Lecture 5. Lie Groups

In this lecture we will make a digression from the development of geometry of manifolds to discuss an very important special case.

5.1 Examples

Recall that a Lie Group is a group with the structure of a smooth manifold such that the composition from $M \times M \to M$ and the inversion from $M \to M$ are smooth maps.

Example 5.1.1 The general linear, special linear, and orthogonal groups. The general linear group $GL(n)$ (or $GL(n, \mathbb{R})$) is the set of non-singular $n \times n$ matrices with real components. The is a Lie group under the usual multiplication of matrices. We showed in Example 2.1.1 that the multiplication on $GL(n)$ is a smooth map from $GL(n) \times GL(n) \to GL(n)$. To show that $GL(n)$ is a Lie group, we nned to show that the inversion is also a smooth map.

$GL(n)$ is an open subset of $M^n \simeq \mathbb{R}^{n^2}$, so is covered by a single chart. With respect to this chart, the inversion is given by the formula

$$\left(M^{-1}\right)_i^j = \frac{1}{n! \det M} \sum_{\sigma,\tau} \mathrm{sgn}\sigma \ \mathrm{sgn}\tau \ M_{\sigma(1)}^{\tau(1)} M_{\sigma(2)}^{\tau(2)} \ldots M_{\sigma(n-1)}^{\tau(n-1)} \delta_{\sigma(n)}^i \delta_j^{\tau(n)}$$

where $\delta_i^j = 1$ if $i = j$ and 0 otherwise, and the sum is over pairs of permutations σ and τ of the set $\{1, \ldots, n\}$. Each component of M^{-1} is therefore given by sums of productss of components of M, divided by the non-zero smooth function $\det M$:

$$\det M = \frac{1}{n!} \sum_{\sigma,\tau} \mathrm{sgn}\sigma \ \mathrm{sgn}\tau \ M_{\sigma(1)}^{\tau(1)} M_{\sigma(2)}^{\tau(2)} \ldots M_{\sigma(n-1)}^{\tau(n-1)} M_{\sigma(n)}^{\tau(n)}$$

and so $\left(M^{-1}\right)_i^j$ is a smooth function of the components of M, and $GL(n)$ is a Lie group.

From Example 3.2.1 we know that the multiplications on $SL(n)$ and $O(n)$ are also smooth; these are both submanifolds of $GL(n)$, so it follows that the restriction of the inversion to each of these is smooth, so $SL(n)$ and $O(n)$ are Lie groups.

Example 5.1.2 The group $O(2)$. Recall that the orthogonal group $O(n)$ (or $O(n,\mathbb{R})$ is the group of $n \times n$ matrices with real components satisfying $M^T M = \mathrm{I}_n$. In particular $O(2)$ is the group of orthogonal 2×2 matrices.

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in $O(2)$ if the rows (or columns) of M form an orthonormal basis:

$$a^2 + b^2 = 1;$$
$$c^2 + d^2 = 1;$$
$$ac + bd = 0.$$

In particular M is determined by its first row $[a\ b]$ and its determinant $\delta = \det M = \pm 1$:

$$M = \begin{bmatrix} a & b \\ -\delta b & \delta a \end{bmatrix}.$$

This gives a natural map $\varphi : O(2) \to S^1 \times \{-1, 1\}$ given by

$$\varphi \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ((a,b),\ ad - bc) \in S^1 \times \{-1, 1\}.$$

Then φ is smooth and has the smooth inverse

$$\varphi^{-1}((a,b),\ \delta) = \begin{bmatrix} a & b \\ -\delta b & \delta a \end{bmatrix}.$$

This shows that $O(2)$ is diffeomorphic to $S^1 \times \{-1, 1\}$. In geometric terms, this map takes an orientation-preserving orthogonal transformation (i.e. a rotation) to its angle of rotation, or an orientation-reversing orthogonal transformation (i.e. a reflection) to twice the angle between the line of reflection and the positive x-axis.

Example 5.1.3 The torus $\mathbb{T}^n$ The Torus $\mathbb{T}^n$ is the product of n copies of $S^1 = \mathbb{R}/\sim$ where $a \sim b \Leftrightarrow a - b \in \mathbb{Z}$, with the group structure given by

$$([a_1], \ldots, [a_n]) + ([b_1], \ldots, [b_n]) = ([a_1 + b_1], \ldots, [a_n + b_n]).$$

This can also be naturally embedded as a subgroup of the group $(\mathbb{C}\backslash\{0\})^n$ (with the usual complex multiplication on each factor) by the map

$$F([x_1], \ldots, [x_n]) = \left(e^{2\pi i x_1}, \ldots, e^{2\pi i x_n}\right).$$

Remark. The previous example is one of an important class of examples of groups (and more generally of manifolds) which arise in the following way: Suppose M is a smooth manifold, and G is a group which acts on M – that is, there is a map $\rho : G \times M \to M$, such that the map $\rho_g : M \to M$ given by $\rho_g(x) = \rho(g, x)$ is a diffeomorphism, and $\rho_g \circ \rho_h = \rho_{hg}$. Equivalently, ρ is a group homomorphism from G to the group of diffeomorphisms of M.

The action of G on M is called *totally discontinuous* if for any $x \in M$ there is a neighbourhood U of x in M such that $\rho_g(U) \cap U = \emptyset$ for $g \neq e$. For example, the action of the group $\mathbb{Z}^n$ on $\mathbb{R}^n$ given by

$$\rho((z_1, \ldots, z_n), (x_1, \ldots, x_n)) = (x_1 + z_1, \ldots, x_n + z_n)$$

is totally discontinuous, but the action of $\mathbb{Z}$ on S^1 given by

$$\rho(k, z) = e^{ik\alpha} z$$

is not totally discontinuous if $\alpha \notin \mathbb{Q}$.

If G acts totally discontinuously on M, then the quotient space $M/\sim$ with $x \sim y \Leftrightarrow y = \rho_g(x)$ for some $g \in G$ can be made a differentiable manifold with the same dimension as M, in a unique way such that the projection from M onto $M/\sim$ is a local diffeomorphism: Let $\mathcal{A}$ be an atlas for M. Construct an atlas $\mathcal{B}$ for $M/\sim$ as follows: For each $[x] \in M/\sim$ choose a representative $x \in M$, and choose a chart $\varphi : W \to Z$ in $\mathcal{A}$ about x. Choose U such that $\rho_g(U) \cap U = \emptyset$ for all $g \neq e$, and take $\tilde{\varphi}([y]) = \varphi(x)$ for each $y \in (U \cap W)/\sim$. $\tilde{\varphi}$ is well-defined since $y \in U \Rightarrow [y] \cap U = \{y\}$, and for the same reason $\tilde{\varphi}$ is $1:1$ on $W \cap U$. The transition maps between these charts are just restrictions of transition maps of charts in $\mathcal{A}$, and so are smooth.

Note that the Klein bottle and the Möbius strip are given by the quotient of the plane by totally discontinuous actions, and the real projective space $\mathbb{R}P^n$ is the quotient of the sphere S^n by the totally discontinuous action of $\mathbb{Z}_2$ generated by the antipodal map.

Example 5.1.4 The group $SO(3)$. $SO(3)$ is the group of orientation-preserving orthogonal transformations of three-dimensional space. Note that an orthogonal transformation is a real unitary transformation, and so is conjugate to a diagonal unitary transformation with spectrum closed under conjugation. In the three-dimensional case this means that the spectrum consists of a pair of conjugate points on the unit circle, together with 1 or -1. In the case of a special orthogonal transformation, the spectrum must be $\{1,\ e^{i\alpha},\ e^{-i\alpha}\}$ for some $\alpha \in [0, \pi]$. In particular such a transformation fixes some vector in $\mathbb{R}^3$, and is a rotation by angle α about this axis. Thus an element in $SO(3)$ is determined by a choice of unit vector v and an angle α.

Example 5.1.5 S^3 as a Lie group. The three-dimensional sphere S^3 can be made into a Lie group as follows: We can think of S^3 as contained in $\mathbb{R}^4$, which we identify with the quaterions $\mathbb{H} = \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}\}$ where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, and $\mathbf{ki} = \mathbf{j}$. We think of x in $\mathbb{H}$ as having a real part x_0 and a vector part $x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ corresponding to a vector in $\mathbb{R}^3$. The multiplication on $\mathbb{H}$ can be rewritten in these terms as

$$(x, \mathbf{x}) \cdot (y, \mathbf{y}) = (xy - \mathbf{x} \cdot \mathbf{y},\ x\mathbf{y} + y\mathbf{x} + \mathbf{x} \times \mathbf{y}). \tag{5.1}$$

Given $x \in \mathbb{H}$, the *conjugate* $\bar{x}$ of x is given by

$$\overline{(x, \mathbf{x})} = (x, -\mathbf{x}).$$

Then the identity (5.1) gives

$$\overline{(x, \mathbf{x})} \cdot (x, \mathbf{x}) = x^2 + \mathbf{x}^2$$

which is the same as the norm of x as an element of $\mathbb{R}^4$. We also have

$$\begin{aligned}\overline{(x, \mathbf{x}) \cdot (y, \mathbf{y})} &= (xy - \mathbf{x} \cdot \mathbf{y},\ -x\mathbf{y} - y\mathbf{x} - \mathbf{x} \times \mathbf{y}) \\ &= (y, -\mathbf{y}) \cdot (x, -\mathbf{x}) \\ &= \overline{(y, \mathbf{y})} \cdot \overline{(x, \mathbf{x})}.\end{aligned}$$

Therefore

$$|x \cdot y|^2 = \overline{x \cdot y} \cdot x \cdot y = \bar{y} \cdot \bar{x} \cdot x \cdot y = |x|^2 |y|^2.$$

In particular, for x and y in S^3 we have $xy \in S^3$. It follows that the restriction of the multiplication in $\mathbb{H}$ to S^3 makes S^3 into a Lie group. This example will reappear later in a more familiar guise.

Remark. We have seen two examples of spheres which are also Lie groups: S^1 is a Lie group, and so is S^3 (one could also say that $S^0 = \{1, -1\}$ is a 0-dimensional Lie group). This raises the question: Are all spheres Lie groups? If not which ones are? The following proposition, which also introduces some important concepts in the theory of Lie groups, will be used to show that the two-dimensional sphere S^2 can't be made into a Lie group:

Proposition 5.1.1 *Let G be a Lie group. Then there exist n smooth vector fields $E_1, \ldots, E_n$ on G each of which is everywhere non-zero, such that $E_1, \ldots, E_n$ form a basis for T_xM for every $x \in M$.*

Proof. Consider the left and right translation maps from G to itself given by $l_g(g') = gg'$ and $r_g(g') = g'g$ for all $g' \in G$. These are smooth maps, and $(l_g)^{-1} = l_{g^{-1}}$ and $(r_g)^{-1} = r_{g^{-1}}$ are smooth, so l_g and r_g are diffeomorphisms from G to G. The derivative of a diffeomorphism is a linear isomorphism, and in particular this gives us isomorphisms $D_e l_g,\ D_e r_g :\ T_eG \to T_gG$.

Choose a basis $\{e_1, \ldots, e_n\}$ for T_eG. Then define $E_i(g) = D_e l_g(e_i)$ for each i and each $g \in G$. The vector fields $E_1, \ldots, E_n$ have the properties claimed in the proposition.

We say that a vector field V on a Lie group G is **left-invariant** if it satisfies $D_e l_g(V_e) = V_g$ for all $g \in G$. It is straightforward to show that the set of left-invariant vector fields form a vector space with dimension equal to the dimension of G (in fact $\{E_1, \ldots, E_n\}$ are a basis). Similarly one can define right-invariant vector fields.

Now return to the 2-sphere S^2. We have the following result from algebraic topology:

Proposition 5.1.2 ("The sphere has no hair") *There is no continuous non-vanishing vector field on S^2.*

Corollary 5.1.3 *There is no Lie group structure on S^2.*

Proof (Sketch):

Suppose V is a non-vanishing continuous vector field on S^2. Then in the chart given by stereographic projection from the north pole, we have in the neighbourhood of the origin (possibly after rotating the sphere about the polar axis)[3]

$$(D\varphi_+(V))(x) = (1,0) + o(|x|) \quad \text{as } x \to 0.$$

Now look at this vector field through the chart φ_-: The transition map is $(\varphi_- \circ \varphi_+^{-1})(x,y) = (x/(x^2+y^2), y/(x^2+y^2))$. Therefore $\tilde{V} = D\varphi_-(V)$ is given by

$$\begin{aligned} \tilde{V}(w,z) &= \left(D_{(w,z)}(\varphi_+ \circ \varphi_-^{-1})\right)^{-1}((1,0) + o(|x|)) \\ &= \begin{bmatrix} z^2 - w^2 & -2wz \\ -2wz & w^2 - z^2 \end{bmatrix} \begin{bmatrix} 1 + o(|(w,z)|^{-1}) \\ o(|(w,z)|^{-1}) \end{bmatrix} \\ &= \begin{bmatrix} z^2 - w^2 \\ -2wz \end{bmatrix} + o(|(w,z)|) \quad \text{as } |(z,w)| \to \infty. \end{aligned}$$

There can be no zeroes of the vector field $\tilde{V}$ on $\mathbb{R}^2$, so we can divide $\tilde{V}$ by its length at each point to obtain a continuous vector field V' which has length 1 everywhere, and such that

$$V'(w,z) = \begin{bmatrix} \frac{z^2-w^2}{w^2+z^2} \\ -\frac{2wz}{w^2+z^2} \end{bmatrix} + o(1) \quad \text{as } |(w,z)| \to \infty$$

For each $r \in \mathbb{R}_+$, consider the restriction of V' to the circle of radius r about the origin. Since V' has length 1 everywhere, this defines for each r a continuous map f_r from the circle S^1 to itself, given by

$$f_r(z) = V'(rz)$$

for each $z \in S^1 \subset \mathbb{R}^2$ and each $r > 0$ in $\mathbb{R}$. If we parametrise S^1 by the standard angle coordinate θ, then we have

$$f_r(\cos\theta, \sin\theta) = (\sin^2\theta - \cos^2\theta, -2\cos\theta\sin\theta) + o(1) = -(\cos 2\theta, \sin 2\theta) + o(1)$$

[3] We use the notation $f(x) = o(g(x))$ as $x \to 0$ to mean $\lim_{x\to 0}\left|\frac{f(x)}{g(x)}\right| = 0$. Another useful notation is $f(x) = O(g(x))$ as $x \to 0$ to mean $\limsup_{x\to 0}\left|\frac{f(x)}{g(x)}\right| < \infty$.

as $r \to \infty$. Thus the limit as $r \to \infty$ of the map f_r is just the map that sends θ to $\pi - 2\theta$. In other words, $f_r(\theta)$ winds twice around the circle backwards as θ traverses the circle forwards once: f_r has winding number -2 for r sufficiently large.

But now consider what happens as $r \to 0$: $f_0(\theta)$ is just the constant vector $V'(0,0) \in S^1$, so $\lim_{r\to 0} f_r(\theta) = \theta_0$, a constant. But this means that for r sufficiently small the function $f_r(\theta)$ does not traverse the circle at all as θ moves around S^1, and f_r has winding number zero for r small. Now we use the result from algebraic topology that the winding number of a map from S^1 to itself is a homotopy invariant. The family of maps $\{f_r\}$ is a continuous deformation from a map of winding number -2 to a map of winding number 0, which is impossible. Therefore there is no such vector field. □

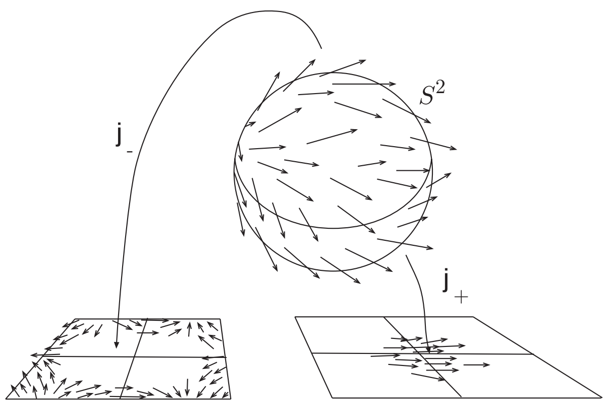


Fig. 5.1: A vector field on S^2, viewed in a region near the centre of the chart φ_+, and in a region near infinity in the chart φ_-.

5.2 Examples of left-invariant vector fields

It is useful to see how to find the left-invariant vector fields for some examples:

Example 5.2.1 More on $\mathbb{R}^n$*.* Let us consider the Lie group $\mathbb{R}^n$ under addition. Then for any $x \in \mathbb{R}^n$ we have $l_x(y) = x + y$, and so $D_0 l_x(v) = v$. Taking the standard basis $\{e_1, \ldots, e_n\}$ for $T_0\mathbb{R}^n \simeq \mathbb{R}^n$, we find that the left-invariant vector fields are given by

$$(E_i)_x = e_i.$$

That is, the left-invariant vector fields are just the constant vector fields.

Example 5.2.2 Matrix groups Let G be a Lie subgroup of $GL(n)$. Then we have $l_M(N) = MN$ for any M and N in G. Differentiating, we get

$$D_{\mathrm{I}} l_M(A) = MA$$

for any $A \in T_{\mathrm{I}}G$. Therefore the left-invariant vector fields have the form

$$A_M = MA$$

where A is a constant matrix in $T_{\mathrm{I}}G$.

Consider the case of the entire group $GL(n)$: In this case, since $GL(n)$ is an open subset of the Euclidean space $M_n \simeq \mathbb{R}^{n^2}$, the tangent space $T_{\mathrm{I}}GL(n)$ is just M_n, and the left-invariant vector fields are given by linear combinations of

$$\left(E_{ij}\right)_M = Me_{ij}$$

where e_{ij} is the matrix the a 1 in the ith row and the jth column, and zero everywhere else.

Next consider the case $G = SL(n)$: What is the tangent space at the origin? Recall that $SL(n)$ is a submanifold of $GL(n)$, given by the level set $\det^{-1}(1)$ of the determinant function. Therefore we can identify the tangent space of $SL(n)$ with the subspace of M_n given by the kernel of the derivative of the determinant:

$$T_M SL(n) = \left\{ A \in M_n : \ \frac{d}{dt} \det(M + tA)\Big|_{t=0} = 0 \right\}.$$

In particular, we have

$$T_{\mathrm{I}} SL(n) = \left\{ A \in M_n : \ \mathrm{t}r(A) = 0 \right\}.$$

Finally, consider $G = O(n)$: This is also a level set of a submersion from M_n, namely the map which sends M to the upper triangular part of $M^T M - I$ (since $M^T M - I$ is symmetric, the entries below the diagonal are superfluous). As before, the tangent space can be identified with the kernel of the derivative of the submersion, and in particular

$$T_{\mathrm{I}} O(n) = \{A \in M_n : \ B^T = -B\},$$

the vector space of antisymmetric matrices.

Example 5.2.3 The group $\mathbb{C}\backslash\{0\}$. The multiplicative group of non-zero complex numbers is an open set in $\mathbb{C} \simeq \mathbb{R}^2$, so we need not worry about charts. We have

$$l_z(w) = zw,$$

and hence $D_1 l_z(v) = zv$ for all $v \in \mathbb{R}^2$ and $z \in \mathbb{C}\backslash\{0\}$. The left invariant vector fields are linear combinations of the two vector fields $(E_1)_z = z$ and $(E_2)_z = iz$.

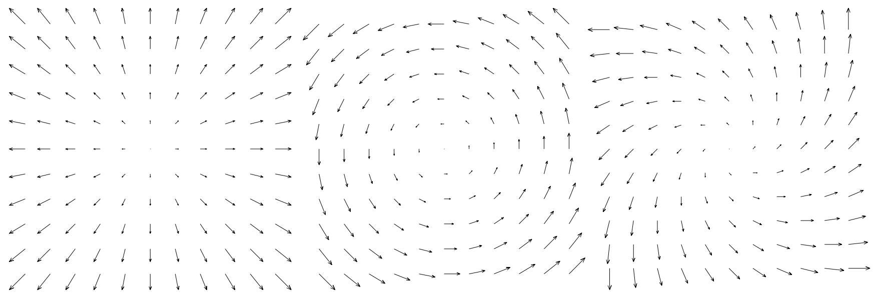


Fig. 5.2: Left-invariant vector fields on $\mathbb{C}\backslash\{0\}$ corresponding to the vectors 1, i, and $1+i$ at the identity.

5.3 1-parameter subgroups

A 1-parameter subgroup Γ of G is a smooth group homomorphism from $\mathbb{R}$ (with addition as the group operation) into G.

Given a one-parameter subgroup $\Gamma : \mathbb{R} \to G$, let $E = \Gamma'(0) \in T_eG$. Then (thinking of a vector as an equivalence class of curves) we have $E = [t \mapsto \Gamma(t)]$. The homomorphism property is that $\Gamma(s)\Gamma(t) = \Gamma(s+t)$. In particular we have $l_{\Gamma(s)}(\Gamma(t)) = \Gamma(s+t)$, and $D_0 I_{\Gamma(s)}(E) = \Gamma'(s)$. In other words, the tangent to the one-parameter subgroup is given by the left-invariant vector field $E_g = D_0 l_g(E)$.

The converse is also true: Suppose E is a left-invariant vector field on G, and $\Gamma : \mathbb{R} \to G$ is a smooth map which satisfies $\Gamma'(t) = E_{\gamma(t)}$. Then Γ defines a one-parameter subgroup.

Proposition 5.3.1 *Let G be a Lie group. To every unit vector $v \in T_eG$ there exists a unique one-parameter subgroup $\Gamma : \mathbb{R} \to G$ with $\Gamma'(0) = v$.*

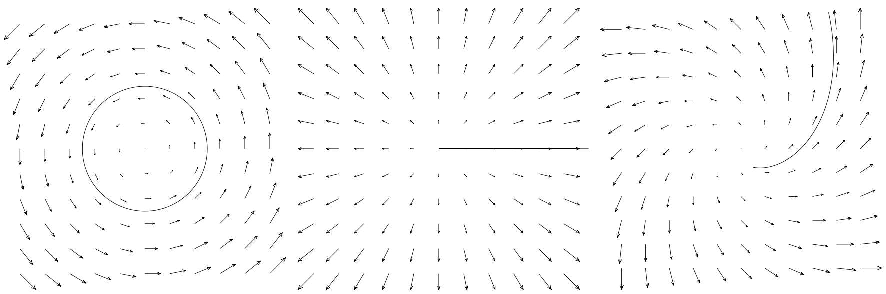
This amounts to solving the ordinary differential equation

$$\frac{d}{dt}\Gamma(t) = D_o l_{\Gamma(t)}(v)$$

with the initial condition $\Gamma(0) = 0$. I will therefore leave the proof until we have seen results on existence and uniqueness of solutions of ordinary differential equations on manifolds in a few lectures from now.

Example 5.3.1: Non-zero complex numbers, revisited. In the case $G = \mathbb{C}\backslash\{0\}$, we have to solve the differential equation

$$\begin{aligned} \dot{z} &= zw \\ z(0) &= 1 \end{aligned}$$



for $w \in \mathbb{C}$ fixed, and this has the solution $z(t) = e^{tw}$. This gives the one-parameter subgroups of $\mathbb{C}\backslash\{0\}$: If $w = 1$ we get the positive real axis, with $\Gamma(t) = e^t$; If $w = i$ we get the unit circle, with $\Gamma(t) = e^{it}$, and for other complex numbers we get logarithmic spirals.

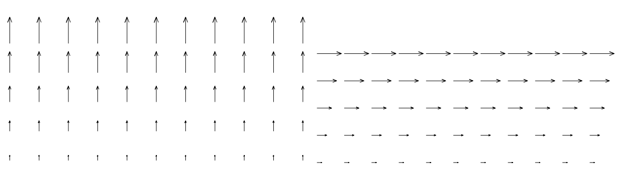
Example 5.3.2: Hyperbolic space. The two-dimensional hyperbolic space $\mathbb{H}^2$ is the upper half-plane in $\mathbb{R}^2$, which we think of as a Lie group by identifying it with the group of matrices

$$\left\{\begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix} : \ \alpha > 0,\ \beta \in \mathbb{R}\right\}$$

by the diffeomorphism $(x, y) \mapsto \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$. The left-invariant vector fields are as follows: To the vector $(1, 0)$ in the tangent space of the identity $(0, 1)$, we associate the vector $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in the tangent space to the matrix group at the identity. The left-invariant vector field corresponding to this is given by

$$(E_1)_{\begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}} = \frac{d}{dt}\begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}.$$

This corresponds to the vector field $(E_1)_{(\beta,\alpha)} = (\alpha, 0)$. Similarly we get a left invariant vector field $(E_2)_{(\beta,\alpha)} = (0, \alpha)$ corresponding to the vector $(0, 1)$.



The one-parameter subgroups are lines through $(0,1)$, with $\Gamma(t) = (ab^{-1}(e^{bt}-1), e^{bt})$.

Example 5.3.3: One-parameter subgroups of matrix groups We saw in Example 5.2.2 that the left-invariant vector fields in subgroups of $GL(n)$ are given by $A_M = MA$ for some fixed matrix $A \in T_I G$. Therefore the one-parameter subgroups are given by the solutions of the differential equation

$$\dot{M} = MA \tag{5.2}$$

with the initial condition $M(0) = \mathrm{I}$. This is exactly the problem which arises in the process of solving a general linear system of ordinary equations: Recall that to solve the system $\dot{v} = Av$ with any initial condition $v(0) = v_0$, one first solves the system (4.1) to obtain a matrix $M(t)$. Then the required solution is $v(t) = M(t)v_0$. The solution of (4.1) is denoted by e^{At}, and is given by the convergent power series

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n.$$

It is a consequence of our construction that the matrix exponential of a matrix $A \in T_i G$ is in G – in particular, if A is traceless then $e^{At} \in SL(n)$, and if A is antisymmetric then $e^{At} \in SO(n)$.

Definition 5.3.1 The exponential map $exp : T_e G \to G$ is the map which sends a vector $v \in T_e G$ to $\Gamma(1)$, where $\Gamma : \mathbb{R} \to G$ is the unique one-parameter subgroup satisfying $\Gamma'(0) = v$.

In particular, in the cases $G = \mathbb{R}_+$, $G = \mathbb{C}\backslash\{0\}$, and $G = GL(n)$, $SL(n)$, or $SO(n)$, the exponential map is the same as the usual exponential map. This is also true for the Quaternions, and for the subgroup S^3 of unit length quaternions.

The theory of regularity for ordinary differential equations (which we will discuss later) implies that the exponential map is smooth. Notice also that if Γ is a one-parameter subgroup with $\Gamma'(0) = v$, then $t \mapsto \Gamma(st)$ is also a one-parameter subgroup, with tangent at the origin given by sv. Therefore $\exp(sv) = \Gamma(s)$. This implies

$$D_0 \exp(v) = \frac{d}{ds} \exp(sv) = \frac{d}{ds}\Gamma(s) = \Gamma'(0) = v.$$

In other words, $D_0 \exp = \mathrm{I}$. In particular this implies that the exponential map is a diffeomorphism in the neighbourhood of the identity.

Example 5.3.4: The exponential map on S^3. The exponential map on S^3 can be computed explicitly. First we derive an explicit formula for the exponential of a quaternion $x = x^0 + x^1\mathbf{i} + x^2\mathbf{j} + x^2\mathbf{k}$. We need to solve the differential equation

$$\dot{w} = wx.$$

First note that $\frac{d}{dt}\bar{w} = \bar{x}\bar{w}$, and so

$$\frac{d}{dt}|w|^2 = \bar{x}\bar{w}w + \bar{w}wx = (x + \bar{x})|w|^2 = 2x_0|w|^2.$$

Therefore $|w(t)| = e^{x_0 t}$. Now consider $u(t) = \frac{w(t)}{|w(t)|}$:

$$\frac{d}{dt}u = \frac{wx}{|w|} - \frac{w}{|w|^2}x_0|w| = \frac{w}{|w|}(x - x_0) = u(x - x_0).$$

Therefore $u(t) = \exp(x - x_0)$, the exponential of the 'vector' part of x. So now assume $x_0 = 0$. Then we can write

$$w^1(t) = \frac{1}{2}\left(\bar{w}\mathbf{i} - \mathbf{i}w\right)$$

and

$$\frac{d}{dt}w^1(t) = \frac{1}{2}\left(-\mathbf{i}wx + \bar{x}\bar{w}\mathbf{i}\right).$$

Differentiating again with respect to t, we get

$$\frac{d^2}{dt^2}w^1(t) = \frac{1}{2}\left(-\mathbf{i}wx^2 + \bar{x}^2\bar{w}\mathbf{i}\right).$$

But now x has no real part, so $\bar{x} = -x$, and $x^2 = \bar{x}^2 = -|x|^2$, giving

$$\frac{d^2}{dt^2}w^1(t) = -|x|^2 w^1(t),$$

and so (since $w^1(0) = 0$ and $(w^1)'(0) = x^1$), $w^1(t) = \frac{x^1}{|x|}\sin(|x|t)$. The other components are similar, so we have (in the case $|x| = 1$)

$$\exp(tx) = \cos t + x\sin t. \tag{5.3}$$

Thus in general we have

$$\exp(x^0 + tv) = e^{x^0}(\cos t + v\sin t)$$

whenever x^0 is real and v is a unit length quaternion with no real part. In particular, the tangent space of S^3 is just the quaternions with no real part, so the exponential is given by Eq. (5.3). The one-parameter subgroups of S^3 are exactly the great circles which pass through the identity.

Exercise 5.3.1 Show that the following map is a group homomorphism and a diffeomorphism from the group S^3 to the group $SU(2)$ of unitary 2×2 matrices with determinant 1:

$$\varphi((x^0, x^1, x^2, x^3)) = \begin{bmatrix} x^0 + ix^3 & x^1 + ix^2 \\ -x^1 + ix^2 & x^0 - ix^3 \end{bmatrix}.$$

Remark. The idea behind this construction is the following: A matrix $U \in SU(2)$ can be chosen by first choosing an eigenvector v (this is given by a choice of any non-zero element of $\mathbb{C}^2$, but is defined only up to multiplication by an arbitrary non-zero complex number, and hence is really a choice of an element of $\mathbb{C}P^1 = \left(\mathbb{C}^2\backslash\{0\}\right)/(\mathbb{C}\backslash\{0\})$. Given this, the other eigenvector is uniquely determined by Gram-schmidt. To complete the specification of U we need only specify the eigenvector of the first eigenvalue, which is an arbitrary element $e^{i\theta}$ of the unit circle S^1. Then the eigenvalue of the other eigenvector is $e^{-i\theta}$, since $\det U = 1$.

Proposition 5.3.2 $\mathbb{C}P^1 \simeq S^2$.

Proof W e have natural charts $\varphi_\pm$ for S^2, with transition map defined on $\mathbb{R}^2\backslash\{0\}$ by $(x,y) \mapsto (x,y)/(x^2+y^2)$. We also have natural charts $\eta_1 : \mathbb{C}P^1\backslash\{[1,0]\} \to \mathbb{R}^2$ given by $\eta_1([z_1,z_2]) = z_1/z_2 \in \mathbb{C} \simeq \mathbb{R}^2$ and $\eta_2 : \mathbb{C}P^1\backslash\{[0,1]\} \to \mathbb{R}^2$ given by $\eta_2([z_1,z_2]) = z_2/z_2$. The transition map between these is then $z \mapsto 1/z : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$. We define a map χ from S^2 to $\mathbb{C}P^1$ by

$$\chi(x) = \begin{cases} \eta_1^{-1}(\varphi_+(x)), & \text{for } x \neq N; \\ \eta_2^{-1}(\overline{\varphi_-(x)}), & \text{for } x \neq S. \end{cases}$$

This makes sense because in the overlap we have

$$\eta_2^{-1} \circ \bar{\varphi}_- = \eta_1^{-1} \circ (\bar{\varphi}_-^{-1} \circ \varphi_+^{-1}) \circ \varphi_+ = \eta_1^{-1} \circ \eta$$

since $\eta_1 \circ \eta_2^{-1} = \bar{\varphi}_- \circ \varphi_+^{-1}$ and these are involutions. Explicitly, we have

$$\chi(x_1,x_2,x_3) = [x_1 + ix_2, 1 - x_3] = [x_1 - ix_2, 1 + x_3]$$

where the second equality holds since

$$\frac{x_1 + ix_2}{1 - x_3} - \frac{1 + x_3}{x_1 - ix_2} = \frac{x_1^2 + x_2^2 - 1 + x_3^2}{(1 - x_3)(x_1 - ix_2)} = 0$$

for $(x_1,x_2,x_3) \in S^2$. □

We then associate the matrix U with a point in S^3 by starting at the identity $(1,0,0,0)$, and following the great circle in the direction $\chi([v]) \in S^2 \simeq T_{(1,0,0,0)}S^3$ for a distance α, and the resulting map is the map φ given in Exercise 5.3.1.

Exercise 5.3.2 Show that the following map from $SU(2)$ to $SO(3)$ is a group homomorphism, and a *local* diffeomorphism, but not a diffeomorphism:

$$\Psi \begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ c_1 + ic_2 & d_1 + id_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(a_1 + d_1) & -b_2 - c_2 & b_1 - c_1 \\ b_2 + c_2 & \frac{1}{2}(a_1 + d_1) & d_2 - a_2 \\ c_1 - b_1 & a_2 - d_2 & \frac{1}{2}(a_1 + d_1) \end{bmatrix}.$$

Consider the composition of this with the map φ in Exercise 5.3.1: Deduce that the group of rotations in space, $SO(3)$, is diffeomorphic to $\mathbb{R}P^3$.

Remark. There is, similarly, a geometric way to understand this map: An element of $SO(3)$ is just a rotation of three-dimensional space, which can be determined by specifying its axis of rotation v (an element of S^2) together with the angle of rotation α about this axis. As before, this corresponds to a point in S^3 by following a great circle from $(1,0,0,0)$ in the direction $v \in S^2 \subset \mathbb{R}^3 \simeq T_{(1,0,0,0)}S^3$ for a distance $\alpha/2$.

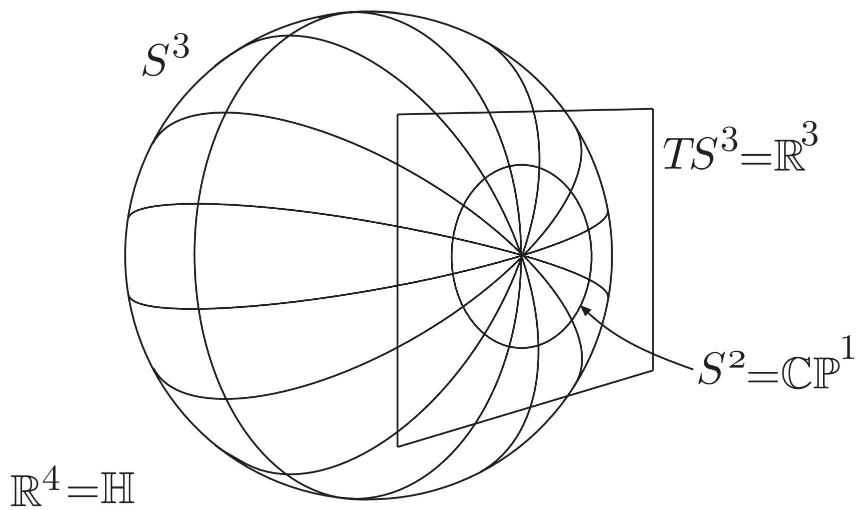


Fig. 5.5: A rotation of $\mathbb{R}^3$ is associated with a point in S^3 by following the great circle through $(1,0,0,0)$ in the direction of the axis of rotation $v \in S^2 \subset TS^3$ for a distance α equal to half the angle of rotation. This is also naturally associated to the matrix $U \in SU(2)$ which sends z to $e^{i\alpha}z$ for any $z \in \mathbb{C}^2$ with $[z] = v \in \mathbb{C}P^1 \simeq S^2 \subset TS^3$.

Notes on Chapter 5:

Lie groups are named after the Norwegian mathematician Sophus Lie (1842–1899). He defined ‘infinitesimal transformations’ (what are now called Lie algebras) first, as part of his work in describing symmetries of partial differential equations, and then went on to develop his theory of ‘continuous transformation groups’.

The classification of compact simple Lie groups was accomplished by Elie Cartan in 1900 (building on earlier work by Killing).

Sophus Lie

Elie Cartan

Wilhelm Killing