Factoring Integers – an Introduction

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1 August 2012

Motivation – Unique Prime Factorisation

Every natural number (integer) $n > 1$ is a product of prime powers

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

and this representation is unique except for the ordering, i.e. it is unique if we assume that $p_1 < p_2 < \cdots p_k$ and the exponents α_i are positive.

$$1001 = 7 \cdot 11 \cdot 13$$

$$500 = 2^2 \cdot 5^3$$

$$2^{32} + 1 = 641 \cdot 6700417$$

Given a large integer n, how can we actually find the prime factors of n?

Historical Example

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Nowadays, the Magma package running on my laptop takes about one second to find Cole's factors of M_{67}. The smallest Mersenne number that is not completely factored is now M_{929}.

Algebraic Factors

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Put $x = 2^k$. This gives

$$2^{3k} + 1 = (2^k + 1)(2^{2k} - 2^k + 1).$$

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Similarly, $10^{67} - 1$ has a factor $10 - 1 = 9$.

This is obvious if you write $10^{67} - 1 = \underbrace{99 \cdots 99}_{67 \text{ digits}}$.

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Put $x = 2^k$:

$$2^{4k+2} + 1 = (2^{2k+1} + 2^{k+1} + 1)(2^{2k+1} - 2^{k+1} + 1).$$

For example, $2^{118} + 1 = (2^{59} + 2^{30} + 1)(2^{59} - 2^{30} + 1)$.

Algebraic and Aurifeuillian factors

Aurifeuillian factors are usually different from algebraic factors. For example, $15^{15}+1$ has algebraic factors $15+1$, 15^3+1, 15^5+1, and Aurifeuillian factors 19231, 142111. Combining this information, we find:

$$15^{15}+1 = 2^4 \cdot 31 \cdot 211 \cdot 1531 \cdot 19231 \cdot 142111.$$

Unfortunately, algebraic and Aurifeuillian factors only apply in very special cases. They don't give a general factoring method.

Modular Arithmetic

Recall that

$$a = b \bmod N$$

means that N divides $(a - b)$. We sometimes write this

$$N \mid (a - b).$$

Exercise. If $p \mid N$, $a = b \bmod N$, and $b = c \bmod p$, then $a = c \bmod p$.
e.g. $17 = 32 \bmod 15$, $32 = 2 \bmod 3$, so $17 = 2 \bmod 3$.

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It's not generally true unless $p \mid N$.
e.g. $17 = 32 \bmod 15$, $32 = 0 \bmod 2$, but $17 \neq 0 \bmod 2$.

Fast Algorithms

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Generally, "fast" = "polynomial time", but the degree of the polynomial and the size of the constant "a" are important in practice.

Fast Modular Exponentiation

The example “Exponentiating Mod Wise” on page 35 of John Hutchinson’s notes illustrates a “fast” algorithm for computing

$$a^b \bmod N.$$

This is feasible even for numbers with hundreds of digits, because the time is (at most) a cubic polynomial in the input size.

Greatest Common Divisor (GCD)

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Application. If $p \mid M$ and $p \mid N$, then $p \mid \gcd(M, N)$ and it could happen that $p = \gcd(M, N)$.

This can be useful when trying to factor N.

Testing Primality

Fermat's *little* Theorem "FLT" (Thm. 19, pg. 44):
If p is prime then

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Sometimes the Theorem is stated as

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but then we need the extra condition $a \neq 0 \bmod p$. Suppose we are given an integer $N > 2$. Before trying to factor N, choose some integer a, where $1 < a < N - 1$, and compute

$$b = a^N \bmod N.$$

If $b \neq a \bmod N$, then N must be composite (otherwise get a contradiction to FLT), so it makes sense to try to factor N.

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If $b \neq a \bmod N$, then N must be composite (otherwise get a contradiction to FLT), so it makes sense to try to factor N. If $b = a \bmod N$ then probably N is prime, so we could be wasting our time trying to factor N. Better to check if N really is prime. This can be done "fast" – see "talks" on my web page.

“Divide and Conquer” Factoring Strategy

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This is an example of the very useful “divide and conquer” strategy – if you can’t immediately solve a problem, try to reduce it to one or more smaller problems.

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From $q = 12^2 - 1$ we get $q = 11 \cdot 13$. Thus $N = 7 \cdot 11 \cdot 13$.

Since it is easy to divide out powers of 2, I'll assume from now on that N is *odd*.

Trial division

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Drawback. The time required is proportional to p. Thus, trial division can be very slow if p is large. Since p is the smallest prime factor of N,

$$p \leq \sqrt{N}.$$

Improvements. We only need to divide by *primes* $2, 3, 5, 7, 11, \ldots$ (but we need a list of these or some way of generating them).

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By the *Prime Number Theorem* this saves a factor of order $\log N$, which is not very significant.

Fermat's Method

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Suppose N is odd and $N = uv$, where $0 < u \leq v$.
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Fermat tries to find integers x and y satisfying the equation $x^2 - N = y^2$, with y small. Thus, he starts with $x = \lceil \sqrt{N} \rceil$, the smallest integer whose square is $\geq N$.

Fermat by hand (or on a calculator)

Suppose $N = 9401$. We find $96^2 < N < 97^2 = 9409$, so start with $x = 97$ and increase x by 1 until (hopefully) we find a value such that $x^2 - N$ is a perfect square.

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It's easy to increase x by 1 and update $x^2 - N$, since $(x+1)^2 - x^2 = 2x + 1$.

x	$2x+1$	$x^2 - N$
97	195	8
98	197	203
99	199	$400 = 20^2$

Thus $N = 99^2 - 20^2 = (99 - 20)(99 + 20) = 79 \cdot 119$.

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It would be much more work to find this by trial division!

Problems with Fermat's Method

- The factors u and v of N that are found by Fermat's method might not be prime, so they have to be factored (but this should not be so hard, since they are smaller than N).
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Fermat takes about

$$x - \sqrt{N} = \frac{u+v}{2} - \sqrt{N} = \frac{u}{2} + \left(\frac{N}{2u} - \sqrt{N}\right) \quad \text{steps},$$

and

$$\frac{N}{2u} > \sqrt{N} \ \text{ if } \ u < \frac{\sqrt{N}}{2}\,.$$

Improving Fermat – the Quadratic Sieve

Fermat's method tries to find x, y such that $x^2 - y^2 = N$, but it would be enough to find x, y such that

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i.e.

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The quadratic sieve (QS) method tries to factor $x_i^2 \bmod N$ for several different x_i, and combine the results to get an equation

$$x^2 = y^2 \bmod N.$$

With luck (at least 50% of the time) this gives nontrivial factors of N.

Small Example of the Quadratic Sieve

Consider $N = 1649$, so $40 < \sqrt{N} < 41$.

$$\begin{aligned} 41^2 &= 1681 = 32 = 2^5 \bmod N \qquad (*) \\ 42^2 &= 1764 = 115 = 5 \cdot 23 \bmod N \\ 43^2 &= 1849 = 200 = 2^3 \cdot 5^2 \bmod N \quad (*) \end{aligned}$$

Multiply the two "relations" marked $(*)$, giving

$$(41 \cdot 43)^2 = 2^8 \cdot 5^2 = (2^4 \cdot 5)^2 \bmod N,$$

i.e.

$$x^2 = y^2 \bmod N,$$

where $x = 41 \cdot 43 = 114 \bmod N$, and $y = 2^4 \cdot 5 = 80$.
We were lucky, because $x \neq \pm y \bmod N$.

$$\gcd(x - y, N) = \gcd(114 - 80, N) = 17,$$

and it's easy to check that $17 \mid N$, in fact $N = 17 \cdot 97$. This would not have been so easy to find using Fermat's method.

How lucky were we?

Suppose $N = p \cdot q$ where p, q are distinct primes, and

$$x^2 = y^2 \bmod N.$$

Thus $$N \mid (x - y)(x + y),$$

so $$p \mid (x - y) \text{ or } p \mid (x + y)$$

and $$q \mid (x - y) \text{ or } q \mid (x + y).$$

There are 4 cases. If $p \mid (x - y)$ and $q \mid (x - y)$ then $N|(x - y)$, so $x = y \bmod N$ and we don't find a factor of N.
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However, in the other two cases we do find a factor. For example, if $p \mid (x - y)$ and $q \mid (x + y)$, we get p from $\gcd(N, x - y)$ and q from $\gcd(N, x + y)$.

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If the 4 cases are equally likely, we have a 50% chance of success.

A Larger Example

Let's try $N = 1098413$. Compute $x^2 - N$ for $x > \sqrt{N}$ and try to factor $x^2 - N$ using only primes in the *factor base* $S = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$:

$$\begin{aligned}
1051^2 &= 2^2 \cdot 7 \cdot 13 \cdot 17 \bmod N \\
1063^2 &= 2^2 \cdot 7^3 \cdot 23 \bmod N \quad (*) \\
1077^2 &= 2^2 \cdot 7 \cdot 13^3 \bmod N \quad (*) \\
1119^2 &= 2^2 \cdot 7 \cdot 17^2 \cdot 19 \bmod N \\
1142^2 &= 7^2 \cdot 13 \cdot 17 \cdot 19 \bmod N \\
1237^2 &= 2^2 \cdot 13 \cdot 19^2 \cdot 23 \bmod \quad (*)
\end{aligned}$$

Multiply the relations $(*)$ to get:

$$(1063 \cdot 1077 \cdot 1237)^2 = (2^3 \cdot 7^2 \cdot 13^2 \cdot 19 \cdot 23)^2 \bmod N,$$

i.e. $$326330^2 = 391638^2 \bmod N,$$

and compute $\gcd(N, 391638 - 326330) = 563$,
giving $$N = 563 \cdot 1951.$$

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In other words, find a *linear dependency* between the rows of the matrix, working over the field $F_2 = \{0, 1\}$ (where the operations are addition and multiplication mod 2).

Example

For example, with $N = 1098413$, the matrix we get is:

$$\begin{array}{c} \begin{array}{ccccccccc} 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 & 23 \end{array} \\ \left[\begin{array}{ccccccccc} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \\ (*) \\ (*) \\ \\ \\ (*) \end{array}$$

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Adding the rows marked $(*)$, using arithmetic mod 2, we get

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which means that these rows are linearly dependent over F_2.

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Finding a linear dependency takes about the same work as solving a system of linear equations, and is feasible even if the matrix is very large.

Related Factoring Methods

- Instead of just considering $x^2 - N$ we can consider several quadratic polynomials $a_i x^2 + b_i x + c_i$ where $b_i^2 - 4a_i c_i = N$. This gives the *Multiple Polynomial Quadratic Sieve* (MPQS), which is faster than the quadratic sieve if the polynomials are chosen correctly.
- Instead of working over the integers mod N, we can work over *number fields*. This gives the *Number Field Sieve* (NFS) which is complicated but the best method known for factoring large N.
- QS, MPQS and NFS take a time which depends mainly on the size of N and is more or less independent of the size of the factors of N (unlike trial division and other methods that we'll consider later).

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$$F_5 = 641 \cdot 6700417.$$

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$$F_9 = 2424833 \cdot c_{148},$$

where c_{148} is a composite number with 148 decimal digits.

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where c_{148} is a composite number with 148 decimal digits.

Using the Number Field Sieve, the factors of c_{148} were found:

$$c_{148} = p_{49} \cdot p_{99},$$

where

$p_{49} = 7455602825647884208337395736200454918783366342657$

and p_{99} is a prime with 99 decimal digits – you can find it by division!

Current Record

The largest number factored so far by NFS is RSA768, which is a number with 768 bits (232 decimal digits). It turned out to be a product of two primes, each having 116 decimal digits (though not close enough to be found by Fermat's method).

```
33478071698956898786044169848212690817704794983713768568912431388982883793878002287614711652531743087737814467999489
```

and

```
36746043666799590428244633799627952632279158164343087642676032283815739666511279233373417143396810270092798736308917
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43138898288379387800228761471165253174308773781446799948 9
```

and

```
36746043666799590428244633799627952632279158164343087642676
03228381573966651127923337341714339681027009279873630891 7
```

It's not yet feasible to factor 1024-bit ($\approx$ 300 digit) numbers, but it might be in a few years' time.

Another Idea – the Pollard "p-1" Method

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If we don't know p but can *guess* a suitable E, we can compute

$$\gcd(2^E - 1, N),$$

and (with some luck) this will give us p.

Guessing E

If all the prime power factors of $p-1$ are $\leq B$, take

$$E = \prod_{p_i^{\alpha_i} \leq B} p_i^{\alpha_i}.$$

Because E might be large (roughly e^B), we don't usually compute E explicitly; instead we compute $2^E \bmod N$ using a loop like:

$$a \leftarrow 2; \quad \text{for } i = 1, 2, \ldots \text{ do } a \leftarrow a^{p_i^{\alpha_i}} \bmod N.$$

Guessing E continued

In practice we don't know the factors of $p-1$ (because we don't know p), but we do know that the time for the computation is proportional to B, so we just take a fairly large value, say $B \approx 1000000$, depending on how much computer time we are willing to use.

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Otherwise, we have to increase B and try again, or try another method (e.g. MPQS).

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It turns out that

$$p-1 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 23 \cdot 31 \cdot 163 \cdot 401 \cdot 617 \cdot 4271 \cdot 13681 \cdot 22877 \cdot 43397 \cdot 203459 \cdot 1396027 \cdot 6995393 \cdot 13456591 \cdot 2110402817,$$

and 2110402817 is small enough (if you have a fast computer).

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and 2110402817 is small enough (if you have a fast computer).

However, this situation is unusual. A 66-digit number is extremely unlikely to have all its prime factors so small. (The chance is roughly 1 in a million.)

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If $p - 1 = 2q$ where q is a prime, then the Pollard "$p - 1$" method is very slow.

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(p, q) is called a "Sophie Germain" pair after Marie-Sophie Germain (1776–1831). There seem to be infinitely many such pairs, e.g. $(5, 2), (7, 3), (11, 5), (23, 11)$, but no one has proved this.

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The problem is similar to the problem of *twin primes*, that is pairs $(p, p + 2)$ where p and $p + 2$ are both prime.

The Elliptic Curve Method (ECM)

The set $G = \{1, 2, \ldots, p-1\}$ forms a *group* of order $p-1$ with the operation "multiplication mod p" if p is a prime.

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ECM is the best method for finding "small" factors p of large numbers N, say factors $p < N^{1/3}$.

The running time of ECM depends mainly on the size of p, and only weakly on the size of N.

ECM Examples

I factored the 10-th and 11-th Fermat numbers using ECM.

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For example,

$$F_{10} = 2^{1024} + 1 = p_8 \cdot p_{10} \cdot p_{40} \cdot p_{252},$$

$$\begin{aligned} p_8 &= 45592577 \\ p_{10} &= 6487031809 \\ p_{40} &= 4659775785220018543264560743076778192897 \\ p_{252} &= 130439874405 \cdots 127014424577 \end{aligned}$$

p_8 and p_{10} are "easy".
p_{40} was found by ECM, and would have been very hard to find by any other method.
p_{252} can be found by division once the other factors are known (of course, we have to check that it is prime).

ECM Examples continued

$$F_{11} = 2^{2048} + 1 = p_6 \cdot p'_6 \cdot p_{21} \cdot p_{22} \cdot p_{564},$$

$$\begin{aligned} p_6 &= 319489 \\ p'_6 &= 974849 \\ p_{21} &= 167988556341760475137 \\ p_{22} &= 35608419064458339205 13 \\ p_{564} &= 1734624471 \cdots 6598834177 \end{aligned}$$

The 21-digit and 22-digit factors were found by ECM; then it is easy to find the 564-digit factor p_{564} (though proving that it is prime is not so easy).

ECM Record

The largest factor found by ECM is a 73-digit factor

$$p_{73} = 180842235317734956454651203551253000127948125985424886045434898945102687$$

of

$$2^{1181} - 1$$

(found by Bos, Kleinjung, Lenstra and Montgomery on 7 March 2010, using a cluster of PlayStation 3 game consoles).

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The largest prime factor of the group order is 10801302048203.

Summary

We've looked at several methods for factoring integers:

- Trial division (simple but slow).
- Fermat's method (also simple, but slow in most cases).
- Quadratic sieve (QS) and MPQS.
- Number field sieve (NFS) – the best general-purpose method.
- Pollard $p-1$ (fast if you are lucky).
- Elliptic curve method (ECM) – the best method for finding "small" factors.

A good strategy for factoring is:

- Check if the number N to be factored is a prime power!
- If not, try to find factor(s) by ECM and divide them out.
- If what remains is not a prime power, try MPQS or NFS.

References

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2. Richard Crandall and Carl Pomerance, *Prime Numbers: A Computational Perspective*, second edition, Springer-Verlag, New York, 2005.

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