

Discrete analogues of Macdonald-Mehta integrals

Richard P. Brent

MSI, ANU and
CARMA, University of Newcastle

7 December 2015

Joint work with
Christian Krattenthaler (Vienna)
and Ole Warnaar (Univ. of Qld.)

Abstract

We consider identities satisfied by discrete analogues of Mehta-like integrals. The integrals are related to Selberg's integral and the Macdonald conjectures. Our discrete analogues have the form

$$S_{\alpha,\beta,\delta}(r, n) := \sum_{k_1, \dots, k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} |k_i^\alpha - k_j^\alpha|^\beta \prod_{j=1}^r |k_j|^\delta \binom{2n}{n+k_j}$$

where $\alpha, \beta, \delta, r, n$ are non-negative integers subject to certain restrictions.

In the ten cases that we consider, it is possible to express $S_{\alpha,\beta,\delta}(r, n)$ as a product of Gamma functions and simple functions such as powers of two. For example, if $1 \leq r \leq n$, then

$$S_{2,2,3}(r, n) = \prod_{j=1}^r \frac{(2n)! j!^2}{(n-j)!^2}.$$

Motivation

For an application of the probabilistic method, Brent and Osborn (2013) needed an explicit formula for the sum

$$S(2, n) := \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} |k^2 - \ell^2| \binom{2n}{n+k} \binom{2n}{n+\ell}.$$

They showed that

$$S(2, n) = 2n^2 \binom{2n}{n}^2.$$

Note: we can assume that $k, \ell \in [-n, n]$ as otherwise the product of binomial coefficients vanishes. Thus, there are $O(n^2)$ nonzero terms in the sum.

Motivation cont.

Ohtsuka conjectured, and Prodinger proved, an analogous triple-sum identity:

$$\begin{aligned} S(3, n) &:= \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |\Delta(k^2, \ell^2, m^2)| \binom{2n}{n+k} \binom{2n}{n+\ell} \binom{2n}{n+m} \\ &= 3n^3(n-1) \binom{2n}{n}^2 2^{2n-1}, \end{aligned}$$

where $\Delta(x, y, z) := (y-x)(z-y)(z-x)$ and $n \geq 2$.

Ole Warnaar suggested generalising this and similar results to r -fold sums that may be regarded as discrete analogues of certain well-known integrals. At a certain point we decided to consult **Christian Krattenthaler**. This resulted in a collaboration between the three of us. In the short time available I will summarise some of the results obtained by this collaboration.

Notation and definitions

The **Vandermonde** determinant is

$$\Delta(k_1, \dots, k_r) := \det(k_i^{j-1})_{1 \leq i, j \leq r} = \prod_{1 \leq i < j \leq r} (k_j - k_i).$$

Suppose $\alpha \geq 0$ and $k = (k_1, \dots, k_r) \in \mathbb{R}^r$. A useful notation is

$$\Delta(k^\alpha) := \Delta(k_1^\alpha, \dots, k_r^\alpha) = \prod_{1 \leq i < j \leq r} (k_j^\alpha - k_i^\alpha).$$

We consider centered binomial sums involving the absolute value of a generalised Vandermonde, specifically

$$S_{\alpha, \beta, \delta}(r, n) := \sum_{k \in \mathbb{Z}^r} |\Delta(k^\alpha)|^\beta \prod_{j=1}^r |k_j|^\delta \binom{2n}{n + k_j}.$$

Here $\alpha, \beta, \delta, r, n$ are non-negative integer parameters.

Our aim is to express these **sums** as **products**.

Examples

The two-fold sum $S(2, n)$ mentioned previously is

$$S_{2,1,0}(2, n) := \sum_{k_1} \sum_{k_2} |\Delta(k^2)| \prod_{j=1}^2 \binom{2n}{n+k_j}.$$

The three-fold sum $S(3, n)$ mentioned previously is

$$S_{2,1,0}(3, n) := \sum_{k_1} \sum_{k_2} \sum_{k_3} |\Delta(k^2)| \prod_{j=1}^3 \binom{2n}{n+k_j}.$$

We saw that these sums can be expressed as products of factorials and powers of two.

Examples

The r -fold generalisation can also be expressed as a product:

$$\begin{aligned} S_{2,1,0}(r, n) &:= \sum_{k_1, \dots, k_r \in \mathbb{Z}} |\Delta(k^2)| \prod_{j=1}^r \binom{2n}{n + k_j} \\ &= \prod_{j=0}^{r-1} \frac{(2n)!}{(n-j)!} \frac{\Gamma(\frac{j+3}{2})}{\Gamma(\frac{3}{2})} \frac{\Gamma(n-j+\frac{3}{2})}{\Gamma(n-\frac{j}{2}+\frac{3}{2})} \frac{\Gamma(\frac{j+1}{2})}{\Gamma(n-\frac{j-1}{2})}. \end{aligned}$$

We assume that $n \geq r - 1$; otherwise the sum vanishes.

Thus, a sum with $O(n^r)$ nonzero terms has been expressed as a product of $O(r)$ Gamma functions (or reciprocals of Gamma functions). We call such a product a *Gamma product*.

Polynomial powers of constants, e.g. 2^{n-r} , are allowed in such products.

Mehta's integral

Mehta's r -fold integral is

$$F_r(\gamma) := \int_{\mathbb{R}^r} |\Delta(x)|^{2\gamma} d\psi(x),$$

where $\psi(x)$ is the r -dimensional Gaussian measure, i.e.

$$d\psi(x) := \frac{\exp(-\frac{1}{2}\|x\|_2^2)}{(2\pi)^{r/2}} dx_1 \cdots dx_r.$$

Mehta and Dyson evaluated $F_r(\gamma)$ for the cases $\gamma \in \{1/2, 1, 2\}$, and conjectured the general result

$$F_r(\gamma) = \prod_{j=1}^r \frac{\Gamma(1 + j\gamma)}{\Gamma(1 + \gamma)}.$$

This was later proved by Bombieri and Selberg, using Selberg's integral (see the survey by Forrester and Warnaar).

Discrete approximation

The finite sum

$$S_{1,2\gamma,0}(r,n) = \sum_{-n \leq k_j \leq n} |\Delta(k)|^{2\gamma} \prod_{j=1}^r \binom{2n}{n+k_j}$$

is a (scaled) discrete approximation to $F_r(\gamma)$. Using

$$\binom{2n}{n+k} \sim \frac{2^{2n}}{\sqrt{n\pi}} e^{-k^2/n}$$

as $n \rightarrow \infty$ with $k = o(n^{2/3})$, we see that

$$\lim_{n \rightarrow \infty} \frac{S_{1,2\gamma,0}(r,n)}{2^{2rn} (n/2)^{\gamma r(r-1)/2}} = F_r(\gamma).$$

Macdonald-Mehta integrals

More generally, we can define a *Macdonald-Mehta integral*

$$\mathcal{F}_{\alpha,\beta,\delta}(r) := \int_{R^r} |\Delta(x^\alpha)|^\beta \prod_{j=1}^r |x_j|^\delta d\psi(x),$$

where $\alpha, \beta = 2\gamma$, and δ are non-negative real parameters.

Then $S_{\alpha,\beta,\delta}(r, n)$ is a (scaled) discrete approximation to $\mathcal{F}_{\alpha,\beta,\delta}(r)$, in the sense that

$$\mathcal{F}_{\alpha,\beta,\delta}(r) = \lim_{n \rightarrow \infty} \frac{S_{\alpha,\beta,\delta}(r, n)}{2^{2rn} (n/2)^{\alpha\beta r(r-1)/4 + \delta r/2}}.$$

The Macdonald-Mehta integrals arise in Macdonald's (ex-)conjecture related to root systems of finite reflection groups.

When does a Gamma product (probably) exist?

Given a finite sum $f(r, n)$, how can we determine if a Gamma product for the sum is likely to exist?

Observe that, if $f(r, n)$ has a Gamma product, then all the prime factors of $f(r, n)$ are “small”. More precisely, they are $O(n)$ as $n \rightarrow \infty$ with r fixed.

Example – Gamma product exists

If $f(r, n) := S_{2,1,0}(r, n)$, we find experimentally that the prime factors of $f(r, n)$ are bounded by $2n$. Following is the output of a small Magma program checking prime factors for $r = 3, n \leq 20$. The program finds the largest prime factor p of $S_{2,1,0}(3, n)$, and prints p and p/n .

```
alpha 2 beta 1 delta 0 r 3
n 2 max p 3 p/n 1.500
n 3 max p 5 p/n 1.667
n 4 max p 7 p/n 1.750
n 5 max p 7 p/n 1.400
n 6 max p 11 p/n 1.833
n 7 max p 13 p/n 1.857
n 8 max p 13 p/n 1.625
n 9 max p 17 p/n 1.889
n 10 max p 19 p/n 1.900
n 11 max p 19 p/n 1.727
n 12 max p 23 p/n 1.917
n 13 max p 23 p/n 1.769
n 14 max p 23 p/n 1.643
n 15 max p 29 p/n 1.933
n 16 max p 31 p/n 1.938
n 17 max p 31 p/n 1.823
n 18 max p 31 p/n 1.722
n 19 max p 37 p/n 1.947
n 20 max p 37 p/n 1.850
Max p/n 1.947
```

It appears that p is a (weakly) monotonic increasing function of n , and it is reasonable to conjecture that $p \leq 2n$.

Example continued

Other positive values of r give similar results. Thus, it is plausible that a Gamma product exists. In fact, it does:

$$f(r, n) = \prod_{j=0}^{r-1} \frac{(2n)!}{(n-j)!} \frac{\Gamma(\frac{j+3}{2})}{\Gamma(\frac{3}{2})} \frac{\Gamma(n-j+\frac{3}{2})}{\Gamma(n-\frac{j}{2}+\frac{3}{2})} \frac{\Gamma(\frac{j+1}{2})}{\Gamma(n-\frac{j-1}{2})}$$

for $n \geq r - 1$ (otherwise the sum vanishes).

From the product it is easy to see that $p \leq 2n$, confirming what we found experimentally.

Example – Gamma product does not exist

We make a small change and set $\delta = 3$. Here is the output:

```
alpha 2 beta 1 delta 3 r 3
n 3 max p 5 p/n 1.667
n 4 max p 7 p/n 1.750
n 5 max p 11 p/n 2.200
n 6 max p 19 p/n 3.167
n 7 max p 29 p/n 4.143
n 8 max p 41 p/n 5.125
n 9 max p 17 p/n 1.889
n 10 max p 71 p/n 7.100
n 11 max p 89 p/n 8.091
n 12 max p 109 p/n 9.083
n 13 max p 131 p/n 10.08
n 14 max p 31 p/n 2.214
n 15 max p 181 p/n 12.07
n 16 max p 31 p/n 1.938
n 17 max p 239 p/n 14.06
n 18 max p 271 p/n 15.06
n 19 max p 61 p/n 3.210
n 20 max p 37 p/n 1.850
Max p/n 15.06
```

Now p is no longer a monotonic increasing function of n , and p/n can be large. Thus, a Gamma product is unlikely to exist.

Ten cases

By checking prime factors, we determined that Gamma products for $S_{\alpha,\beta,\delta}(r,n)$ were likely to exist in the **ten** cases

$$\alpha, \beta \in \{1, 2\}, \quad 0 \leq \delta \leq 2\alpha + \beta - 3,$$

and unlikely to exist in any other cases where α, β are positive integers.

α	β	δ
1	1	0
1	2	0, 1
2	1	0, 1, 2
2	2	0, 1, 2, 3

The ten cases naturally fall into three families indicated by the **colour-coding**. Why just ten cases? The methods used to prove the ten cases may provide some clues.

The three families

For $\alpha = 2$ we have **seven** cases given by

$$S_{2,2\gamma,\delta}(r,n) = \prod_{j=0}^{r-1} \frac{(2n)!}{\Gamma(n-j+\chi)} \frac{\Gamma(1+j\gamma+\gamma)}{\Gamma(1+\gamma)} \\ \times \prod_{j=0}^{r-1} \frac{\Gamma(n-j-\gamma+\chi+1)}{\Gamma(n-j\gamma-\gamma+\chi+1)} \frac{\Gamma(j\gamma + \frac{\delta+1}{2})}{\Gamma(n-j\gamma - \frac{\delta-3}{2} - \chi)},$$

where $\chi := \chi[\delta = 0] = \max(0, 1 - \delta)$.

For example, we already gave the case $S_{2,1,0}(r,n)$ which corresponds to $\gamma = \frac{1}{2}$, $\delta = 0$.

The three families continued

For $\alpha = 1, \delta = 0$ we have **two** cases given by

$$S_{1,2\gamma,0}(r, n/2) = \prod_{j=1}^r \frac{2^{n-2\gamma(j-1)} n! (n-j+\gamma+1)! \Gamma(1+j\gamma)}{(n-j+1)! (n-(j-2)\gamma)! \Gamma(1+\gamma)}.$$

The **one** remaining case is

$$S_{1,2,1}(r, n) = r! \prod_{j=1}^{\lceil r/2 \rceil} \frac{(2n)! (j-1)!^2}{(n-j)! (n-j+1)!} \prod_{j=1}^{\lfloor r/2 \rfloor} \frac{(2n)! (j-1)! j!}{(n-j)!^2}.$$

This is the only case where the product involves “floor” and/or “ceiling” functions.

Some generalisations and analogues

We briefly mention some generalisations and analogues. They are of independent interest, and some of the generalisations are necessary for the known proofs of the primary identities.

- ▶ K-generalisations.
- ▶ q -analogues.
- ▶ Sums over $\mathbb{Z} + \frac{1}{2}$.

These classes are not mutually exclusive. We can have Kq -generalisations, etc.

The next few slides give some examples.

K-generalisations

If $\beta = 2$ we have identities with an extra integer variable m symmetric with n . Consider the case $(\alpha, \beta, \delta) = (1, 2, 0)$.

For $m, n, r \in \mathbb{Z}$, $m, n \geq r/2 > 0$, we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}^r} \Delta(k)^2 \prod_{j=1}^r \binom{2m}{m+k_j} \binom{2n}{n+k_j} \\ = \prod_{j=1}^r \frac{j! (2m)! (2n)! (2m+2n-2r+j+1)!}{(2m+j-r)! (2n+j-r)! (m+n+j-r)!^2}. \end{aligned}$$

Dividing each side by $\binom{2m}{m}^r$, taking the limit as $m \rightarrow \infty$, and simplifying, we obtain the primary identity

$$S_{1,2,0}(r, n) := \sum_{k \in \mathbb{Z}^r} \Delta(k)^2 \prod_{j=1}^r \binom{2n}{n+k_j} = \prod_{j=1}^r \frac{2^{2n-2j+2} j! (2n)!}{(2n+j-r)!}.$$

q -functions

Assume that $0 < q < 1$ and m, n are integers such that $0 \leq m \leq n$. Then the q -shifted factorial, q -binomial coefficient, q -gamma function and q -factorial are defined by:

$$(a; q)_n := \prod_{k=1}^n (1 - aq^{k-1}), \quad (a; q)_\infty := \prod_{k=1}^{\infty} (1 - aq^{k-1}),$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q^{n-m+1}; q)_m}{(q; q)_m},$$

$$\Gamma_q(x) := (1 - q)^{1-x} \frac{(q; q)_\infty}{(q^x; q)_\infty},$$

$$[n]_q! := \Gamma_q(n + 1) = \prod_{k=1}^n \frac{1 - q^k}{1 - q}.$$

Quick introduction to Schur functions

For $x = (x_1, \dots, x_n)$ and $\lambda = (\lambda_1, \lambda_2, \dots)$ a partition of length at most n , the Schur function $s_\lambda(x)$ may be defined by

$$s_\lambda(x) := \frac{\det_{1 \leq i, j \leq n} (x_i^{\lambda_j + n - j})}{\det_{1 \leq i, j \leq n} (x_i^{n - j})}.$$

The Schur functions form a basis for the ring Λ_n of symmetric functions in n variables x_1, \dots, x_n . They occur in the representation theory of the symmetric group S_n and of the general linear group $\mathrm{GL}_n(\mathbb{C})$.

Specialisations of Schur functions

The *principal specialisation* of the Schur function $s_\lambda(x)$ arises on substituting $x_j = q^{j-1}$, $1 \leq j \leq n$.

For example, if λ is a partition of length at most n and largest part at most r , which we write as $\lambda \subseteq (r^n)$, then

$$s_\lambda(1, q, \dots, q^{n-1}) = q^{\sum_{i \geq 1} (i-1)\lambda_i} \prod_{i=1}^r \begin{bmatrix} n+r-1 \\ \lambda'_i + r - i \end{bmatrix}_q \begin{bmatrix} n+r-1 \\ r-i \end{bmatrix}_q^{-1} \prod_{1 \leq i < j \leq r} \frac{1 - q^{\lambda'_i - \lambda'_j + j - i}}{1 - q^{j-i}}.$$

Here λ' is the conjugate partition of λ (obtained by reflecting the Young diagram of λ in the main diagonal).

Another useful specialisation (the *odd specialisation*) is $x_j = q^{j-1/2}$, i.e. consider $s_\lambda(q^{1/2}, q^{3/2}, \dots, q^{n-1/2})$.

q -analogues of the primary identities

Seven of the ten primary identities have q -analogues, in the sense that there are identities involving q which give the corresponding primary identity in the limit as $q \rightarrow 1$.

We generally have to divide both sides by a suitable power of $(1 - q)$ before taking the limit, in order to ensure that the limit is finite.

The three exceptions are cases $(1, 2, 1)$, $(2, 1, 0)$, $(2, 1, 2)$, where we do not know any q -analogue (they may exist – it is not easy to rule them out).

Example

Let $0 < q < 1$, r a positive integer and n an integer or half-integer such that $n \geq (r-1)/2$. Then we have a q -analogue of $S_{1,1,0}(r, n)$:

$$\begin{aligned} & \sum_{k_1, \dots, k_r = -n}^n \prod_{1 \leq i < j \leq r} |1 - q^{k_i - k_j}| \prod_{j=1}^r q^{(k_j + n - r + j)^2 / 2} \left[\begin{matrix} 2n \\ n + k_j \end{matrix} \right]_q \\ &= (1-q)^{\binom{r}{2}} \frac{r!}{[r]_{q^{1/2}}!} \prod_{j=1}^r (-q^{1/2}; q^{1/2})_j (-q^{j/2+1}; q)_{2n-r} \\ & \quad \times \prod_{j=1}^r \frac{\Gamma_q(1 + \frac{1}{2}j)}{\Gamma_q(\frac{3}{2})} \frac{\Gamma_q(2n+1) \Gamma_q(2n-j+\frac{5}{2})}{\Gamma_q(2n-j+2) \Gamma_q(2n-\frac{1}{2}j+2)}. \end{aligned}$$

Dividing both sides by $(1-q)^{\binom{r}{2}}$ and taking the limit as $q \rightarrow 1$ yields the primary identity

$$S_{1,1,0}(r, n) = 2^{2rn - \binom{r}{2}} \prod_{j=1}^r \frac{\Gamma(1 + \frac{1}{2}j)}{\Gamma(\frac{3}{2})} \frac{\Gamma(2n+1) \Gamma(2n-j+\frac{5}{2})}{\Gamma(2n-j+2) \Gamma(2n-\frac{1}{2}j+2)}.$$

Comments on the proof of the q -analogue

The q -analogue identity on the previous slide follows from some more or less well-known results on [odd specialisations of Schur functions](#), combined with [MacMahon's formula](#) for the generating function of symmetric plane partitions that fit in a box of size $n \times n \times r$ (proved by Andrews and Macdonald). For more details, see the longer version of this talk available on my website.

K_q -generalisations

Some of the K -generalisations have q -analogues, which we call K_q -generalisations. For example, if $\alpha = \beta = 2$, $1 \leq \delta \leq 3$, then

$$\begin{aligned} & \sum_{k_1, \dots, k_r} \left(q^{2\binom{r+1}{3} + (\delta'+1)\binom{r+1}{2}} \prod_{1 \leq i < j \leq r} [k_j - k_i]_q^2 [k_i + k_j]_q^2 \right. \\ & \quad \times \prod_{j=1}^r q^{k_j^2 - (2j + \delta')k_j} \left(\frac{1 + q^{k_j}}{2} \right) \left| [k_j]_q^\delta \right| \left[\begin{matrix} 2n \\ n + k_j \end{matrix} \right]_q \left[\begin{matrix} 2m \\ m + k_j \end{matrix} \right]_q \Bigg) \\ & = \prod_{j=1}^r \frac{[2n]_q!}{[n-j]_q! [n-j-\delta']_q!} \frac{[2m]_q!}{[m-j]_q! [m-j-\delta']_q!} \\ & \quad \times \prod_{j=1}^r j[j-1]_q! [j+\delta']_q! \frac{[m+n-j-r-\delta']_q!}{[m+n-j+1]_q!}, \end{aligned}$$

where $\delta' := (\delta - 3)/2$, and we interpret $[x]_q!$ as $\Gamma_q(x+1)$ if $x \notin \mathbb{Z}$.

Sums over $\mathbb{Z} + \frac{1}{2}$

For most of the primary identities there is a corresponding identity where the parameter n is a half-integer, and we sum over half-integers k_j rather than integers. Thus, the binomial coefficients $\binom{2n}{n+k_j}$ are well-defined.

In the half-integer cases (1, 2, 1) and (2, 2, 3) we get not one Gamma product but a **sum of $O(r)$ Gamma products**.

For example, in case (1, 2, 1) we have: for all positive half-integers n and positive integers $r \leq 2n + 1$,

$$\sum_{k_1, \dots, k_r \in \mathbb{Z} + \frac{1}{2}} \Delta(k)^2 \prod_{i=1}^r |k_i| \binom{2n}{n+k_i} \\ = F \times r! \prod_{i=1}^{\lceil r/2 \rceil} \frac{(2n)! (i-1)!^2}{(n-i+\frac{1}{2})!^2} \prod_{i=1}^{\lfloor r/2 \rfloor} \frac{(2n)! i!^2}{(n-i+\frac{1}{2})!^2},$$

$$\text{where } F = \sum_{s=0}^{\lfloor r/2 \rfloor} \frac{(n-s+1)_s}{(-16)^{\lfloor r/2 \rfloor - s} s!} \binom{2\lfloor r/2 \rfloor - 2s}{\lfloor r/2 \rfloor - s}^2.$$

Comments on our proofs

We have proved all of the primary identities and most of the generalisations. Our proofs fall into several categories.

- ▶ Proofs via enumeration of non-intersecting lattice paths (this approach was suggested by Helmut Prodinger).
- ▶ Proofs via determinantal formulas that do not seem to have a natural interpretation in terms of lattice paths, but follow from elliptic hypergeometric transformation formulas.
- ▶ Proofs via Okada-type formulas for the multiplication of Schur functions indexed by partitions of rectangular shape.
- ▶ Other proofs involving Schur functions.

Proofs via non-intersecting lattice paths

We can use the method described in **Christian Krattenthaler's** talk this morning to prove some of our primary identities that involve squares of the Vandermonde, i.e. $\beta = 2$.

In fact, this method naturally gives the corresponding K-generalisation, from which we can easily deduce the primary identity.

In some cases we can also prove these identities by a different method. For example, the case $(1, 2, 0)$ can be proved rather easily using a known hypergeometric sum ($x, y \in \mathbb{R}$)

$$\begin{aligned} \sum_{0 \leq k_1, \dots, k_r \leq n} \Delta(k)^2 \prod_{i=1}^r \binom{n}{k_i} (x)_{k_i} (y)_{n-k_i} \\ = \prod_{j=1}^r j! (n-j+2)_{j-1} (x)_{j-1} (y)_{j-1} (x+y+j+r-2)_{n-r+1}. \end{aligned}$$

More ...

A longer version of this talk, with more details and references,
is available from my website

<http://maths.anu.edu.au/~brent/talks.html>.

*Tony, we expect to have more identities to report
on your 80th birthday!*