Lecture 16. Curvature

In this lecture we introduce the curvature tensor of a Riemannian manifold, and investigate its algebraic structure.

16.1 The curvature tensor

We first introduce the curvature tensor, as a purely algebraic object: If X, Y, and Z are three smooth vector fields, we define another vector field $R(X,Y)Z$ by

$$R(X,Y)Z = \nabla_Y\left(\nabla_X Z\right) - \nabla_Y\left(\nabla_Y Z\right) - \nabla_{[Y,X]}Z.$$

Proposition 16.1.1 *$R(X,Y)Z$ is a tensor of type* $(3,1)$.

Proof. R is tensorial in the first two arguments, because we can write

$$R(X,Y)Z = \left(\nabla\nabla Z\right)(Y,X) - \left(\nabla\nabla Z\right)(X,Y),$$

and each of the terms of the right is a tensor in X and Y. This leaves one further calculation:

$$\begin{aligned}
R(X,Y)(fZ) &= \nabla_Y\left(f\nabla_X Z + X(f)Z\right) - \nabla_X\left(f\nabla_Y Z + Y(f)Z\right) \\
&\quad - [X,Y](f)Z - f\nabla_{[X,Y]}Z \\
&= f\nabla_Y\nabla_X Z + Y(f)\nabla_X Z + YX(f)Z + X(f)\nabla_Y Z \\
&\quad - f\nabla_X\nabla_Y Z - X(f)\nabla_Y Z - XY(f)Z - Y(f)\nabla_X Z \\
&\quad - [X,Y](f)Z - f\nabla_{[X,Y]}Z \\
&= f\left(\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{[X,Y]}Z\right) \\
&\quad + \left(YX(f) - XY(f) - [X,Y](f)\right)Z \\
&= fR(X,Y)Z.
\end{aligned}$$

□

Remark. Note that this calculation does not use the compatibility of the connection with the metric, only the symmetry of the connection. Thus any (symmetric) connection gives rise to a curvature tensor. However, we will only be interested in the case of the Levi-Civita connection from now on.

As usual we can write the curvature tensor in terms of its components in any coordinate tangent basis:

$$R = R_{ikj}{}^{l} dx^i \otimes dx^k \otimes dx^j \otimes \partial_l.$$

Then an application of the metric index-lowering operator gives a tensor of type $(4, 0)$ defined by

$$R(u, v, w, z) = g(\nabla_v \nabla_u w - \nabla_u \nabla_v w - \nabla_{[v,u]} w, z).$$

The components of this are $R_{ijkl} = R_{ijk}{}^{p} g_{pl}$.

Proposition 16.1.2 (Symmetries of the curvature tensor)

(1). $R_{ikjl} + R_{kijl} = 0$;
(2). $R_{ikjl} + R_{kjil} + R_{jikl} = 0$;
(3). $R_{ikjl} + R_{iklj} = 0$;
(4). $R_{ikjl} = R_{jlik}$.

The second identity here is called the *first Bianchi identity*.

Proof. The first symmetry is immediate from the definition of curvature. For the second, work in a coordinate tangent basis:

$$\begin{aligned} R_{ikjl} + R_{kjil} + R_{jikl} &= g\left(\nabla_k \nabla_i \partial_j - \nabla_i \nabla_k \partial_j, \partial_l\right) \\ &\quad + g\left(\nabla_j \nabla_k \partial_i - \nabla_k \nabla_j \partial_i, \partial_l\right) \\ &\quad + g\left(\nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k, \partial_l\right) \\ &= g\left(\nabla_k \left(\nabla_i \partial_j - \nabla_j \partial_i\right), \partial_l\right) \\ &\quad + g\left(\nabla_j \left(\nabla_k \partial_i - \nabla_i \partial_k\right), \partial_l\right) \\ &\quad + g((\nabla_i \left(\nabla_j \partial_k - \nabla_k \partial_j\right), \partial_l) \\ &= 0 \end{aligned}$$

by the symmetry of the connection.

The third symmetry is a consequence of the compatibility of the connection with the metric:

$$\begin{aligned} 0 &= \partial_i \partial_j g_{kl} - \partial_j \partial_i g_{kl} \\ &= \partial_i \left(g(\nabla_j \partial_k, \partial_l) + g(\partial_k, \nabla_j \partial_l)\right) \\ &\quad - \partial_j \left(g(\nabla_i \partial_k, \partial_l) + g(\partial_k, \nabla_i \partial_l)\right) \\ &= g(\nabla_i \nabla_j \partial_k, \partial_l) + g(\nabla_i \partial_k, \nabla_j \partial_l) + g(\nabla_j \partial_k, \nabla_i \partial_l) + g(\partial_k, \nabla_i \nabla_j \partial_l) \\ &\quad - g(\nabla_j \nabla_i \partial_k, \partial_l) - g(\nabla_j \partial_k, \nabla_i \partial_l) - g(\nabla_i \partial_k, \nabla_j \partial_l) - g(\partial_k, \nabla_j \nabla_i \partial_l) \\ &= R_{jikl} + R_{jilk}. \end{aligned}$$

Finally, the last symmetry follows from the previous ones:

$$\begin{aligned}
R_{ikjl} &=^{(2)} -R_{kjil} - R_{jikl} \\
&=^{(3)} R_{kjli} + R_{jilk} \\
&=^{(2)} -R_{jlki} - R_{lkji} - R_{iljk} - R_{ljik} \\
&=^{(3),(1)} 2R_{jlik} + R_{lkij} + R_{ilkj} \\
&=^{(2)} 2R_{jlik} - R_{kilj} \\
&=^{(1),(3)} 2R_{jlik} - R_{ikjl}.
\end{aligned}$$

□

Note that if M is a one-dimensional Riemannian manifold, then the curvature is zero (since it is antisymmetric). This reflects the fact that any one-dimensional manifold can be locally parametrised by arc length, and so is locally isometric to any other one-dimensional manifold. The curvature tensor is invariant under isometries.

Next consider the two-dimensional case: Any component of R in which the first two or the last two indices are the same must vanish, by symmetries (1) and (3). There is therefore only one independent component of the curvature: If we take $\{e_1, e_2\}$ to be an orthonormal basis for T_xM, then we define the *Gauss curvature* of M at x to be $K = R_{1212}$. This is independent of the choice of basis: Any other one is given by $e_1' = \cos\theta e_1 + \sin\theta e_2$ and $e_2' = -\sin\theta e_1 + \cos\theta e_2$, so we have

$$\begin{aligned}
R_{1'2'1'2'} &= R(\cos\theta e_1 + \sin\theta e_2, -\sin\theta e_1 + \cos\theta e_2, e_1', e_2') \\
&= \cos^2\theta R(e_1, e_2, e_1', e_2') - \sin^2\theta R(e_2, e_1, e_1', e_2') \\
&= R(e_1, e_2, e_1', e_2') \\
&= R(e_1', e_2', e_1, e_2) \\
&= R(e_1, e_2, e_1, e_2) \\
&= R_{1212}.
\end{aligned}$$

More generally, we see that (in any dimension), if $\{e_i\}$ are orthonormal, then R_{ijkl} depends only the (oriented) two-dimensional plane generated by e_i and e_j, and the one generated by e_k and e_l.

16.2 Sectional curvatures

The last observation motivates the following definition:

If Σ is a two-dimensional subspace of T_xM, then the *sectional curvature* of Σ is $K(\sigma) = R(e_1, e_2, e_1, e_2)$, where e_1 and e_2 are any orthonormal basis for Σ. This is indepedent of basis, by the calculation above.

Proposition 16.2.1 *The curvature tensor is determined by the sectional curvatures.*

Proof. We will give an explicit expression for a component R_{ijkl} of the curvature tensor, in terms of sectional curvatures. We work with an orthonormal basis $\{e_1, \ldots, e_n\}$ at a point of M.

For convenience we will refer to the oriented plane generated by e_1 and e_j by the notation $e_i \wedge e_j$. We compute the sectional curvature of the plane $\frac{1}{2}(e_i + e_k) \wedge (e_j + e_l)$:

$$\begin{aligned}
K\left(\frac{(e_i+e_k)\wedge(e_j+e_l)}{2}\right) &= \frac{1}{4}R(e_i+e_k, e_j+e_l, e_i+e_k, e_j+e_l)\\
&= \frac{1}{4}K(e_i\wedge e_j) + \frac{1}{4}K(e_i\wedge e_l) + \frac{1}{4}K(e_j\wedge e_k) + \frac{1}{4}K(e_k\wedge e_l)\\
&\quad + \frac{1}{2}R_{ijil} + \frac{1}{2}R_{ijkj} + \frac{1}{2}R_{ilkl} + \frac{1}{2}R_{klkj}\\
&\quad + \frac{1}{2}R_{ijkl} + \frac{1}{2}R_{kjil}.
\end{aligned}$$

Now add the same expression with e_k and e_l replaced by $-e_k$ and $-e_l$:

$$\begin{aligned}
R_{ijkl} + R_{kjil} = K\left(\frac{(e_i+e_k)\wedge(e_j+e_l)}{2}\right) + K\left(\frac{(e_i-e_k)\wedge(e_j-e_l)}{2}\right)\\
- \frac{1}{2}K(e_i\wedge e_j) - \frac{1}{2}K(e_i\wedge e_l) - \frac{1}{2}K(e_j\wedge e_k) - \frac{1}{2}K(e_k\wedge e_l).
\end{aligned}$$

Finally, subtract the same expression with e_i and e_j interchanged: On the left-hand side we get

$$R_{ijkl} + R_{kjil} - R_{jikl} - R_{kijl} = 2R_{ijkl} - R_{jkil} - R_{kijl} = 3R_{ijkl}$$

by virtue of the Bianchi identity. Thus we have

$$\begin{aligned}
R_{ijkl} &= \frac{1}{3}K\left(\frac{(e_i+e_k)\wedge(e_j+e_l)}{2}\right) + \frac{1}{3}K\left(\frac{(e_i-e_k)\wedge(e_j-e_l)}{2}\right)\\
&\quad - \frac{1}{3}K\left(\frac{(e_j+e_k)\wedge(e_i+e_l)}{2}\right) - \frac{1}{3}K\left(\frac{(e_j-e_k)\wedge(e_i-e_l)}{2}\right)\\
&\quad - \frac{1}{6}K(e_j\wedge e_l) - \frac{1}{6}K(e_i\wedge e_k) + \frac{1}{6}K(e_i\wedge e_l) + \frac{1}{6}K(e_j\wedge e_k).
\end{aligned}$$

□

16.3 Ricci curvature

The Ricci curvature is the symmetric $(2,0)$-tensor defined by contraction of the curvature tensor:

$$R_{ij} = \delta_l^k R_{ikj}{}^l = g^{kl} R_{ikjl}.$$

This can be interpreted in terms of the sectional curvatures: Given a unit vector v, choose an orthonormal basis for TM with $e_n = v$. Then we have

$$R(v,v) = \sum_{i=1}^{n} R(e_i, v, e_i, v) = \sum_{i=1}^{n=1} R_{inin} = \sum_{i=1}^{n} K(v \wedge e_i).$$

Thus the Ricci curvature in direction v is an average of the sectional curvatures in 2-planes containing v.

16.4 Scalar curvature

The scalar curvature is given by a further contraction of the curvature:

$$R = g^{ij} R_{ij} = g^{ij} g^{kl} R_{ikjl}.$$

$R(x)$ then (except for a constant factor depending on n) the average of the sectional curvatures over all 2-planes in T_xM.

16.5 The curvature operator

The full algebraic structure of the curvature tensor is elucidated by constructing a vector space on which it acts as a bilinear form.

At each point x of M we let $\Lambda^2 T_xM$ be the vector space obtained by dividing the space $T_xM \otimes T_xM$ by the relation

$$u \otimes v \sim -v \otimes u.$$

This is a vector space of dimension $n(n-1)/2$, with basis elements

$$e_i \wedge e_j = [e_i \otimes e_j]$$

for $i < j$. More generally, if u and v are any two vectors in T_xM, we denote

$$u \wedge v = [u \otimes v].$$

This is called the wedge product of the vectors u and v.

In particular, if u and v are orthogonal and have unit length, then we identify $u \wedge v \in \Lambda^2 T_xM$ with the two dimensional oriented plane in T_xM

generated by u and v. The construction of $\Lambda^2 T_x M$ simply extends the set of two-dimensional planes in $T_x M$ to a vector space, to allow formal sums and scalar multiples of them. We refer to the space $\Lambda^2 T_x M$ as the space of 2-planes at x (even though not everything can be interpreted as a plane in $T_x M$), and the corresponding bundle is the 2-plane bundle of M. We extend the metric to $\Lambda^2 TM$ by taking $\{e_i \wedge e_j | \ 1 \leq i < j \leq n\}$ to be an orthonormal basis for $\Lambda^2 T_x M$ whenever $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $T_x M$.

A 2-plane which can be expressed in the form $u \wedge v$ for some vectors u and v is called a *simple* 2-plane, and such a plane corresponds to a subspace of $T_x M$.

Exercise 16.5.1 Show that every 2-plane is simple if $n = 2$ or $n = 3$, but not if $n \geq 4$.

The importance of the 2-plane bundle is the following:

Proposition 16.5.2 *The curvature tensor defines a symmetric bilinear form on the space of 2-planes $\Lambda^2 T_x M$, by*

$$R(A^{ij} e_i \wedge e_j, B^{kl} e_k \wedge e_l) = A^{ij} B^{kl} R_{ijkl}.$$

Here the sum is over all i and j with $i < j$, and all k and l with $k < l$.

In particular, this curvature operator, since symmetric, can be diagonalised. It is important to note that the eigenvalues of the curvature operator need not be sectional curvatures! The sectional curvatures are the values of the curvature operator on simple 2-planes, but there is no reason why the eigen-vectors of the curvature operator should be simple 2-planes. In particular, it is possible to have all the sectional curvatures positive (or negative) at a point, while not having all of the eigenvalues of the curvature operator positive (negative).

In the special case of three dimensions, however, every 2-plane is simple, and so the eigenvalues of the curvature operator are sectional curvatures. In this case we refer to the eigenvectors of the curvature operator as the principal 2-planes, and the eigenvalues the principal sectional curvatures.

16.6 Calculating curvature

Suppose we are given a metric g and wish to calculate the curvature. In principle we have all the ingredients to do this, but in practice this can get very messy:

First, in local coordinates we can write down the connection ceofficients, which are smooth functions on the coordinate domain, such that

$$\nabla_{\partial_i}\partial_j = {\Gamma_{ij}}^k \partial_k.$$

From these we can calculate the second derivatives:

$$\begin{aligned}\nabla_k \nabla_i \partial_j &= \nabla_k \left({\Gamma_{ij}}^l \partial_l\right) \\ &= \partial_k {\Gamma_{ij}}^l \partial_l + {\Gamma_{ij}}^l {\Gamma_{kl}}^p \partial_p.\end{aligned}$$

This gives the expression for the curvature:

$$R_{ikj}{}^l = \partial_k {\Gamma_{ij}}^l - \partial_i {\Gamma_{kj}}^l + {\Gamma_{ij}}^q {\Gamma_{kq}}^l - {\Gamma_{kj}}^q {\Gamma_{iq}}^l.$$

Let us consider a simple case, namely when the parametrisation is conformal, so that the metric takes the very simple form

$$g_{ij} = f\delta_{ij}$$

for some function f. Then the inverse metric is also easy to compute:

$$g^{ij} = f^{-1}\delta^{ij}.$$

Therefore a connection coefficient is given by

$$\begin{aligned}{\Gamma_{ij}}^k &= \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) \\ &= \frac{1}{2} f^{-1}\delta^{kl}\left(\partial_i f \delta_{jl} + \partial_j f \delta_{il} - \partial_l f \delta_{ij}\right) \\ &= \frac{1}{2}\left(\delta_j^k \partial_i \log f + \delta_i^k \partial_j \log f - \delta_{ij}\delta^{kl}\partial_l \log f\right).\end{aligned}$$

This gives the following expression for the curvature tensor components, where we write $u = \log\sqrt{f}$:

$$\begin{aligned}R_{ikj}{}^l &= \delta_j^l \partial_k \partial_i u + \delta_i^l \partial_k \partial_j u - \delta_{ij}\delta^{lp}\partial_k \partial_p u \\ &\quad - \delta_j^l \partial_i \partial_k u - \delta_k^l \partial_i \partial_j u + \delta_{kj}\delta^{lp}\partial_i \partial_p u \\ &\quad + \left(\delta_j^q \partial_i u + \delta_i^q \partial_j u - \delta_{ij}\delta^{qp}\partial_p u\right)\left(\delta_q^l \partial_k u + \delta_k^l \partial_q u - \delta_{kq}\delta^{lm}\partial_m u\right) \\ &\quad - \left(\delta_j^q \partial_k u + \delta_k^q \partial_j u - \delta_{kj}\delta^{qp}\partial_p u\right)\left(\delta_q^l \partial_i u + \delta_i^l \partial_q u - \delta_{iq}\delta^{lm}\partial_m u\right) \\ &= \delta_i^l \partial_k \partial_j u - \delta_{ij}\delta^{lp}\partial_k \partial_p u - \delta_k^l \partial_i \partial_j u + \delta_{kj}\delta^{lp}\partial_i \partial_p u \\ &\quad + |Du|^2\left(\delta_{jk}\delta_i^l - \delta_{ij}\delta_k^l\right) \\ &\quad + \delta_k^l \partial_i u \partial_j u + \delta_{ij}\delta^{lq}\partial_q u \partial_k u - \delta_i^l \partial_j u \partial_k u - \delta_j k \delta^{lq}\partial_q u \partial_i u.\end{aligned}$$

Taking the trace over k and l gives the Ricci curvature:

$$R_{ij} = -\delta_{ij}\left(\Delta u + (n-2)|Du|^2\right) - (n-2)\left(\partial_i \partial_j u - \partial_i u \partial_j u\right),$$

where $\Delta u = \delta^{kl}\partial_k \partial_l u$ is the Laplacian of u. Finally, multiplying by $f^{-1}\delta^{ij}$ gives the scalar curvature:

$$R = -(n-1)f^{-1}\left(2\Delta u + (n-2)|Du|^2\right).$$

Example 16.6.1 We will compute the curvature of the left-invariant metric $g_{ij} = y^{-2}\delta_{ij}$ on the upper half-plane $\mathbb{H}^n$ in $\mathbb{R}^n$, with coordinates $(x_1, \ldots, x_{n-1}, y)$, $y > 0$. If this case we have $f = y^{-2}$, so $u = -\log y$. Therefore $\partial_i u = -y^{-1}\delta_i^n$, and $\partial_i\partial_j u = y^{-2}\delta_i^n\delta_j^n$. Also $|Du|^2 = y^{-2}$. The equation above therefore gives

$$\begin{aligned} R_{ikjl} = y^{-4}\big(&\delta_{il}\delta_j^n\delta_k^n + \delta_{jk}\delta_i^n\delta_l^n - \delta_{ij}\delta_k^n\delta_l^n - \delta_{kl}\delta_i^n\delta_j^n \\ &+ (\delta_{jk}\delta_{il} - \delta_{ij}\delta_{kl}) \\ &+ \delta_{kl}\delta_i^n\delta_j^n + \delta_{ij}\delta_k^n\delta_l^n - \delta_{jk}\delta_i^n\delta_l^n - \delta_{il}\delta_k^n\delta_l^n\big) \\ = y^{-4}&\left(\delta_{jk}\delta_{il} - \delta_{ij}\delta_{kl}\right). \end{aligned}$$

An orthonormal basis is given by $\{ye_i\}$, which gives for any sectional curvature

$$K = -1.$$

Thus hyperbolic space has constant negative curvature.

16.7 Left-invariant metrics

Another situation in which is it possible to conveniently write down the curvature of a metric is when it arises as a left-invariant metric for a Lie group.

Let G be a Lie group, with a left-invariant metric g, and an orthonormal basis of left-invariant vector fields $E_1, \ldots, E_n$. Write $[E_i, E_j] = c_{ij}{}^k E_k$. Then we have

$$\nabla_{E_i} E_j = \frac{1}{2}\left(c_{ij}{}^l + c^l{}_{ij} + c^l{}_{ji}\right) E_l,$$

and so

$$\nabla_k \nabla_i E_j = \frac{1}{4}\left(c_{ij}{}^l + c^l{}_{ij} + c^l{}_{ji}\right)\left(c_{kl}{}^p + c^p{}_{kl} + c^p{}_{lk}\right) E_p.$$

Also, we have

$$\nabla_{[E_k, E_i]} E_j = c_{ki}{}^l \nabla_l E_j = \frac{1}{2} c_{ki}{}^l \left(c_{lj}{}^p + c^p{}_{lj} + c^p{}_{jl}\right) E_p.$$

Combining these:

$$\begin{aligned} R_{ikj}{}^p = &\frac{1}{4}\left(c_{ij}{}^l + c^l{}_{ij} + c^l{}_{ji}\right)\left(c_{kl}{}^p + c^p{}_{kl} + c^p{}_{lk}\right) \\ &- \frac{1}{4}\left(c_{kj}{}^l + c^l{}_{kj} + c^l{}_{jk}\right)\left(c_{il}{}^p + c^p{}_{il} + c^p{}_{li}\right) \\ &- \frac{1}{2} c_{ki}{}^l \left(c_{lj}{}^p + c^p{}_{lj} + c^p{}_{jl}\right). \end{aligned}$$

Example 16.7.1 This gives us another method to compute the curvature of the metric on the upper half-plane: We think of this as the Lie group G of matrices of the form

$$\begin{bmatrix} v_n & v_1 & v_2 & \dots & v_{n-1} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

with $v_n > 0$. We identify this with the upper half-space of $\mathbb{R}^n$ by associating the above matrix with the point $(v_1, \dots, v_n)$. We take the left-invariant metric for which the usual basis at the identity $(0, \dots, 0, 1)$ is orthonormal. The corresponding left-invariant vector fields are:

$$(E_i)_v = \begin{bmatrix} 0 & 0 & \dots & 0 & v_n & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \sim v_n e_i$$

for $1 \leq i \leq n-1$, and

$$(E_n)_v = \begin{bmatrix} v_n & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \sim v_n e_n.$$

Therefore the corresponding metric is $g_{ij} = x_n^{-2}\delta_{ij}$. The structure coefficients ${c_{ij}}^k$ can be computed as follows: If $1 \leq i, j \leq n-1$, then $[E_i, E_j] = 0$; If $1 \leq i \leq n-1$, then

$$[E_i, E_n] = -v_n E_i.$$

Therefore

$${c_{ij}}^k = -\delta_j^n \delta_i^k + \delta_i^n \delta_j^k.$$

This gives (since with respect to the basis $\{E_i\}$, $g_{ij} = \delta_{ij}$)

$$\begin{aligned} R_{ikjl} = &-\frac{1}{4}\left(\delta_j^n\delta_{kp} - \delta_k^n\delta_{jp} + \delta_k^n\delta_{jp} - \delta_p^n\delta_{jk} + \delta_j^n\delta_{kp} - \delta_p^n\delta_{jk}\right) \\ &\times \left(\delta_p^n\delta_{il} - \delta_i^n\delta_{pl} + \delta_i^n\delta_{pl} - \delta_l^n\delta_{ip} + \delta_p^n\delta_{il} - \delta_l^n\delta_{ip}\right) \\ &+ \frac{1}{4}\left(\delta_j^n\delta_{ip} - \delta_i^n\delta_{jp} + \delta_i^n\delta_{jp} - \delta_p^n\delta_{ji} + \delta_j^n\delta_{ip} - \delta_p^n\delta_{ji}\right) \\ &\times \left(\delta_p^n\delta_{kl} - \delta_k^n\delta_{pl} + \delta_k^n\delta_{pl} - \delta_l^n\delta_{kp} + \delta_p^n\delta_{kl} - \delta_l^n\delta_{kp}\right) \\ &+ \frac{1}{2}\left(\delta_k^n\delta_{ip} - \delta_i^n\delta_{kp}\right)\left(\delta_j^n\delta_{lp} - \delta_p^n\delta_{jl} + \delta_p^n\delta_{jl} - \delta_l^n\delta_{jp} + \delta_j^n\delta_{jp}\right) \\ = &-\delta_j^n\delta_k^n\delta_{il} + \delta_j^n\delta_l^n\delta_{ik} + \delta_{il}\delta_{jk} - \delta_i^n\delta_l^n\delta_{jk} \\ &+ \delta_j^n\delta_i^n\delta_{kl} - \delta_j^n\delta_l^n\delta_{ik} - \delta_{kl}\delta_{ij} + \delta_k^n\delta_l^n\delta_{ij} \\ &+ \delta_k^n\delta_j^n\delta_{il} - \delta_k^n\delta_l^n\delta_{ij} - \delta_i^n\delta_j^n\delta_{kl} + \delta_i^n\delta_l^n\delta_{kj} \\ = &-\delta_{kl}\delta_{ij} + \delta_{jk}\delta_{il}. \end{aligned}$$

Hence the curvature operator has all eigenvalues equal to -1, and all of the sectional curvatures are -1.

16.8 Bi-invariant metrics

The situation becomes even simpler if we have a bi-invariant metric:

Proposition 16.8.1 *If g is a bi-invariant metric on a Lie group G, then for any left-invariant vector fields X, Y, and Z,*

$$g([X,Y],Z) = g([Z,X],Y).$$

Proof. g must be left-invariant, so that for any $h \in G$,

$$g_h(D_e l_h(X), D_e l_h(Y)) = g_e(X,Y).$$

Similarly, g must be right-invariant, so

$$g_h(D_e r_h(X), D_e r_h(Y)) = g_e(X,Y).$$

Together these imply that

$$g_e((D_e r_h)^{-1} D_e l_h(X), (D_e r_h)^{-1} D_e l_h(Y)) = g_e(X,Y).$$

We will denote by Ad_h the isomorphism of T_eG given by $(D_e r_h)^{-1} D_e l_h$. Then Ad is a representation of the group G on the vector space T_eG, called the *adjoint* representation: If γ has tangent vector X, then

$$\begin{aligned} Ad_k Ad_h(X) &= \frac{d}{dt} \left(r_k\right)^{-1} l_k \left(r_h\right)^{-1} l_h(\gamma(t))\Big|_{t=0} \\ &= \frac{d}{dt} \left(kh\gamma(t)h^{-1}k^{-1}\right)\Big|_{t=0} \\ &= Ad_{kh}(X). \end{aligned}$$

So we have $g(Ad_h(X), Ad_h(Y)) = g(X,Y)$, or g is Ad-invariant. But now differentiate this equation with $h = e^{tZ}$, at $t = 0$. In doing so we differentiate the maps Ad_h.

Definition 16.8.2 The derivative at the identity of the map $Ad : G \to GL(T_eG)$ is denoted $ad : T_eG \to L(T_eG, T_eG)$.

The map ad is a representation of the Lie algebra $\mathfrak{g}$ of G: It is a linear map such that

$$[ad(X), ad(Y)] = ad([X,Y]),$$

where on the left hand side the Lie bracket is to be interpreted as the commutator of the linear transformations $Ad(X)$ and $Ad(Y)$.

Now when we differentiate the Ad-invariance condition, we get:

$$0 = g(ad(Z)(X), Y) + g(X, ad(Z)(Y)).$$

Lemma 16.8.3

$$ad(X)Y = [X, Y].$$

Proof. By definition we have

$$ad(X)Y = \frac{d}{dt}\left(Ad_{e^{tX}}Y\right)|_{t=0} = \frac{d}{dt}\frac{d}{ds}\left(e^{tX}e^{sY}e^{-tX}\right)\Big|_{s=t=0}.$$

This makes sense because $\frac{d}{ds}\left(e^{tX}e^{sY}e^{-tX}\right)|_{s=0}$ is a vector in T_eG for each t, and so can be differentiated with respect to t. The identification between this and the Lie bracket comes from a more general statement:

Lemma 16.8.4 *Let M be a smooth manifold, X and Y in $\mathcal{X}(M)$, and $x \in M$. Then if $\Psi_{X,t}$ and $\Psi_{Y,t}$ are local flows of the vector fields X and Y near x,*

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\left(\Psi_{Y,t}\Psi_{X,s}\Psi_{Y,-t}(x)\right)\Big|_{s=0}\Big|_{t=0} = [X, Y](x).$$

Proof. In local coordinates near x (say, with $x^k = 0$), we have

$$X^k(z) = X^k(x) + z^i\partial_i X^k(x) + O(z^2).$$

Therefore by definition of the local flow,

$$\begin{aligned}(\Psi_{X,t}(z))^k &= z^k + t(X^k(x) + z^i\partial_i X^k(x) + O(z^2)) + O(t^2)\\ &= z^k + tX^k(x) + tz^i\partial_i X^k(x) + O(tz^2, t^2).\end{aligned}$$

From this we can compute:

$$(\Psi_{Y,-t}(x))^k = -tY^k(x) + O(t^2),$$

and

$$(\Psi_{X,s}\Psi_{Y,-t}(x))^k = -tY^k(x) + sX^k(x) - stY^i(x)\partial_i X^k(x) + O(st^2, s^2)$$

and finally,

$$\begin{aligned}(\Psi_{Y,t}\Psi_{X,s}\Psi_{Y,-t}(x))^k &= -tY^k(x) + sX^k(x) - stY^i(x)\partial_i X^k(x) + O(st^2, s^2)\\ &\quad + tY^k(x) + t(-tY^i(x) + sX^i(x))\partial_i Y^k(x) + O(st^2, t^2)\\ &= sX^k(x) - st\left(Y^i(x)\partial_i X^k(x) - X^i(x)\partial_i Y^k(x)\right)\\ &\quad + O(s^2, t^2).\end{aligned}$$

Differentiating with respect to s and t when $s = t = 0$ gives

$$\partial_t\partial_s\left(\Psi_{Y,t}\Psi_{X,s}\Psi_{Y,-t}(x)\right)^k\Big|_{s=t=0} = X^i(x)\partial_i Y^k(x) - Y^i(x)\partial_i X^k(x)$$
$$= \left([X,Y](x)\right)^k.$$

□

In the present case, we have

$$e^{tX} = \Psi_{X,t}(e);$$

And by left-invariance,

$$\frac{d}{dt}\Psi_{X,t}(h) = X_h = D_e l_h X_e = \frac{d}{dt}\left(he^{tX}\right),$$

so that

$$\Psi_{X,t}(h) = he^{tX}.$$

Therefore

$$e^{tX}e^{sY}e^{-tX} = \Psi_{X,-t}\left(e^{tX}e^{sY}\right) = \Psi_{X,-t}\Psi_{Y,s}\Psi_{X,t}(e).$$

So Lemma 16.8.4 gives the result. □

This completes the proof of Proposition 16.8.1. □

Corollary 16.8.5 *If G is a Lie group and g a bi-invariant Riemannian metric, then*

$$R_{ijkl} = \frac{1}{4}c_{ijp}c_{klp} = \frac{1}{4}\left(c_{ikp}c_{jlp} - c_{jkp}c_{ilp}\right).$$

Proof. By Proposition 16.8.1 we have $c_{ijk} = c_{jki}$, and we also have the symmetry $c_{ijk} = -c_{jik}$. Therefore

$$\nabla_{E_i}E_j = \frac{1}{2}g^{kl}\left(c_{ijl} + c_{lij} + c_{lji}\right)E_k$$
$$= \frac{1}{2}g^{kl}\left(c_{ijl} + c_{ijl} + c_{jil}\right)E_k$$
$$= \frac{1}{2}g^{kl}c_{ijl}E_k.$$

Therefore

$$R_{ikjl} = g\left(\nabla_k\nabla_i E_j - \nabla_i\nabla_k E_j - \nabla_{[E_k,E_i]}E_j, E_l\right)$$
$$= \frac{1}{4}c_{kpl}c_{ijp} - \frac{1}{4}c_{ipl}c_{kjp} - \frac{1}{2}c_{kip}c_{pjl}$$

Now we note that the Jacobi identity (cf. Lecture 6) gives

$$\begin{aligned}0 &= [E_i, [E_j, E_k]] + [E_j, [E_k, E_i]] + [E_k, [E_i, E_j]] \\ &= c_{ipl}c_{jkp} + c_{jpl}c_{kip} + c_{kpl}c_{ijp}\end{aligned}$$

which gives the result, including the equality between the two expressions. □

Example 16.8.6 We will apply this to a simple example, namely the Lie group S^3: Here the metric for which i, j and k are orthonormal at the identity is easily seen to be bi-invariant, and we have the structure coefficients

$$[i, j] = ij - ji = 2k;$$

and similarly $[j, k] = 2i$ and $[k, i] = 2j$. This gives, if we label $E_1 = i$, $E_2 = j$, and $E_3 = k$,

$$c_{123} = c_{231} = c_{312} = 2$$

and

$$c_{213} = c_{321} = c_{132} = -2,$$

and all others are zero. Thus we have

$$R_{1212} = \frac{1}{4}c_{12p}c_{12p} = \frac{1}{4}c_{123}^2 = 1;$$

and similarly $R_{1313} = R_{2323} = 1$. Also

$$R_{1213} = \frac{1}{4}c_{12p}c_{13p} = 0,$$

and similarly $R_{1223} = R_{1323} = 0$. Therefore the curvature operator is just the identity matrix with respect to the basis $E_1 \wedge E_2$, $E_2 \wedge E_3$, $E_3 \wedge E_1$, and all the eigenvalues are equal to 1. In particular all the sectional curvatures are equal to 1.