The Mean Square Error in the Prime Number Theorem does not have a Limit

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7 Oct 2020

Outline

A version of the prime number theorem is $\psi(x) \sim x$ as $x \to \infty$.
RH implies that $\psi(x) - x \ll x^{1/2} \log^2 x$.
Let

$$I(X) := \int_X^{2X} (\psi(x) - x)^2 \, dx.$$

RH implies that $I(X) \asymp X^2$, i.e. $X^2 \ll I(X) \ll X^2$.

In this talk I will outline a proof that $\lim_{X\to\infty} I(X)/X^2$ does not exist.

I will briefly mention upper and lower bounds on $I(X)/X^2$ for large X. There won't be time to discuss proofs of these results, but they are available in the arXiv preprint
https://arxiv.org/abs/2008.06140.

This is joint work with Dave Platt and Tim Trudgian.

Note on use of the Riemann Hypothesis

In order to simplify the presentation, we shall assume the Riemann Hypothesis (RH) in this talk.

Most of the results (including the result on non-existence of a limit) are independent of RH, but the proofs are different (and usually trivial) if RH is false.

For example, $I(X)/X^2$ is unbounded if RH is false, so in this case we can not hope to prove that $I(X)/X^2$ has a finite limit.

For details of which results depend on RH, see our arXiv preprint.

Classical results

The (second) Chebyshev function is $\psi(x) = \sum_{n \leqslant x} \Lambda(n)$, where $\Lambda(n)$ is the von Mangoldt function.

One form of the prime number theorem is $\psi(x) \sim x$ as $x \to \infty$.

We consider the error term $\psi(x) - x$, or sometimes $(\psi(x) - x)/x^{1/2}$.

Some classical results are:

Helge von Koch (1901) proved $\psi(x) - x \ll x^{1/2} \log^2 x$.

Littlewood (1914) proved that $(\psi(x) - x)/x^{1/2}$ is unbounded, more precisely

$$\psi(x) - x = \Omega_{\pm}(x^{1/2} \log \log \log x).$$

The explicit formula for $\psi(x)$

Our proofs depend on the "explicit" formula for $\psi(x)$:

$$\psi(x) - x = -\sum_{|\Im(\rho)| \leqslant T} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 x}{T}\right)$$

for $T \geqslant T_0$, $x \geqslant X_0$, $x \geqslant T$. Here ρ is a nontrivial zero of $\zeta(s)$.

See, for example, Montgomery and Vaughan, Theorem 12.5.

The mean square error

From computational results, as well as the explicit formula for $\psi(x)$, it is plausible that $\psi(x) - x$ is "usually" of order $x^{1/2}$.

This suggests considering the mean square error

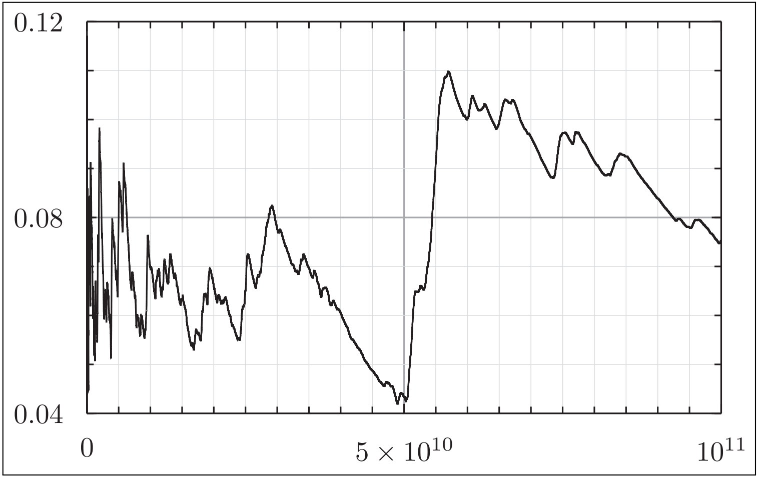
$$I(X) := \int_X^{2X} (\psi(x) - x)^2 \, dx,$$

which we expect to be of order

$$\int_X^{2X} x \, dx \asymp X^2.$$

The next slide shows the behaviour of $I(X)/X^2$ for $X \leqslant 10^{11}$.

$I(X)/X^2$ for $X \leqslant 10^{11}$, sampled every 10^5



Upper and lower bounds on $I(X)/X^2$ (summary)

Cramér (1922) showed that $I(X)/X^2$ is bounded.

(New) explicit bounds, for all sufficiently large X, are

$$\frac{1}{5374} \leqslant \frac{I(X)}{X^2} \leqslant 0.8603.$$

The lower bound improves on Stechkin and Popov (1996), who obtained a constant $1/40000$ (small but positive).

The upper bound improves on Pintz (1982), who stated that $I(X)/X^2 \leqslant 1$ for all sufficiently large X. As far as we know, no proof of this upper bound has ever appeared (until now).

It would require another talk to give details of these results, but they are available in our arXiv preprint.

The non-existence result – outline of proof

We outline a proof that $\lim_{X\to\infty} I(X)/X^2$ does not exist.

It is easier to work with

$$J(X) := \int_0^X (\psi(x) - x)^2 \, dx.$$

and deduce results for $I(X) = J(2X) - J(X)$.

It is not hard to show that $\lim_{X\to\infty} I(X)/X^2$ exists if and only if $\lim_{X\to\infty} J(X)/X^2$ exists.

The nonexistence of $\lim_{X\to\infty} J(X)/X^2$ follows from

$$\liminf_{X\to\infty} 2J(X)/X^2 \leqslant c_1$$

and

$$\limsup_{X\to\infty} 2J(X)/X^2 \geqslant c_2,$$

where c_1 and c_2 are certain constants with $c_1 < c_2$.

The constants c_1 and c_2

Let $\rho = \frac{1}{2} + i\gamma$ be a generic nontrivial zero of $\zeta(s)$, with multiplicity m_ρ. Then

$$c_1 := \sum_{\text{distinct } \rho} \frac{m_\rho^2}{|\rho|^2} < 0.047$$

(see Montgomery and Vaughan, §13.1), and

$$c_2 := \sum_{\rho_1, \rho_2} \frac{2}{\rho_1 \overline{\rho_2}(1 + \rho_1 + \overline{\rho_2})} > 0.104,$$

with the usual convention for multiple zeros (if they exist).

Both sums are absolutely convergent, and it is easy to obtain the upper bound on c_1. The lower bound on c_2 is more difficult.

Observe that c_2 is real, and the “diagonal” terms in c_2 (i.e. those with $\rho_1 = \rho_2$) sum to c_1.

The liminf result (sketch)

Let

$$G(X) := \int_1^X \left(\frac{\psi(x) - x}{x^{1/2}}\right)^2 \frac{dx}{x}\,.$$

From Montgomery and Vaughan, Thm. 13.6 and Ex. 13.1.1.3,

$$G(X) \sim c_1 \log X \text{ as } X \to \infty.$$

Now $G(X)/\log X$ can be regarded as a logarithmically weighted mean square of $(\psi(x) - x)/x^{1/2}$, while $2J(X)/X^2$ is a linearly weighted mean square of the same function.

Thus, it is plausible, and not hard to prove (using integration by parts), that

$$\liminf_{X\to\infty} 2J(X)/X^2 \leqslant c_1 \leqslant \limsup_{X\to\infty} 2J(X)/X^2.$$

The limsup result (sketch)

Fix $\varepsilon > 0$. From the explicit formula, with $X \geqslant T$ sufficiently large (depending on ε),

$$\int_T^X (\psi(x) - x)^2\,dx = \int_T^X \sum_{|\gamma_1|,|\gamma_2| \leqslant T} \frac{x^{1+i(\gamma_1-\gamma_2)}}{\rho_1\overline{\rho_2}}\,dx + E_1(X,T),$$

where $E_1(X,T)$ is a manageable error term. Thus

$$\frac{J(X)}{X^2} = \sum_{|\gamma_1|,|\gamma_2| \leqslant T} \frac{X^{i(\gamma_1-\gamma_2)}}{\rho_1\overline{\rho_2}(2+i(\gamma_1-\gamma_2))} + E_2(X,T),$$

where $E_2(X,T)$ is also manageable (details omitted).
Now, using Dirichlet's theorem, there exist arbitrarily large X such that all the $X^{i\gamma}$ for $|\gamma| \leqslant T$ are close to unity, and

$$\frac{2J(X)}{X^2} \geqslant c_2 - \varepsilon.$$

The final step

We have to prove that $c_2 > c_1$. Recall that

$$c_2 = \sum_{\rho_1,\rho_2} \frac{2}{\rho_1\overline{\rho_2}(1+\rho_1+\overline{\rho_2})}\,.$$

Since c_2 is real, we only need to consider the real parts of terms in the sum. Define a truncated sum of real parts

$$S(Y) := \sum_{|\gamma_1|,|\gamma_2|\leqslant Y} \Re\left(\frac{2}{\rho_1\overline{\rho_2}(1+\rho_1+\overline{\rho_2})}\right).$$

Although the terms in this sum can have either sign, the negative terms are dominated by the (positive) diagonal terms, so we can show that $S(Y)$ is monotonic non-decreasing. Thus, for all $Y \geqslant 0$, the finite sum $S(Y)$ gives a lower bound on c_2.

To prove that $c_2 > c_1$, it is sufficient to take $Y = 70$, i.e. to consider the contribution of the $(2\times)17$ smallest nontrivial zeros of $\zeta(s)$.

A sharper bound on c_2

If we take $Y = 74920.83$, i.e. we sum over the smallest $(2\times)10^5$ nontrivial zeros of $\zeta(s)$, we obtain

$$c_2 \geqslant S(Y) > 0.104.$$

Final remarks

Make a change of variables $x = e^u$, $X = e^U$ and define $f(u) := (\psi(x) - x)/x^{1/2}$.
It is known [Montgomery and Vaughan, Thm. 13.6] that

$$\lim_{U\to\infty} \frac{1}{U} \int_0^U f(u)^2\, du = c_1$$

(and the limit does exist). Now

$$\int_U^{U+\log 2} f(u)^2\, du = \int_X^{2X} \frac{(\psi(x) - x)^2}{x} \frac{dx}{x} \asymp \frac{I(X)}{X^2}\,.$$

Thus, an explanation of why the limit of $I(X)/X^2$ does not exist is that, on a log scale, we are averaging over too short an interval.

References

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