An asymptotic expansion inspired by Ramanujan*†

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Abstract

Corollary 2, Entry 9, Chapter 4 of Ramanujan's first notebook claims that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{nk} \left(\frac{x^k}{k!}\right)^n \sim \ln x + \gamma$$

as $x \to \infty$. This is known to be correct for the case $n = 1$, but incorrect for $n \geq 3$. We show that the result is correct for $n = 2$. We also consider the order of the error term, and discuss a different, correct generalisation of the case $n = 1$.

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1 Introduction

Much of Ramanujan's work was not published during his lifetime, but was summarised in his Notebooks. These were printed in facsimile in 1957 [14], and edited editions have been published by Berndt [2, 3, 4].

Many of Ramanujan's results were obtained in a formal manner, and he did not state sufficient conditions for their validity. For example, Example 3, Entry 9, Chapter 4 (page 97 of [2]) is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\varphi(k) - \varphi(-k))}{k} = \varphi'(0) \tag{1}$$

but clearly some conditions on the function $\varphi(z)$ are required. Berndt has given sufficient conditions, but they do not always hold for Ramanujan's applications of (1). In fact, Ramanujan does not claim exact equality in (1), but writes that the left side is "nearly" equal to the right side. Thus, some sort of approximation or asymptotic equality is intended.

To illustrate the use of (1), take $\varphi(z) = x^z/\Gamma(z+1)$, where $x > 0$ is real. Proceeding formally, we obtain

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{x^k}{k!} = \ln x + \gamma \ , \tag{2}$$

where $\gamma = -\Gamma'(1)$ is Euler's constant.

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If equality in (2) is interpreted as asymptotic equality (usually denoted by " $\sim$ ") as $x \to \infty$, then the result is correct. In fact, a classical result (also given on page 167 of [3]) is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{x^k}{k!} - \ln x - \gamma = \int_x^{\infty} \frac{e^{-t}}{t} dt = O(e^{-x}/x) \ . \tag{3}$$

Ramanujan's Corollary 2, Entry 9, Chapter 4 (page 98 of [2]) is that, for positive integer n,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{x^k}{k!}\right)^n \sim n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{x^k}{k!} \ .$$

In view of (3), this is equivalent to

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{nk} \left(\frac{x^k}{k!}\right)^n \sim \ln x + \gamma \tag{4}$$

It is plausible that Ramanujan derived this result from (1) in the same formal manner that we derived (2) above, but taking $\varphi(z) = (x^z/\Gamma(z+1))^n$.

Berndt [2] shows that (4) is false for $n \geq 3$; in fact, the function defined by the left side of (4) changes sign infinitely often, and grows exponentially large as $x \to \infty$. However, Berndt leaves the case $n = 2$ open.

The aim of this paper is to show that (4) is true in the case $n = 2$. Theorem 1 in Section 3 gives an exact expression for the error in (4) as an integral involving the Bessel function $J_0(x)$, and Corollary 1 deduces an asymptotic expansion. The most significant term is $O(x^{-3/2})$ as $x \to \infty$.

Theorem 1 is a special case[1] of a formula given on page 48 of Luke [9]. However, the connection with Ramanujan does not seem to have been noticed before.

In Corollary 2, Entry 2, Chapter 3 of his first Notebook, Ramanujan shows that the function on the left side of (2) can be written as

$$e^{-x} \sum_{k=0}^{\infty} H_k \frac{x^k}{k!} \ ,$$

where $H_k = \sum_{j=1}^{k} 1/j$ is a harmonic number ($H_0 = 0$). Thus

$$\sum_{k=0}^{\infty} H_k \frac{x^k}{k!} \Big/ \sum_{k=0}^{\infty} \frac{x^k}{k!} \sim \ln x + \gamma \ . \tag{5}$$

In Section 4 we indicate how Ramanujan might have generalised (5) in much the same way that he attempted to generalise (2).

2 Notation and Preliminary Results

In this section we give some preliminary results on integrals involving J_0. These results may be found in the literature, but for completeness we sketch their proofs.

Recall that

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k!k!} \ . \tag{6}$$

$J_0(z)$ is an entire function, but we are only concerned with its behaviour on the positive real axis. For small positive x, $J_0(x) = 1 + O(x^2)$. For large positive x, Hankel's asymptotic

[1] Also given in formula 11.1.20, Chapter 11 of Abramowitz and Stegun [1] (the chapter was written by Luke).

expansion [11, 15] gives $J_0(x) = O(x^{-1/2})$. These observations are sufficient to show that the integral occurring in Lemma 2 below is absolutely convergent.

In the proof of the following Lemma, $(a)_k = a(a+1)\cdots(a+k-1) = \Gamma(a+k)/\Gamma(a)$, and

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

is a hypergeometric function.

Lemma 1 *For* $0 < \mu < \frac{3}{2}$,

$$\int_0^{\infty} t^{\mu-1} J_0(t)dt = \frac{2^{\mu-1}\Gamma(\mu/2)}{\Gamma(1-\mu/2)} \ .$$

Proof. The integral[2] is known as Weber's infinite integral [16]. We sketch a proof in the case $\mu < \frac{1}{2}$, which is all that is needed below.

Let α be a complex variable. We first evaluate

$$I(\alpha) = \int_0^{\infty} e^{-\alpha t} t^{\mu-1} J_0(t)dt$$

for $R(\alpha) > 1$. Integrating term by term, the power series (6) gives

$$I(\alpha) = \alpha^{-\mu}\Gamma(\mu) F\left(\frac{\mu}{2}, \frac{\mu+1}{2}; 1; -\alpha^{-2}\right) \ . \qquad (7)$$

By analytic continuation, the result (7) holds[3] in the right half-plane $R(\alpha) > 0$. Using a well-known hypergeometric function identity[4], we deduce that

$$I(\alpha) = \frac{\Gamma(\mu)}{\Gamma(1-\frac{\mu}{2})} \left(\frac{\Gamma(\frac{1}{2}) F(\frac{\mu}{2}, \frac{\mu}{2}; \frac{1}{2}; -\alpha^2)}{\Gamma(\frac{\mu+1}{2})} + \frac{\alpha\Gamma(-\frac{1}{2}) F(\frac{\mu+1}{2}, \frac{\mu+1}{2}; \frac{3}{2}; -\alpha^2)}{\Gamma(\frac{\mu}{2})} \right) \ .$$

Since $J_0(t) = O(t^{-1/2})$ as $t \to \infty$, our assumption that $\mu < \frac{1}{2}$ makes it is easy to justify changing the order of integration and taking the limit as $\alpha \to 0+$. Thus

$$\int_0^{\infty} t^{\mu-1} J_0(t)dt = \lim_{\alpha \to 0+} I(\alpha) = \frac{\Gamma(\mu)\Gamma(\frac{1}{2})}{\Gamma(1-\frac{\mu}{2})\Gamma(\frac{\mu+1}{2})} \ .$$

The Lemma now follows from the duplication formula for the Gamma function. □

Lemma 2

$$\int_0^{\infty} \left(\frac{e^{-t/2} - J_0(t)}{t} \right) dt = 0 \ . \qquad (8)$$

Proof. A slightly more general result is given in equation 6.622.1 of Gradshteyn and Ryzhik [7], and attributed to Nielsen [10]. We show that (8) follows easily from Lemma 1.

Let μ be a small positive parameter. From Lemma 1,

$$\int_0^{\infty} (e^{-t/2} - J_0(t)) t^{\mu-1} dt = 2^{\mu}\Gamma(\mu) - \frac{2^{\mu-1}\Gamma(\mu/2)}{\Gamma(1-\mu/2)} \ . \qquad (9)$$

Since $\Gamma(1) = 1$, $\Gamma'(1) = -\gamma$, and $\mu\Gamma(\mu) = \Gamma(1+\mu)$, the right side of (9) is

$$\frac{2^{\mu}}{\mu} \left((1-\gamma\mu) - \left(\frac{1-\gamma\mu/2}{1+\gamma\mu/2} \right) + O(\mu^2) \right) = O(\mu) \ .$$

The result follows on letting $\mu \to 0+$. □

[2]More precisely, the generalisation with $J_0(t)$ replaced by $J_\nu(t)$: see Sec. 13.24 of Watson [15].

[3]A generalisation of (7) is given in Sec. 13.2 of Watson [15], and is attributed to Hankel [8] and Gegenbauer [6].

[4]See equation (10.16), Chapter 5 of Olver [11].

3 Ramanujan's Corollary for $n = 2$

Our main result is the following Theorem, which proves that (4) is valid for $n = 2$.

Theorem 1 *Let*

$$e(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k} \left(\frac{x^k}{k!} \right)^2 - \ln x - \gamma \ .$$

Then, for real positive x,

$$e(x) = \int_{2x}^{\infty} \frac{J_0(t)}{t} dt \ .$$

Proof. Proceeding as on page 99 of [2], using the fact[5] that

$$\gamma = \int_0^1 \frac{1-e^{-t}}{t} dt - \int_1^{\infty} \frac{e^{-t}}{t} dt \ ,$$

we have

$$\begin{aligned} e(x) &= \int_0^x \frac{1-J_0(2t)}{t} dt - \int_1^x \frac{dt}{t} - \int_0^1 \frac{1-e^{-t}}{t} dt + \int_1^{\infty} \frac{e^{-t}}{t} dt \\ &= \int_0^x \frac{e^{-t}-J_0(2t)}{t} dt + \int_x^{\infty} \frac{e^{-t}}{t} dt \ . \end{aligned}$$

Now, from Lemma 2 with a change of variable,

$$\int_0^x \frac{e^{-t}-J_0(2t)}{t} dt = \int_x^{\infty} \frac{J_0(2t)-e^{-t}}{t} dt \ ,$$

so

$$e(x) = \int_x^{\infty} \frac{J_0(2t)-e^{-t}}{t} dt + \int_x^{\infty} \frac{e^{-t}}{t} dt = \int_x^{\infty} \frac{J_0(2t)}{t} dt \ ,$$

and Theorem 1 follows by a change of variable. □

Corollary 1 *Let $e(x)$ be as in Theorem 1. Then, for large positive x, $e(x)$ has an asymptotic expansion whose leading terms are given by*

$$e(x) = \frac{1}{2\pi^{1/2}x^{3/2}} \left(\cos\left(2x + \frac{\pi}{4}\right) + \frac{13\sin\left(2x+\frac{\pi}{4}\right)}{16x} + O(x^{-2}) \right) \ .$$

Proof. Using integration by parts and the fact that $xJ_0''(x) + J_0'(x) + xJ_0(x) = 0$ (a special case of Bessel's differential equation), it is easy to deduce from Theorem 1 that

$$e(x) = \frac{J_0'(2x)}{2x} + \frac{J_0(2x)}{2x^2} - 4\int_{2x}^{\infty} \frac{J_0(t)}{t^3} dt \ .$$

Continuing in the same way, we obtain an asymptotic expansion

$$e(x) \ \sim \ \frac{J_0'(2x)}{2x} \sum_{k=0}^{\infty} (-1)^k \frac{k!k!}{x^{2k}} \ + \ \frac{J_0(2x)}{2x^2} \sum_{k=0}^{\infty} (-1)^k \frac{k!(k+1)!}{x^{2k}} \ .$$

The Corollary follows from Hankel's asymptotic expansions for $J_0(z)$ and $J_0'(z) = -J_1(z)$. □

[5] See page 103 of [2].

4 A Different Generalisation

An obvious generalisation of (5) is

$$\sum_{k=0}^{\infty} H_k \left(\frac{x^k}{k!}\right)^n \Bigg/ \sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right)^n \sim \ln x + \gamma \tag{10}$$

as $x \to \infty$.

It is easy to show that (10) is valid for all positive integer n. An essential difference between (4) and (10) is that there is a large amount of cancellation between terms on the left side of (4), but there is no cancellation in the numerator and denominator on the left side of (10). The function $(x^k/k!)^n$ acts as a smoothing kernel with a peak at $k \simeq x - 1/2$. Since

$$H_k = \ln k + \gamma + O(1/k) \ ,$$

the result (10) is not surprising. What may be surprising is the speed of convergence. Brent and McMillan [5] show that[6]

$$\sum_{k=0}^{\infty} H_k \left(\frac{x^k}{k!}\right)^n \Bigg/ \sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right)^n = \ln x + \gamma + O(e^{-c_n x}) \tag{11}$$

as $x \to \infty$, where

$$c_n = \begin{cases} 1, & \text{if } n = 1; \\ 2n\sin^2(\pi/n), & \text{if } n \geq 2. \end{cases}$$

In the case $n = 2$, (11) has error $O(e^{-4x})$. Brent and McMillan used this case with $x \simeq 17,400$ to compute γ to more than 30,000 decimal places. From Corollary 1, the same value of x in (4) would give less than 8-decimal place accuracy[7].

The case $n = 3$ of (11) is interesting because $\max c_n = c_3 = 4.5$. However, no one seems to have used $n > 2$ in a serious computation of γ.

Although Ramanujan [3, 13] gave many rapidly-convergent series and other good approximations for π, he does not seem to have given series which are particularly useful for approximating γ, except for (3) and (5) above. In his paper [12] on series for γ, he gives several interesting series, of which the simplest[8] is

$$\gamma = 1 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)(2k+1)} \ ,$$

but these series all involve the Riemann zeta function or related functions, so they are not very convenient for computational purposes.

Our analysis has assumed that n in (4) and (11) is a positive integer. It would be interesting to consider the behaviour of the functions occurring in these equations for positive but non-integral values of n, especially in the range $1 < n < 2$.

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[6]We note an error on page 310 of [5]: in the definition of $V_p(z)$, "$z/k!$" should be "$z^k/k!$".

[7]More than 15,000 decimal places would have to be used in the computation to compensate for cancellation of terms $\Omega_\pm(e^{2x}/x^2)$ in (4) !

[8]Due to Glaisher: see [12].

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