Lecture 20. The Gauss-Bonnet Theorem

In this lecture we will prove two important global theorems about the geometry and topology of two-dimensional manifolds. These are the Gauss-Bonnet theorem and the Poincaré-Hopf theorem.

Let us begin with a special case:

Suppose M is a compact oriented 2-dimensional manifold, and assume that there exists a vector field V on M which is nowhere zero. Now let g be any Riemannian metric on M. Then we can produce an orthonormal frame globally on M by taking $e_1 = V/g(V,V)^{1/2}$, and taking e_2 to be the unique unit vector orthogonal to e_1 such that e_1 and e_2 are an oriented basis at each point. The dual basis ω_1, ω_2 is then also globally defined, and we have a globally defined 1-form ω_{12} on M such that $d\omega_1 = \omega_{12} \wedge \omega_2$ and $d\omega_2 = -\omega_{12} \wedge \omega_1$. Then we have $-K(g)\omega_1 \wedge \omega_2 = d\omega_{12}$.

Stokes' Theorem then applies to give

$$-\int_M K(g) dVol(g) = \int_M \Omega_{12} = 0.$$

So we have the remarkable conclusion that the integral of the Gauss curvature over M is zero for every Riemannian metric on M.

Remark. In fact the existence of a non-vanishing vector field implies M is diffeomorphic to a torus $S^1 \times S^1$. A corollary of the computation above is that S^2 does not admit a nonvanishing vector field (as we have already seen), since if it did the integral of Gauss curvature would be zero for any metric, but we know that the standard metric on S^2 has Gauss curvature 1.

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The result we proved above is a special case of the famous Gauss-Bonnet theorem. The general case is as follows:

Theorem 20.1 *The Gauss-Bonnet Theorem Let M be a compact oriented two-dimensional manifold. Then for any Riemannian metric g on M,*

$$\int_M K(g)\, dVol(g) = 2\pi\chi(M)$$

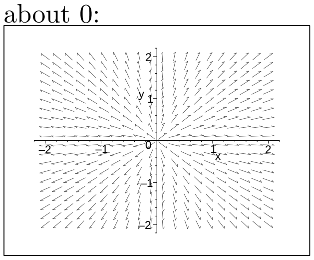
where $\xi(M)$ is the Euler characteristic of M.

Here the Euler characteristic is an integer associated to M which depends only on the topological type of M. If M admits a triangulation (i.e. a decomposition into diffeomorphic images of closed triangles, such that the intersection of any two is either empty, a common vertex, or a common edge) then the Euler characteristic is equal to $V+F-E$, where V is the number of vertices in the triangulation, F is the number of faces, and E is the number of edges.

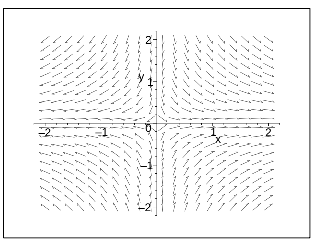
The second important theorem we will prove gives an alternative method of computing the Euler characteristic. To state the result we first need to introduce some new definitions.

Definition 20.2 Let M be a two-dimensional oriented manifold, and V a vector field on M. Suppose $y \in M$, and assume that there exists an open set U containing y such that there are no zeroes of V in $U \backslash \{y\}$. Then the index $I(V, y)$ of V about y is equal to the winding number of V around the boundary of a simply connected neighbourhood of y in U.

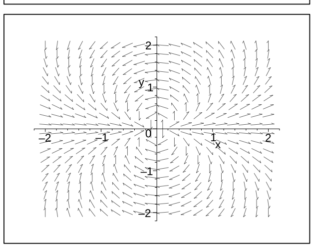
Example 20.3 Here are some vector fields in the plane with nontrivial index about 0:



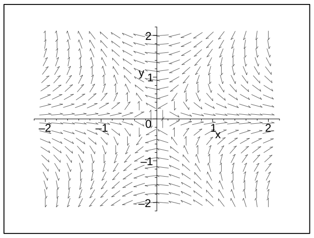
$V=(x, y)$, $I(V, 0)=+1$



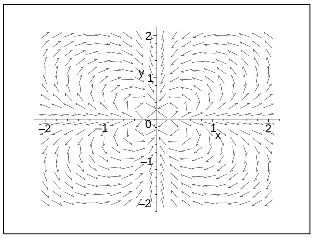
$V=(x,-y)$, $I(V, 0)=-1$



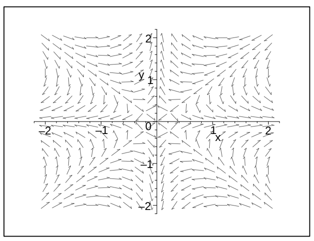
$V=(x^2-y^2, 2xy)$, $I(V, 0)=+2$



$V = (x^2 - y^2, -2xy)$, $I(V, 0) = -2$



$V = (x^3 - 3xy^2, 3x^2y - y^3)$, $I(V, 0) = +3$



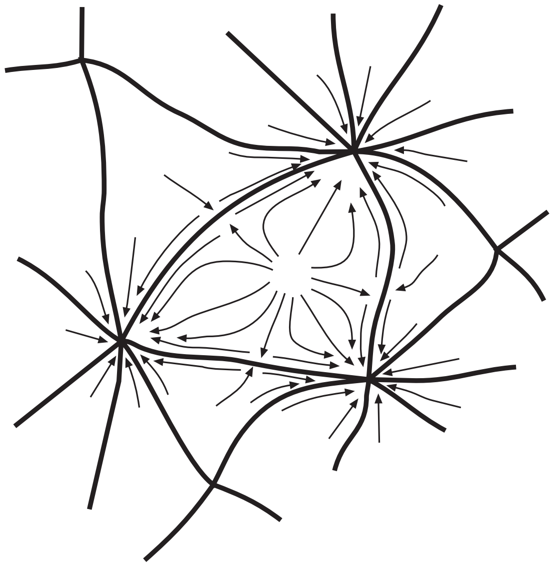
$V = (x^3 - 3xy^2, y^3 - 3x^2y)$, $I(V, 0) = -3$

Now we can state the Poincaré-Hopf theorem:

Theorem 20.4 *Let M be a compact oriented two-dimensional manifold, and let V be any vector field on M which has all zeroes isolated — that is, if y is any zero of V then there is a neithbourhood U of y in M such that V is nonzero on $U \backslash \{y\}$. Label the zeroes of V as $x_1, \ldots, x_N$. Then*

$$\sum_{i=1}^{N} I(V, x_i) = \chi(M).$$

We will prove that the sum of the indices of the vector field V at its zeroes is independent of V, and the result can then be taken to define the Euler characteristic. To see that this definition of the Euler characteristic agrees with the one given above in terms of triangulations, suppose that we are given a triangulation of M. Then choose a vector field as sketched below with zeroes at the centre of each face, the middle of each edge, and at each vertex. This vector field points outwards from the centre of each face, so has index 1 there, and points inwards at each vertex, so also has index 1 there. At the middle of each edge the vector field has index -1. Therefore the sum of the indices equals $F + V - E$, agreeing with our previous definition of the Euler characteristic.



We will prove the Gauss-Bonnet theorem and the Poincaré-Hopf theorem at the same time, by showing that for any Riemannian metric g on M and any vector field V with isolated zeroes, we have

$$\int_M K(g)\, dVol(g) = 2\pi \sum_{i=1}^{N} I(V, x_i).$$

If we keep V fixed and vary g, we deduce that the left-hand side is independent of g, and if we keep g fixed and vary V, we deduce that the right-hand side is independent of V, so we can define the Euler characteristic and deduce both results.

So let V and g be given, and let $x_1, \ldots, x_N$ be the zeroes of V. About each of these zeroes we can choose a small neighbourhood U_i of x_i (say, given by the image of a ball under some chart) such that V is nonvanishing on $\tilde{M} = M \backslash \cup_{i=1}^{N} U_i$. Let γ_i be the boundary of U_i, parametrised anticlockwise in some oriented chart.

Now $\tilde{M}$ is a compact oriented manifold with boundary, and V is nonvanishing on $\tilde{M}$. Define $e_1 = V/g(V,V)^{1/2}$, and take e_2 to be the unit vector orthogonal to e_1 which is given by a positive right-angle rotation of e_1. Denote the dual frame by ω_1 and ω_2, and let ω_{12} be the corresponding connection one-form. Then we have

$$\Omega_{12} = d\omega_{12},$$

and Stokes' theorem gives

$$-\int_{\tilde{M}} K(g)\, dVol(g) = \int_{\tilde{M}} \Omega_{12} = \int_{\partial \tilde{M}} \omega_{12}.$$

The boundary of $\tilde{M}$ is the union of the circles γ_i, parametrized clockwise. Therefore

$$-\int_{\tilde{M}} K(g)\, dVol(g) = -\sum_{i=1}^{N} \int_{\gamma_i} \omega_{12}.$$

Now on each of the regions U_i we can choose a non-vanishing frame $\bar{e}_1^{(i)}$, $\bar{e}_2^{(i)}$. This gives a corresponding connection one-form $\bar{\omega}_{12}^{(i)}$, and we have for each i

$$-\int_{U_i} K(g)\, dVol(g) = \int_{U_i} \Omega_{12} = \int_{U_i} d\bar{\omega}_{12}^{(i)} = \int_{\gamma_i} \bar{\omega}_{12}^{(i)}.$$

Combining these results, we get

$$-\int_{M} K(g)\, dVol(g) = \sum_{i=1}^{N} \int_{\gamma_i} \bar{\omega}_{12}^{(i)} - \omega_{12}.$$

To interpret the integral of $\bar{\omega}_{12}^{(i)} - \omega_{12}$ around γ_i, we need to consider the relationship between the frames e_1 and $\bar{e}_1^{(i)}$. Since these are both unit vectors, they are related by a rotation. Therefore there exists a map from γ_i to $SO(1) \simeq S^1 \subset \mathbb{R}^2$, say $x \mapsto z = (\alpha, \beta) \in S^1$, such that

$$\begin{aligned}
\bar{e}_1^{(i)} &= \alpha e_1 + \beta e_2 \\
\bar{e}_2^{(i)} &= -\beta e_1 + \alpha e_2.
\end{aligned}$$

Then we have

$$\begin{aligned}
\bar{\omega}_1^{(i)} &= \alpha \omega_1 + \beta \omega_2 \\
\bar{\omega}_2^{(i)} &= -\beta \omega_1 + \alpha \omega_2.
\end{aligned}$$

Since $\alpha^2 + \beta^2 = 1$, we have $\beta^{-1} d\alpha = -\alpha^{-1} d\beta = d\theta$. Applying the exterior derivative we find

$$\begin{aligned}
d\bar{\omega}_1^{(i)} &= d\alpha \wedge \omega_1 + \alpha d\omega_1 + d\beta \wedge \omega_2 + \beta d\omega_2 \\
&= -(\omega_{12} - \beta^{-1} d\alpha) \wedge \beta\omega_1 + (\omega_{12} + \alpha^{-1} d\beta) \wedge \alpha\omega_2 \\
&= (\omega_{12} + d\theta) \wedge \bar{\omega}_2^{(i)}.
\end{aligned}$$

and similarly

$$d\bar{\omega}_2^{(i)} = -(\omega_{12} + d\theta) \wedge \bar{\omega}_1^{(i)}.$$

It follows that $\bar{\omega}_{12}^{(i)} - \omega_{12} = d\theta$, and so

$$\int_{\gamma_i} \bar{\omega}_{12}^{(i)} - \omega_{12} = \int_{\gamma_i} d\theta$$

is 2π times the winding number of the map $z : \gamma_i \to S^1$. This is equal to the difference betweeen $I(\bar{e}_1^{(i)}, x_i)$ and $I(e_1, x_i)$. But $\bar{e}_1^{(i)}$ is nonvanishing in U_i, so $I(\bar{e}_1^{(i)}, x_i) = 0$, and by construction $I(e_1, x_i) = I(V, x_i)$. Thus

$$\int_{\gamma_i} \bar{\omega}_{12}^{(i)} - \omega_{12} = -I(V, x_i).$$

We have proved

$$-\int_M K(g)\, dVol(g) = -2\pi \sum_{i=1}^{N} I(V, x_i),$$

and the proofs of the Gauss Bonnet and Poincaré-Hopf theorems are complete.