Lecture 15. de Rham cohomology

In this lecture we will show how differential forms can be used to define topological invariants of manifolds. This is closely related to other constructions in algebraic topology such as simplicial homology and cohomology, singular homology and cohomology, and Čech cohomology.

15.1 Cocycles and coboundaries

Let us first note some applications of Stokes' theorem: Let ω be a k-form on a differentiable manifold M. For any oriented k-dimensional compact submanifold Σ of M, this gives us a real number by integration:

$$\omega : \Sigma \mapsto \int_\Sigma \omega.$$

(Here we really mean the integral over Σ of the form obtained by pulling back ω under the inclusion map).

Now suppose we have two such submanifolds, Σ_0 and Σ_1, which are (smoothly) homotopic. That is, we have a smooth map $F : \Sigma \times [0,1] \to M$ with $F|_{\Sigma\times\{i\}}$ an immersion describing Σ_i for $i = 0, 1$. Then $d(F_*\omega)$ is a $(k+1)$-form on the $(k+1)$-dimensional oriented manifold with boundary $\Sigma \times [0,1]$, and Stokes' theorem gives

$$\int_{\Sigma\times[0,1]} d(F_*\omega) = \int_{\Sigma_1} \omega - \int_{\Sigma_1} \omega.$$

In particular, if $d\omega = 0$, then $d(F_*\omega) = F_*(d\omega) = 0$, and we deduce that $\int_{\Sigma_1} \omega = \int_{\Sigma_0} \omega$.

This says that k-forms with exterior derivative zero give a well-defined functional on homotopy classes of compact oriented k-dimensional submanifolds of M.

We know some examples of k-forms with exterior derivative zero, namely those of the form $\omega = d\eta$ for some $(k-1)$-form η. But Stokes' theorem then gives that $\int_\Sigma \omega = \int_\Sigma d\eta = 0$, so in these cases the functional we defined on homotopy classes of submanifolds is trivial.

This leads us to consider the space of 'non-trivial' functionals on homotopy classes of submanifolds: Each of these is defined by a k-form ω with exterior derivative zero, but is unchanged if we add the exterior derivative of an arbitrary $(k-1)$-form to ω.

We call a k-form with exterior derivative zero a k-cocycle, and a k-form which is an exterior derivative of a form is called a k-coboundary. The space of k-cocycles on M is a vector space, denoted $Z^k(M)$, and the space of k-coboundaries is then $d\Omega^{k-1}(M)$, which is contained in $Z^k(M)$.

15.2 Cohomology groups and Betti numbers

We define the k-th de Rham cohomology group of M, denoted $H^k(M)$, to be

$$H^k(M) = \frac{Z^k(M)}{d\Omega^{k-1}(M)}.$$

Thus an element of $H^k(M)$ is defined by any k-cocycle ω, but is unchanged by changing ω to $\omega + d\eta$ for any $(k-1)$-form η, which agrees with the notion we produced before of a 'nontrivial' functional on homotopy classes of submanifolds.

An element of $H^k(M)$ is called a *cohomology class*, and the cohomology class containing a k-cocycle ω is denoted $[\omega]$. Thus

$$[\omega] = \{\omega + d\eta : \ \eta \in \Omega^{k-1}(M)\}.$$

Since the exterior derivative and Stokes' theorem do not depend in any way on the presence of a Riemannian metric on M, the cohomology groups of M depend only on the differentiable structure on M. It turns out that they in fact depend only on the topological structure of M, and not on the differentiable structure at all — any two homeomorphic manifolds have the same cohomology groups[5]. The groups $H^k(M)$ are therefore topological invariants, which can be used to distinguish manifolds from each other: If two manifolds have different cohomology groups, they cannot be homeomorphic (let alone diffeomorphic).

The k-the cohomology group $H^k(M)$ is a real vector space. The dimension of this vector space is called the kth *Betti number* of M, and denoted $b_k(M)$.

[5] The de Rham theorem states that the de Rham cohomology groups are isomorphic to the singular or Čech cohomology groups with real coefficients, and these are defined in purely topological terms. It is also a consequence of this theorem that the cohomology groups are finite dimensional.

15.3 The group $H^0(M)$

The group $H^0(M)$ is relatively easy to understand: The space $Z^0(M)$ is just the space of functions on M with derivative zero, which is the space of locally constant functions. We interpret Ω^{-1} as the trivial vector space. Therefore $H^0(M) \simeq Z^0(M) = \mathbb{R}^N$ where N is the number of connected components of M. Thefore $b_0(M)$ is equal to the number of connected components of M.

15.4 The group $H^1(M)$

The group $H^1(M)$ is closely related to the fundamental group $\pi_1(M)$. We will examine some aspects of this relationship:

Proposition 15.4.1 *Suppose M is connected. If $\omega \in Z^1(M)$ and $[\omega] \neq 0$ in $H^1(M)$, then there exists a smooth curve $\gamma : S^1 \to M$ such that*

$$\omega(\gamma) := \int_{S^1} \gamma^*\omega \neq 0.$$

Proof. We will prove that if $\omega(\gamma) = 0$ for every smooth map $\gamma : S^1 \to M$, then $[\omega] = 0$ in $H^1(M)$.

On each connected component of M choose a 'base point' x_0. We define a function $f \in C^\infty(M) \simeq \Omega^0(M)$ by setting $f(x_0) = 0$ and extending to other points of M according to

$$f(x) = \int_{[0,1]} \gamma^*\omega$$

for any $\gamma : [0,1] \to M$ with $\gamma(0) = x_0$ and $\gamma(1) = x$. This is well-defined, since if γ_1 and γ_2 are two such curves, then the curve $\gamma_1\#(-\gamma_2)$ obtained by concatenating γ_1 and $-\gamma_2$ (i.e. γ_2 with orientation reversed) gives a map from S^1 to M, so by assumption

$$0 = \int_{S^1} (\gamma_1\#(-\gamma_2))^*\omega = \int_{S^1} \gamma_1^*\omega - \int_{S^1} \gamma_2^*\omega,$$

and so the value of $f(x)$ is independent of the choice of γ. Finally, $df = \omega$, since in a chart φ about x,

$$df(\partial_i) = \partial_i f = \frac{d}{dt}\int_0^{1+t} \gamma^*\omega = \omega(\partial_i)$$

where $\gamma(1+t) = \varphi^{-1}(\varphi(x) + te_i)$.

This shows that $\omega = df$, so that $[\omega] = 0$ in $H^1(M)$. $\square$

Corollary 15.4.2 *If M has finite fundamental group then $H^1(M) = 0$. In particular if M is simply connected, then $H^1(M) = 0$.*

Proof. Let $\omega \in \Omega^1(M)$ with $d\omega = 0$. Then for any closed loop $\gamma : S^1 \to M$, we have $[\gamma]^n = 0$ in $\pi^1(M)$ for some integer n. Therefore we have a homotopy $F : S^1 \times [0,1] \to M$ from $\gamma\#\gamma \ldots \#\gamma$ to the constant loop c, and Stokes' theorem gives

$$0 = \int_{S^1\times[0,1]} F^* d\omega = \int_{S^1} (\gamma\# \ldots \#\gamma)^*\omega - \int_{S^1} c^*\omega = n\omega(\gamma).$$

Since this is true for all closed loops γ, Proposition 15.4.1 applies to show $[\omega] = 0$ in $H^1(M)$, and so $H^1(M) = 0$. □

The same argument tells us something more: In fact $H^1(M)$ is a subspace of the dual space of the vector space $G \otimes \mathbb{R}$, where G is the abelianisation of $\pi_1(M)$, which is the abelian group given by taking $\pi_1(M)$ and imposing the extra relations $aba^{-1}b^{-1} = 1$ for all elements a and b. In particular, $b_1(M)$ is no greater than the smallest number of generators of $\pi_1(M)$. In fact it turns out (at least for compact manifolds) that $H^1(M)$ is isomorphic to the torsion-free part of the abelianisation of $\pi_1(M)$, as described above. We will not prove this here.

15.5 Homotopy invariance

In this section we will prove a remarkable topological invariance property of cohomology groups: They do not change when the space is continuously deformed.

More precisely, suppose M and N are two manifolds, and F is a smooth map from M to N. Then the pullback of forms induces a homomorphism of cohomology groups: If $\omega \in \Omega^k(N)$ is a cocycle, then so is $F^*\omega \in \Omega^k(M)$, since $d(F^*\omega) = F^*(d\omega)$. Also, is $\omega = d\eta$ then $F^*\omega = F^*d\eta = d(F^*\eta)$, so this map is well-defined on cohomology.

Proposition 15.5.1 *Let $F : M \times [0,1] \to N$ be a smooth map, and set $f_t(x) = F(x,t)$ for each $t \in [0,1]$. Then f_t^* is independent of t.*

Proof. Let $\omega \in \Omega^k(N)$ be a cocycle. Then we can write

$$F^*\omega = \omega_0 + dt \wedge \omega_1$$

where $\omega_0 \in \Omega^k(M)$ and $\omega_1 \in \Omega^{k-1}(M)$ for each t. Then $f_t^*\omega = \omega_0$ for each t. Since $F^*\omega$ is a cocycle, we have

$$0 = dF^*\omega = dt \wedge \left(\frac{\partial\omega_0}{\partial t} - d_M\omega_1\right) + \ldots$$

and therefore

$$f_1^*\omega - f_0^*\omega = \omega_0(1) - \omega_0(0) = \int_0^1 \frac{\partial \omega_0}{\partial t} dt = \int_0^1 d_M \omega_1 dt = d_M \int_0^1 \omega_1 dt.$$

Therefore $f_1^*\omega$ and $f_0^*\omega$ represent the same cohomology class. □

Corollary 15.5.2 *If the smooth map $f : M \to N$ is a homotopy equivalence (that is, there exists a continuous map $g : N \to M$ such that $f \circ g$ and $g \circ f$ are both homotopic to the identity) then f^* is an isomorphism.*

15.6 The Poincaré Lemma

We will compute the cohomology groups for a simple example: A subset B in $\mathbb{R}^n$ is star-shaped (with respect to the origin) if for every point $y \in B$, the interval $\{ty : t \in [0,1]\}$ is in B.

Proposition 15.6.1 *(The Poincaré Lemma). Let B be a star-shaped open set in $\mathbb{R}^n$. Then $H^k(B) = \{0\}$ for $k = 1, \ldots, n$.*

Proof. We need to show that for $k > 0$ every k-cocycle is a k-coboundary. In other words, given a k-form ω on B with $d\omega = 0$, we need to find a $(k-1)$-form η such that $\omega = d\eta$. We will do this with k replaced by $k+1$.

Write $\omega = \omega_{i_0 \ldots i_k} dx^{i_0} \wedge \ldots \wedge dx^{i_k}$. For $y \in B$ write $y = y^j e_j$ and define

$$\eta_{i_1 \ldots i_k} = y^j \int_0^1 t^k (\omega_{ty})_{j i_1 \ldots i_k}\, dt.$$

This defines a k-form η on B. We compute the exterior derivative of η at y:

$$\begin{aligned}
(d\eta)_{i_1 \ldots i_k} &= \sum_{p=0}^{k} (-1)^p \partial_{i_p} \eta_{i_0 \ldots \hat{i_p} \ldots i_k} \\
&= \sum_{p=0}^{k} (-1)^p \int_0^1 t^k (\omega_{ty})_{i_p i_0 \ldots \hat{i_p} \ldots i_k} dt \\
&\quad + \sum_{p=0}^{k} (-1)^p y^j \int_0^1 t^{k+1} ((\partial_{i_p}\omega)_{ty})_{j i_0 \ldots \hat{i_p} \ldots i_k} dt.
\end{aligned}$$

We rewrite the last term using the fact that $d\omega = 0$: This means

$$\begin{aligned}
0 &= (d\omega)_{j i_0 \ldots i_k} \\
&= \partial_j \omega_{i_0 \ldots i_k} - \sum_{p=0}^{k} (-1)^p \partial_{i_p} \omega_{j i_0 \ldots \hat{i_p} \ldots i_k}.
\end{aligned}$$

This gives (by the antisymmetry of the components of ω)

$$\begin{aligned}(d\eta)_{i_1\dots i_k} &= (k+1)\int_0^1 t^k(\omega_{ty})_{i_0\dots i_k}\,dt \\ &\quad + y^j\int_0^1 t^{k+1}((\partial_j\omega)_{ty})_{i_0\dots i_k}\,dt \\ &= (k+1)\int_0^1 t^k(\omega_{ty})_{i_0\dots i_k}\,dt \\ &\quad + \int_0^1 t^{k+1}\frac{\partial}{\partial t}(\omega_{ty})_{i_0\dots i_k}\,dt \\ &= (\omega_y)_{i_0\dots i_k}\end{aligned}$$

by the fundamental theorem of calculus. Thus $d\eta = \omega$, and $H^{k+1}(B) = \{0\}$. □

Remark. Now that we have seen the explicit proof of the Poincaré Lemma, I remark that there is a very simple proof using the homotopy invariance result of Proposition 15.4.1: First, the cohomology of $\mathbb{R}^0$ is trivial to compute. Second, there is a smooth homotopy equivalence $f : B \to \mathbb{R}^0 = \{0\}$ defined by $f(x) = 0$: If we take $g : \mathbb{R}^0 \to B$ to be given by $g(0) = 0$, then we have $f \circ g$ equal to the identity on $\mathbb{R}^0$, and $g \circ f(x) = 0$ on B. The latter is homotopic to the identity under the homotopy $F : B \times [0,1] \to B$ given by $F(x,t) = (1-t)x$. Corollary 15.5.2 applies.

15.7 Chain complexes and exact sequences

In this section we will discuss some algebraic aspects of cohomology.

The algebraic situation we are dealing with is the following: We have a complex Ω^* consisting of a sequence of real vector spaces Ω^k, together with linear operators $d : \Omega^k \to \Omega^{k+1}$ satisfying $d^2 = 0$. In any such situation we can define the cohomology groups of the complex as $H^k(\Omega) = \ker d_k/\mathrm{im} d_{k-1}$. We will call such a complex a co-chain complex, the elements of the complex as co-chains, co-chains in the kernel of d as cocycles, and those in the image of d as coboundaries.

Suppose we have two chain complexes A^* and B^*. A chain map f from A^* to B^* is given by a sequence of linear maps f^k from A^k to B^k such that $df^k = f^{k+1}d$ for any k.

A chain map induces a homomorphism of cohomology groups: If $\omega \in A^k$ with $d\omega = 0$, then $f^k\omega \in B^k$ with $df^k\omega = 0$. If we take another representative of the same cohomology class, say $\eta = \omega + d\mu$, then $f^k\eta = f^k\omega + df^{k-1}\mu$ is in the same cohomology class as $f^k\omega$. Therefore we have a well-defined homomorphism of cohomology groups, which we also denote by f.

Of particular interest here is the situation where we have three chain complexes, say A^*, B^* and C^*, with chain maps $f : A^* \to B^*$ and $g : B^* \to C^*$ forming a short exact sequence — that is, for each k, f^k is injective, g^k is surjective, and the kernel of the map g^k coincides with the image of the map f^k.

It follows that we have a sequence of maps f from $H^k(A)$ to $H^k(B)$ and g from $H^k(B)$ to $H^k(C)$, and that the kernel of g coincides with the image of f. Let us consider those cohomology classes in $H^k(C)$ which are in the image of g. If ω is a C-cocycle, then we know that $\omega = g\eta$ for some cochain η in B^k, by the assumption of surjectivity of g. However, we cannot deduce that η is a cocycle. However we can deduce that $gd\eta = df\eta = d\omega = 0$, and since the kernel of g coincides with the image of f it follows that $d\eta = f\mu$ for some cochain $\mu \in A^{k+1}$. Then we have $fd\mu = df\mu = dd\eta = 0$, and the injectivity of f implies $d\mu = 0$. Therefore μ represents a cohomology class in $H^{k+1}(A)$. In the case where ω is the image of a cocycle in B under g, we have $d\eta = 0$ and hence $\mu = 0$. Conversely, if $[\mu] = 0$ then $\mu = d\sigma$, hence $d(\eta - f\sigma) = 0$, and $\omega = g(\eta - f\sigma)$, so $[\omega] = g[\eta - f\sigma]$ is in the image of g.

This suggests that we have a homomorphism from $H^k(C)$ to $H^{k+1}(A)$ with kernel coinciding with the image of g. To verify this we need to show that the cohomology class of μ does not depend on our choice of η or on our choice of representative of the cohomology class of ω.

Independence of the choice of η is easy to check: η can be replaced by $\eta + f\sigma$ for arbitrary $\sigma \in A^k$. Therefore $d\eta$ is replaced by $d\eta + df\sigma = d\eta + fd\sigma$, and μ is placed by $\mu + d\sigma$ which is in the same cohomology class as μ.

Independence of the choice of representative in the cohomology class of ω also follows easily: If we replace ω by $\omega + d\alpha$, then η is replaced by $\eta + d\beta$, and $d\eta$ is unchanged, so μ is unchanged.

The homomorphism we have constructed is called the connecting homomorphism. Finally, we note that the image of the connecting homomorphism coincides with the kernel of f: If μ arises from some cohomology class ω, then we have by construction $f\mu = d\eta$, so $[f\mu] = 0$ in $H^{k+1}(B)$. Conversely, if $[f\mu] = 0$, then $f\mu = d\eta$ for some η, and then μ is given by applying the connecting homomorphism to $[g\eta]$ (note that $dg\eta = gd\eta = gf\mu = 0$, so $g\eta$ does represent a cohomology class).

We have therefore produced from the short exact sequence of chain complexes a long exact sequence of cohomology groups:

$$\ldots \to H^k(A) \to H^k(B) \to H^k(C) \to H^{k+1}(A) \to \ldots$$

In the next few sections we will see some example of these long exact sequences in cohomology and their applications.

15.8 The Meyer-Vietoris sequence

Next we want to discuss a way to compute the cohomology of complicated manifolds by cutting them up into simpler pieces. Suppose M is a manifold which is the union of two open subsets U and V, and suppose that we know the cohomology groups of U, V and the intersection $U \cap V$. We want to relate the cohomology groups of M to these. We will do this by constructing an exact sequence relating the cohomology groups of M, U, V and $U \cap V$.

Let ω be a k-cochain on M. Then the restriction of ω to U and to V are also k-cochains. This defines a chain map i from $\Omega^k(M)$ to $\Omega^k(U) \oplus \Omega^k(V)$, given by

$$i(\omega) = (\omega|_U, \omega|_V).$$

Similarly, if α and β are k-cochains on U and V respectively, then we can consider their restrictions to the intersection $U \cap V$, and these are again k-cocycles. We consider the map j from $\Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$ given by

$$(\alpha, \beta) \mapsto \alpha|_{U\cap V} - \beta|_{U\cap V}.$$

This is again a chain map.

These two chain maps define a short exact sequence: The map i is injective, j is surjective, and the image of i coincides with the kernel of j.

By the argument of the previous section, this short exact sequence of cochain complexes gives rise to a long exact sequence of cohomology groups. This long exact sequence of cohomology groups is called the *Meyer-Vietoris sequence* for de Rham cohomology:

$$H^0(M) \xrightarrow{i} H^0(U) \oplus H^0(V) \xrightarrow{j} H^0(U \cap V) \xrightarrow{\eta} H^1(M) \xrightarrow{i} \dots$$
$$\dots H^k(M) \xrightarrow{i} H^k(U) \oplus H^k(V) \xrightarrow{j} H^k(U \cap V) \xrightarrow{\eta} H^{k+1}(M) \xrightarrow{i} \dots$$

15.9 Compactly supported cohomology

The algebraic discussions of section 15.7 allow us to extend our notion of cohomology to more general situations where we have chain complexes. Here we introduce the notion of compactly supported cohomology:

Let M be a smooth manifold. Then we denote by $\Omega^k_c(M)$ the space of differential k-forms on M with compact support. This is a subspace of $\Omega^k(M)$ which is closed under exterior differentiation, and hence forms a cochain complex. The cohomology of this complex is called the compactly supported cohomology of M, and denoted $H^k_c(M)$.

Next we will give another useful example of a long exact sequence relating compactly supported cohomology to the usual cohomology.

Suppose M is a smooth manifold, and Σ is a submanifold within M. Then we have a natural chain map from $H^*(M)$ to $H^*(\Sigma)$ given by pulling back forms via the inclusion map i. This map is surjective. The kernel consists of those forms ω on M which have $i^*\omega = 0$. This is again a chain complex, since $i^*(d\omega) = di^*\omega = 0$ if $i^*\omega = 0$. This gives us a short exact sequence relating the cohomologies of M, Σ, and the chain complex $\Omega_0^k(M,\Sigma) = \{\omega \in \Omega^*(M) : \ i^*\omega = 0\}$.

We will now show that the cohomology of the latter is isomorphic to the compactly supported cohomology of $M\backslash\Sigma$.

To see this, we note that $\Omega_c^k(M\backslash\Sigma) \subset \Omega_0^k(M,\Sigma)$. We denote by C^k the quotient space $\Omega_0^k(M,\Sigma)/\Omega_c^k(M\backslash\Sigma)$. We define an operator $d : C^k \to C^{k+1}$ by $d[\omega] = [d\omega]$. If $\eta \in \Omega_c^k(M\backslash\Sigma)$, then $d(\omega+\eta) = d\omega + d\eta \in d\omega + \Omega_c^{k+1}(M\backslash\Sigma)$, so this operator is well defined and satisfies $d^2 = 0$. Therefore the complex C is a cochain complex, and we have a short exact sequence of chain complexes

$$0 \to \Omega_c^*(M\backslash\Sigma) \to \Omega_0^*(M,\Sigma) \to C \mapsto 0.$$

This induces a long exact sequence in cohomology:

$$\ldots \to H_c^k(M\backslash\Sigma) \to H_0^k(M,\Sigma) \to H^k(C) \to H_c^{k+1}(M\backslash\Sigma) \ldots$$

We will prove that $H^k(C) = 0$ for all k, and the long exact sequence above then implies that $H_c^k(M\backslash\Sigma) \simeq H_0^k(M,\Sigma)$.

Suppose $\omega \in \Omega_0^k(M,\Sigma)$ satisfies $d[\omega] = 0$, that is,

$$d\omega = \eta$$

for some $\eta \in \Omega_c^{k+1}(M\backslash\Sigma)$. We want to show that $[\omega] = d[\sigma]$ for some $[\sigma] \in C^{k-1}$, which means we want to show that $\omega - d\sigma \in \Omega_c^k(M\backslash\Sigma)$.

Since Σ is a smooth compact submanifold, the nearest-point projection p (defined using any Riemannian metric on M) is a smooth map from a neighbourhood T of Σ in M to Σ, and is homotopic to the identity map on T. We can assume that $du = 0$ on T since $du \in \Omega_c^{k+1}(M\backslash\Sigma)$. Therefore by the homotopy invariance we have

$$\omega - p^*\omega = dv$$

for some $v \in \Omega_0^{k-1}(T,\Sigma)$. But we also have $p = i \circ p$ and so $p^*\omega = p^*i^*\omega = 0$ since $\omega \in \Omega_0^k(M,\Sigma)$, and we have $\omega = dv$. Now let φ be a smooth function on M which is identically 1 in a neighbourhood of Σ, but identically zero in a neighbourhood of $M\backslash T$. Then $\varphi v \in \Omega_0^{k-1}(M,\Sigma)$, and $\omega - d(\varphi v) \in \Omega_c^k(M\backslash\Sigma)$. Therefore $[\omega] - d[\varphi v] = [\omega - d(\varphi v)] = 0$ in C^k, and $H^k(C) = 0$ as claimed.

Therefore we have a long exact sequence in cohomology:

$$\ldots H_c^k(M\backslash\Sigma) \to H^k(M) \to H^k(\Sigma) \to H_c^{k+1}(M\backslash\Sigma) \to \ldots$$

15.10 Cohomology of spheres

We will use the Meyer-Vietoris sequence to deduce the cohomology groups of the spheres S^n for any n. We start with the circle S^1: We can think of this as a union of two intervals U and V, such that $U \cap V$ is a union of two disjoint intervals.

Now we can apply the Meyer-Vietoris sequence to compute the cohomology of $S^1 = U \cup V$: The sequence becomes

$$0 \mapsto \mathbb{R} \mapsto \mathbb{R}^2 \mapsto \mathbb{R}^2 \mapsto H^1(S^1) \mapsto 0$$

which implies that $H^1(S^1) = \mathbb{R}$ (this could be computed directly by seeing what the cocycles are on S^1 explicitly). Clearly $H^k(S^1) = \{0\}$ for $k > 1$ because S^1 is a 1-manifold.

Now consider the cohomology of the sphere S^2: We observe that $S^2 = U \cup V$ where U and V are diffeomorphic to disks and $U \cap V$ is diffeomorphic to $S^1 \times (0,1)$. By homotopy invariance $S^1 \times (0,1)$ has the same cohomology as S^1. So the sequence in this case becomes:

$$0 \mapsto \mathbb{R} \mapsto \mathbb{R}^2 \mapsto \mathbb{R} \mapsto H^1(S^2) \mapsto 0 \mapsto \mathbb{R} \mapsto H^2(M) \mapsto 0.$$

It follows that $H^1(S^2) = \{0\}$ and $H^2(S^2) = \mathbb{R}$.

Proceeding in the same way for higher dimensions, we find $H^k(S^n) = \mathbb{R}$ if $k = 0$ or $k = n$ and $H^k(S^n) = 0$ otherwise.

15.11 Compactly supported cohomology of $\mathbb{R}^n$

Proposition 15.11.1

$$H_c^k(\mathbb{R}^n) = \begin{cases} 0, & k < n \\ \mathbb{R}, & k = n. \end{cases}$$

Proof. For $k = 0$ the result is immediate because constants are not compactly supported in $\mathbb{R}^n$.

For $k = 1$ and $n = 1$ the result is also immediate: If $\omega = \omega_1 dx^1$, then $\omega = df$ implies $\int_{\mathbb{R}} \omega_1 = 0$, and conversely.

We will use the long exact sequence from section 15.9 together with the results on cohomology groups of spheres from section 15.10: The sphere S^n contains an equatorial S^{n-1} as a submanifold, and the complement $S^n \backslash S^{n-1}$ is diffeomorphic to two copies of $\mathbb{R}^n$. Therefore the long exact sequence becomes (for $n > 1$)

$$\begin{aligned}
&0 \to \mathbb{R} \to \mathbb{R}\\
\to &H_c^1(\mathbb{R}^n)^2 \to 0 \to 0\\
&\dots\\
\to &H_c^{n-1}(\mathbb{R}^n)^2 \to 0 \to \mathbb{R}\\
\to &H_c^n(\mathbb{R}^n)^2 \to \mathbb{R} \to 0
\end{aligned}$$

It follows that $H_c^k(\mathbb{R}^n) = 0$ for $k = 1, \dots, n-1$ and $H_c^n(\mathbb{R}^n) = \mathbb{R}$. □

Furthermore, it is immediate that the n-coboundaries of compactly supported cohomology are precisely those that have integral zero: This contains the n-coboundaries (by Stokes' theorem), and has codimension 1.

15.12 The group $H^n(M)$

If M is a compact manifold of dimension n, then we always know what the nth cohomology group is:

Proposition 15.8.1 *If M is a compact connected manifold of dimension n, then $H^n(M) = \mathbb{R}$ if M is orientable, and $H^n(M) = \{0\}$ if M is not orientable.*

Proof. First suppose M is oriented. Choose an atlas for M consisting of coordinate regions U_α, $\alpha = 1, \dots, N$, each of which is diffeomorphic to $\mathbb{R}^n$. Let $\{\rho_\alpha\}$ be a smooth partition of unity with ρ_α supported in U_α.

Define a map $\xi : \Omega^n(M) \to \mathbb{R}^N$ by

$$\xi(\omega) = (\int_M \rho_1\omega, \dots, \int_M \rho_N\omega).$$

Now consider the subspace X of $\mathbb{R}^N$ defined by

$$X = \{\xi(dv) |\ v \in \Omega^{n-1}(M)\}.$$

If ω is exact, then clearly $\xi(\omega) \in X$. Conversely, if $\xi(\omega) \in X$ then we have $v \in \Omega^{n-1}(M)$ such that $\int_M \rho_\alpha(\omega - dv) = 0$ for every α. Now $\rho_\alpha(\omega - dv)$ is a compactly supported form in $\mathbb{R}^n$, with integral zero, and hence by Proposition 15.11.1 there exists $v_\alpha \in \Omega_c^{n-1}(U_\alpha)$ such that $\rho_\alpha(\omega - dv) = dv_\alpha$. Summing over α, we find

$$\omega - dv = \sum_\alpha \rho_\alpha(\omega - dv) = \sum_\alpha dv_\alpha$$

and hence

$$\omega = d(v + \sum_\alpha v_\alpha)$$

and ω is exact.

The subspace X is defined by a finite collection of equations $c_{jk}x_k = 0$, $j = 1, \dots, K$. Therefore an n-form ω on M is exact if and only if

$$\int_M (c_{jk}\rho_k)\omega = 0$$

for $j = 1, \dots, K$. Suppose that $c_{jk}\rho_k$ is non-constant for some j. Then in one of the regions U_α we can find an n-form ω supported in U_α with integral zero such that $\int_{U_\alpha} c_{jk}\rho_k\omega$ is non-zero. But then it follows that ω is not exact, contradicting Proposition 15.11.1. Therefore $c_{jk}\rho_k$ is constant for each j, and $\omega \in \Omega^n(M)$ is exact if and only if $\int_M \omega = 0$.

Next suppose M is not orientable. Let $\tilde{M}$ be the double cover of M, which we can define as follows: Define an equivalence relation on $\Lambda^n T_x^*M\backslash\{0\}$ for each $x \in M$ by taking $\omega \sim \eta$ iff $\omega = \lambda\eta$ for some $\lambda > 0$. The quotient space at each point consists of two points, and the quotient bundle $P\Lambda^n T^*M = \{(x, [\omega]) : \ \omega \in \Lambda^n T_x^*M\}$ is a $\mathbb{Z}_2$-bundle over M. Fix $x \in M$ and $\omega \neq 0$ in $\Lambda^n T_x^*M$. Then we take $\tilde{M}$ to be the connected component of $(x, [\omega])$ in $P\Lambda^n T^*M$. If M is orientable then there is a global non-vanishing section of $\Lambda^n T^*M$, so $\tilde{M}$ is diffeomorphic to M, while if M is not orientable then $\tilde{M}$ covers M twice (if M is connected), and there is a natural projection π from $\tilde{M}$ to M given by $\pi(x, [\omega]) = x$. $\tilde{M}$ is always orientable, since $T_(x, [\omega])\tilde{M} \simeq T_x M$, hence $\Lambda^n T^*_{(x,[\omega])}\tilde{M} \simeq \Lambda^n T_x^*M$, and so $[\omega] \in P\Lambda^n T_x^*M$ gives a global section of $P\Lambda^n T^*\tilde{M}$.

In the case where M is not orientable, there is a natural involution i of $\tilde{M}$ induced by the map $\omega \mapsto -\omega$ of $\Lambda^n T^*M$, and this is orientation-reversing. Since $\pi \circ i = \pi$, a differential n-form $\tilde{\omega}$ on $\tilde{M}$ arises from pull-back by π of a differential form ω on M if and only if $i^*\tilde{\omega} = \tilde{\omega}$. But then we have

$$\int_{\tilde{M}} \tilde{\omega} = -\int_{\tilde{M}} i^*\tilde{\omega} = -\int_{\tilde{M}} \tilde{\omega}$$

since i is orientation-reversing, and hence $\int_{\tilde{M}} \tilde{\omega} = 0$. It follows from the case we have already considered that $\tilde{\omega} = d\eta$ for some $\eta \in \Omega^{n-1}(\tilde{M})$. Then let $\tilde{\eta} = (\eta + i^*\eta)/2$. Then we have $i^*\tilde{\eta} = \tilde{\eta}$, so $\tilde{\eta} = \pi^*\eta'$ for some $\eta' \in \Omega^{n-1}(M)$, and $d\tilde{\eta} = (d\eta + di^*\eta)/2 = (\tilde{\omega} + i^*\tilde{\omega})/2 = \tilde{\omega}$. It follows that $d\eta' = \omega$, so ω is exact. Therefore $H^n(M) = 0$, as claimed. $\square$

15.13 Cohomology of surfaces

In this section we will use the results we have developed above about cohomology groups to compute the cohomology of compact surfaces.

We already know the cohomology groups of S^2. Next we will compute the cohomology groups of the torus $\mathbb{T}^2 = S^1 \times S^1$. This can be written as the union of open sets U and V, where U and V are each diffeomorphic to $S^1 \times \mathbb{R}$, and $U \cap V$ is diffeomorphic to two copies of $S^1 \times \mathbb{R}$. We therefore know $H^0(\mathbb{T}^2) = H^0(U) = H^0(V) = \mathbb{R}$, $H^0(U \cap V) = \mathbb{R}^2$, $H^1(U) = H^1(V) = \mathbb{R}$ and $H^1(U \cap V) = \mathbb{R}^2$, and $H^2(U) = H^2(V) = H^2(U \cap V) = 0$, $H^2(\mathbb{T}^2) = \mathbb{R}$. Thus we have the long exact sequence

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^2 \to H^1(\mathbb{T}^2) \to \mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R} \to 0 \to 0$$

which implies that $H^1(\mathbb{T}^2) = \mathbb{R}^2$.

We will proceed by induction on the genus of the surface: A surface M_{g+1} of genus $g+1$ can be written as the union of sets A and B, where A is diffeomorphic to $\mathbb{T}^2 \backslash \{p\}$, B is diffeomorphic to $M_g \backslash \{q\}$, and $A \cap B$ is diffeomorphic to $S^1 \times \mathbb{R}$. To use this we first need to find the cohomology groups of $\mathbb{T}^2 \backslash \{p\}$ and $M_g \backslash \{p\}$.

Proposition 15.13.1 *Let M be a compact oriented manifold of dimension $n > 1$, $p \in M$. Then $H^n(M \backslash \{p\}) = 0$.*

Proof. Let ω be an n-form on $M \backslash \{p\}$. Then we can write $\omega = \omega_0 + \omega_1$, where ω_0 is compactly supported in $M \backslash \{p\}$ and has integral equal to zero, and ω_1 is supported in a region diffeomorphic to $S^1 \times (0,1)$, and is identically zero on $S^1 \times (0, 1/2)$.

ω_0 extends to a form on M with integral zero, so there exists $\eta_0 \in \Omega^{n-1}(M)$ such that $\omega_0 = d\eta_0$.

By the proof of the Poincaré Lemma, there also exists a form $\eta_1 \in \Omega^{n-1}(S^1 \times (0,1))$, vanishing on $S^1 \times (0, 1/2)$, such that $\omega_1 = d\eta_1$.

Therefore $\omega = d\eta_0 + d\eta_1$ is exact, and $H^n(M \backslash \{p\}) = 0$. □

From this we can deduce the cohomology of $M_g \backslash \{p\}$ as follows: M_g is the union of U and V, where $U \simeq M_g \backslash \{p\}$, $V \simeq \mathbb{R}^2$, and $U \cap V \simeq S^1 \times \mathbb{R}$.

From this we obtain the long exact sequence

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R} \to H^1(M_g) \to H^1(M_g \backslash \{p\}) \to \mathbb{R} \to \mathbb{R} \to 0$$

which implies that $H^1(M_g \backslash \{p\}) \simeq H^1(M_g)$.

Finally, we can apply the Meyer-Vietoris sequence to A and B as above, obtaining

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R} \to H^1(M_{g+1}) \to H^1(M_g) \oplus \mathbb{R}^2 \to \mathbb{R} \to \mathbb{R} \to 0$$

which implies that $H^1(M_{g+1}) \simeq H^1(M_g) \oplus \mathbb{R}^2$.

By induction, we deduce that the cohomology groups of the surface of genus g are given by $H^0(M_g) = \mathbb{R}$, $H^1(M_g) = \mathbb{R}^{2g}$, and $H^2(M_g) = \mathbb{R}$.