x \in M$ is a continuous map from an open set $U \subseteq M$ to a (relatively) open subset V of $\mathbb{R}^n_+ = \{(x^1, \ldots, x^n) : \ x^n \geq 0$ with $\varphi(x) \in \mathbb{R}^{n-1} \times \{0\}$.

Here the open subsets of $\mathbb{R}^n_+$ are the sets of the form $W \cap \mathbb{R}^n_+$ where $W \subseteq \mathbb{R}^n$ is open.

Definition 14.1.2 A smooth *boundary atlas* $\mathcal{A}$ for a topological space M is a collection of maps $\varphi_\alpha : U_\alpha \to V_\alpha$ each of which is either a chart or a boundary chart for M, such that $\cup U_\alpha = M$ and such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a smooth map between open sets of $\mathbb{R}^n_+$ for each α and β.

Definition 14.1.3 A smooth *manifold with boundary* is a topological space M equipped with an equivalence class of smooth boundary atlases, where two boundary atlases are equivalent if their union is again a boundary atlas.

If M is a manifold with boundary, then the *boundary* ∂M of M is the subset of M consisting of all those points $x \in M$ for which there is a boundary chart about x.

Proposition 14.1.1 *If M is a smooth manifold with boundary of dimension n, then ∂M is a smooth manifold of dimension $n-1$, with atlas given by the restriction to ∂M of all boundary charts for M.*

Proof. Let $\varphi : U \to V$ and $\eta : W \to Z$ be boundary charts for M, and let $U_0 = \varphi^{-1}(\mathbb{R}^n \times \{0\} \cap V)$ and $W_0 = \eta^{-1}(\mathbb{R}^n \times \{0\} \cap Z)$. Assume that $U_0 \cap W_0$ is non-empty. Then the associated charts for ∂M are $\varphi_0 = \varphi|_{U_0}$ and $\eta_0 = \eta|_{W_0}$. The transition map $\eta_0 \circ \varphi_0^{-1}$ is given by the restriction of the smooth map $\eta \circ \varphi^{-1}$ to $\mathbb{R}^n \times \{0\}$, and is therefore smooth. □

14.2 Induced orientation on the boundary

Suppose M^n is an oriented manifold with boundary — that is, M is equipped with a smooth boundary atlas $\mathcal{A}$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is an orientation-preserving map for all α and β.

Proposition 14.2.1 *∂M is an orientable manifold of dimension $n-1$.*

Proof. Let $\mathcal{A}$ be an oriented boundary atlas for M. Then the corresponding atlas for ∂M is automatically oriented: Any pair of overlapping oriented boundary charts for M map Rls^{n+} to $\mathbb{R}^n_+$, and the derivative of a transition map on the boundary must have the form

$$D\left(\eta \circ \varphi^{-1}\right) = \begin{bmatrix} D\left(\eta_0 \circ \varphi_0^{-1}\right) & * \\ 0 & a \end{bmatrix}$$

where $a = \langle D(\eta \circ \varphi^{-1})(e_{n+1}), e_{n+1}\rangle > 0$. Therefore $\eta_0 \circ \varphi_0^{-1}$ is orientation-preserving, and the atlas is oriented. □

Note that the orientation can be understood geometrically as follows: An n-tuple of linearly independent vectors $u_1, \ldots, u_n$ tangent to ∂M is called positively oriented if the $(n+1)$-tuple $u_1, \ldots, u_n, \partial_{n+1}$ is oriented in M for any boundary chart.

The orientation constructed on ∂M in the proof is called the *induced orientation* on ∂M from the orientation on M (this is a matter of convention — we could just as well have chosen the opposite orientation).

In the proof of the Proposition above we ignored the case $n = 1$ — the boundary of a 1-dimensional manifold is a 0-dimensional manifold (i.e. a collection of points). What does it mean to define an orientation on a zero-dimensional manifold? Our original definition clearly makes no sense in that case. However the equivalent definition in Proposition 13.6.1 does make sense: We will say a 0-manifold N is oriented if it is equipped with a function (i.e. a 0-form) from N to $\mathbb{Z}_2 = \{-1, 1\}$ (this corresponds to the remarks after the proof of Proposition 13.6.1: The orientation bundle in this case is just $N \times \mathbb{Z}_2$. In this case we also have to allow boundary charts for 1-manifolds which map to $(-\infty, 0]$ as well as $[0, \infty)$ (in higher dimensions we can always transform charts into any half-plane via an orientation-preserving map to map into the upper half-plane, but not if $n = 1$).

14.3 More on partitions of unity

Now we want to extend our results on partitions of unity on manifolds to the slightly more general setting of manifolds with boundary.

Proposition 14.3.1 *Let M be a differentiable manifold with boundary. Then there exists a partition of unity on M subordinate to any boundary atlas for M.*

The proof of this result is identical to our result on existence of partitions of unity on differentiable manifolds, except that we have to include functions with support $B_r^n(0) \times [0, r)$. These can be constructed easily from the smooth compactly supported functions we already know.

14.4 Integration of forms on oriented manifolds

Now we can give some further meaning to differential forms by defining what is meant by integration of differential forms on oriented manifolds. A key point to keep in mind here is that none of our definitions depend on us having a metric on the manifold, so we do not in general have any notion of volume of surface area or length. Nevertheless the structure of differential forms is exactly what is required to produce a well-defined notion of integration.

Let M^n be a compact, oriented differentiable manifold with boundary, and let $\omega \in \Omega^n(M)$. Then we define the integral of ω over M, denoted $\int_M \omega$, as follows: Let $\{\rho_\alpha : \alpha \in \mathcal{I}\}$ be a partition of unity subordinate to an oriented boundary atlas for M, so that for each α there exists an oriented chart (either a regular chart or a boundary chart) $\varphi_\alpha : U_\alpha \to V_\alpha$ for M, such that $\mathrm{supp}\rho_\alpha \subset U_\alpha$.

Define

$$\int_M \omega = \sum_{\alpha \in \mathcal{I}} \int_{V_\alpha} \left((\varphi_\alpha^{-1})_*(\rho_\alpha \omega)\right)(e_1, \ldots, e_n)\, dx^1 \ldots dx^n.$$

To put this into words: We write ω as a sum $\sum_\alpha \rho_\alpha \omega$ of forms which are supported in charts. Each of these can be integrated over the chart by integrating the smooth function obtained by plugging in the coordinate tangent vectors for that chart. Then we add the resulting numbers together to get the integral of ω.

We need to check that this does not depend on the choice of partition of unity. Suppose $\{\phi_\beta : \beta \in \mathcal{J}\}$ is any other partition of unity for M, with corresponding oriented coordinate charts $\eta_\beta : W_\beta \to Z_\beta$. Then we have

$$\sum_\alpha \int_{V_\alpha} \left((\varphi_\alpha^{-1})_*(\rho_\alpha\omega)\right)(e_1,\ldots,e_n)\,dx^1\ldots dx^n$$
$$= \sum_{\alpha,\beta} \int_{\varphi_\alpha(U_\alpha\cap W_\beta)} \left((\varphi_\alpha^{-1})_*(\rho_\alpha\phi_\beta\omega)\right)(e_1,\ldots,e_n)\,dx^1\ldots dx^n.$$

Fix α and β. Then by definition of the pull-back, we have for any form σ

$$\left((\varphi_\alpha^{-1})_*\sigma\right)(e_1,\ldots,e_n) = \left((\eta_\beta^{-1})_*\sigma\right)(D(\eta_\beta\circ\varphi_\alpha^{-1})(e_1),\ldots,D(\eta_\beta\circ\varphi_\alpha^{-1})(e_n)).$$

We apply the following useful Lemma:

Lemma 14.4.1 *Let ω be an alternating n-tensor, and L a linear map. Then*

$$\omega(Le_1,\ldots,Le_n) = (\det L)\omega(e_1,\ldots,e_n).$$

Proof. We can assume that ω is non-zero. Consider the map from $GL(n)$ to $\mathbb{R}$ defined by

$$\tilde{\det} : L \mapsto \frac{\omega(Le_1,\ldots,Le_n)}{\omega(e_1,\ldots,e_n)}.$$

The denominator is non-zero by assumption.

The multilinearity and antisymmetry of ω imply $\tilde{\det}$ is linear in each row, is unchanged by adding one row to another, and has the value 1 if $L = I$. These are the axioms that define the determinant, so $\tilde{\det} = \det$. □

It follows that

$$\left((\varphi_\alpha^{-1})_*\sigma\right)(e_1,\ldots,e_n) = \det(D(\eta_\beta\circ\varphi_\alpha^{-1}))\left((\eta_\beta^{-1})_*\sigma\right)(e_1,\ldots,e_n).$$

We also know, since the charts are oriented, that the determinant on the right-hand side is positive. The change of variables formula therefore gives

$$\int_{\varphi_\alpha(U_\alpha\cap W_\beta)} \left((\varphi_\alpha^{-1})_*(\rho_\alpha\phi_\beta\omega)\right)(e_1,\ldots,e_n)\,dx^1\ldots dx^n$$
$$= \int_{\varphi_\alpha(U_\alpha\cap W_\beta)} |\det(D(\eta_\beta\circ\varphi_\alpha^{-1}))|\left((\eta_\beta^{-1})_*\sigma\right)(e_1,\ldots,e_n)\,dx^1\ldots dx^n$$
$$= \int_{\eta_\beta(U_\alpha\cap W_\beta)} \left((\eta_\beta^{-1})_*\sigma\right)(e_1,\ldots,e_n)\,dx^1\ldots dx^n.$$

Consequently

$$\sum_\alpha \int_{V_\alpha} \left((\varphi_\alpha^{-1})_*(\rho_\alpha\omega)\right)(e_1,\ldots,e_n)\,dx^1\ldots dx^n$$
$$= \sum_\beta \int_{Z_\beta} \left((\eta_\beta^{-1})_*(\phi_\beta\omega)\right)(e_1,\ldots,e_n)\,dx^1\ldots dx^n,$$

so the integral of ω is well-defined.

14.5 Stokes' theorem

Now we are in a position to prove the fundamental result concerning integration of forms on manifolds, namely Stokes' theorem. This will also give us a geometric interpretation of the exterior derivative.

Proposition 14.5.1 *Let M^n be a compact differentiable manifold with boundary, and let $\omega \in \Omega^{n-1}(M)$. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

where the integral on the right-hand side is taken using the induced orientation on $\partial\Omega$, integrating the restriction of ω to ∂M (i.e. the pull-back of ω by the inclusion map).

In particular, if M is a compact manifold without boundary, then the integral of the exterior derivative of any $(n-1)$-form is zero.

Proof. Let $\{\rho_\alpha\}$ be a partition of unity on M, with each ρ_α supported in a chart $\varphi_\alpha : U_\alpha \to V_\alpha$. We can write

$$\begin{aligned}
\int_M d\omega &= \sum_\alpha \int_M d(\rho_\alpha \omega) \\
&= \sum_\alpha \int_{V_\alpha} ((\varphi_\alpha^{-1})_*(d(\rho_\alpha\omega)))(e_1, \dots, e_n) dx^1 \dots dx^n \\
&= \sum_\alpha \int_{V_\alpha} d\left((\varphi_\alpha^{-1})_*(\rho_\alpha\omega)\right)(e_1, \dots, e_n)\, dx^1 \dots dx^n.
\end{aligned}$$

For each α there are two possibilities: The chart φ_α is either a regular chart or a boundary chart.

In the first case, V_α is an open set in $\mathbb{R}^n$. Write ω in components in the chart φ_α:

$$\omega = \sum_{j=1}^n \omega_j dx^1 \wedge \hat{dx^j} \wedge \dots \wedge dx^n.$$

Then the integrand in the corresponding integral becomes

$$\sum_{j=1}^n \frac{\partial(\rho_\alpha \omega_j)}{\partial x^j}.$$

Applying Fubini's theorem and the fundamental theorem of calculus, and noting that $\rho_\alpha = 0$ on the boundary of our domain, we find that the resulting integral is zero.

In the second case the value of the corresponding integral is

$$\begin{aligned}\sum_{j=1}^{n}\int_{V_\alpha}\frac{\partial(\rho_\alpha\omega_j)}{\partial x^j}dx^1\dots dx^n &= \int_{\mathbb{R}^{n-1}\times\{0\}}\int_{-\infty}^{0}\frac{\partial(\rho_\alpha\omega_n)}{\partial x^n}dx^n\,dx^1\dots dx^{n-1}\\ &= \int_{\mathbb{R}^{n-1}\times\{0\}}\rho_\alpha\omega_n dx^1\dots dx^{n-1}\\ &= \int_{\partial M}\rho_\alpha\omega.\end{aligned}$$

Note that verifying the last line here involves checking that the orientation on ∂M is correct. Summing over α and noting that $\sum_\alpha \rho_\alpha = 1$, we obtain the result. □

Turning the result of Stokes' theorem around, we can interpret the exterior derivative in the following way: Let ω be a k-form in a manifold M. Fix linearly independent vectors $v_1, \dots, v_{k+1}$ in T_xM, and choose any chart φ about x. Write $v_j = v_j^l\partial_l$ in this chart. For r small we can define smooth maps x_r from the k-dimensional sphere S^k into M, by

$$x_r(z^i e_i) = \varphi^{-1}(\varphi(x) + rz^i v_i^k e_k).$$

Using Stokes' theorem, we can deduce

$$d\omega(v_1, \dots, v_{k+1}) = \lim_{r\to 0}\frac{1}{r^{k+1}|B^{k+1}|}\int_{S^k}(x_r)_*\omega$$

where $|B^{k+1}|$ is the volume of the unit ball in $\mathbb{R}^{k+1}$. In this sense the exterior derivative measures the 'boundary integral per unit volume' of a form (where 'volume' is measured in comparison to that of the parallelepiped generated by $v_1, \dots, v_{k+1}$, not using any notion of measure or metric on the manifold).

This is easy to understand in the case of a 0-form (i.e. a function). Then the 'boundary integral' becomes 'difference in values at the endpoints', while 'per unit volume' means 'per unit time along a curve with velocity v_1'. So this just recaptures the usual notion of the directional derivative of a function in terms of difference quotients.

Example 14.5.2 (The case of 1-manifolds)

Let f be a 0-form on a compact 1-manifold M. Note that M is a union of circles $\{S_i^1\}$ and closed intervals I_i with endpoints x_i^+ and x_-^i. An orientation on M amounts to choosing a direction ('left' or 'right') on each component of M, and the induced orientation on ∂M (i.e. the endpoints $x_i^\pm$) is given by assigning an endpoint the value 1 if the orientation direction of M points out of M there, -1 if it points inwards. Each of the closed interval components I_i of M therefore has one endpoint with orientation $+1$ (say x_i^+) and the other with orientation -1. Stokes' theorem becomes

$$\int_M df = \sum_i f(x_i^+) - f(x_i^-).$$

Example 14.5.3 (Regions in $\mathbb{R}^2$)

Let V be a vector field on an bounded open set U in $\mathbb{R}^2$ with smooth boundary curves. We can write $V = V^i e_i$. There is a corresponding 1-form ω defined by

$$\omega(v) = \langle V, v\rangle$$

for all vectors v. Explicitly, this means $\omega = \omega_i dx^i$ where $\omega_i = \omega(e_i) = \langle V, e_i\rangle = V^i$. The exterior derivative is then

$$d\omega = \frac{\partial V^i}{\partial x^j} dx^j \wedge dx^i = (\frac{\partial V^2}{\partial x^1} - \frac{\partial V^1}{\partial x^2}) dx^1 \wedge dx^2,$$

which we recognize as the curl of the vector field V times $dx^1 \wedge dx^2$. The integral of $d\omega$ over U is then

$$\int_U d\omega = \int_U \mathrm{curl} V \, dx^1 dx^2$$

and the integral of ω around the boundary is

$$\int_{\partial U} \omega = \int_{\partial U} \langle V, T\rangle$$

where T is the unit tangent vector to ∂U, taken to run anticlockwise on those parts of the boundary that lie on the 'outside' of U, and clockwise on parts that are 'inside' U. Stokes' theorem tells us that these two are equal. This recaptures the classical Stokes' theorem in the plane.

Example 14.5.4 (Vector fields in space).

The same argument as above applies if V is a vector field in $\mathbb{R}^3$ and M is a two-dimensional submanifold with boundary: There is a corresponding 1-form ω defined as above, and this can be restricted (i.e. pulled back by the inclusion map) to M. The exterior derivative of the resulting form is the curl of V in the normal direction to M, times the volume form on M. So applying out general Stokes' theorem in this case gives that the flux of the curl of the vectgor field V through the surface M is equal to the circulation of V around the boundary of M, which is the classical Stokes' theorem.

There is another way to associate the vector field V with a form: If $V = V^i e_i$ is a vector field, then we can take ω to be a 2-form defined by

$$\omega_{ij} = \varepsilon_{ijk} V^k$$

where ε is the alternating tensor, defined by

$$\varepsilon_{ijk} = \begin{cases} 1 \text{ if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 \text{ if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0 \text{ otherwise.} \end{cases}$$

Explicitly this gives

$$\omega = V^3 dx^1 \wedge dx^2 - V^2 dx^1 \wedge dx^3 + V^1 dx^2 \wedge dx^3.$$

Taking the exterior derivative gives

$$d\omega = \sum_{i=1}^{3} \frac{\partial V^i}{\partial x^i} dx^1 \wedge dx^2 \wedge dx^3.$$

This is just the divergence of the vector field V times the volume form. Stokes' theorem then says that the integral of $d\omega$ over a region U (i.e. the integral of the divergence of V over U) is equal to the integral of the 2-form ω over ∂U. The latter is equal to the integral of $\langle V, \mathbf{n} \rangle$ over ∂U, where $\mathbf{n}$ is the outward-pointing unit normal vector. This is just the classical Gauss theorem (or divergence theorem).