Thinking and Doing Mathematics in the 21st Century

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16 November 2018

Abstract

We consider aspects of thinking and doing mathematics in the early 21st century, and the contrast with the early 20th century. In particular, we comment on the influence of computers and the Internet since the second half of the 20th century. To illustrate, we give a case study from the speaker's own work.

Analogy with Quantum Mechanics

$$\text{Mathematical thinking} \implies \text{concepts, models, computations, conjectures, theorems, proofs (“outcomes”)}$$

$$\text{Wave function } \Psi \implies \text{momentum, position, etc (“observables”)}$$

On the left we have something mysterious. It exists, but is hard (or impossible) to pin down.

On the right we have specific things.

In quantum mechanics, they are “observables”.

In mathematics, they are the desired outcomes.

Scope of this talk

I will talk mainly about the outcomes (observables) that arise from mathematical thinking.

Mathematical thinking, interpreted as what goes on in a mathematician's head, may not have changed much since the 19th century. What has changed dramatically is the set of tools that we can use to augment our thinking, and thus expand the set of feasible outcomes.

$$\text{thinking} + \text{tools} \implies \text{concepts, models, computations,}$$
$$\text{conjectures, theorems, proofs}$$

We can think of 21st century mathematician(s) plus their tools (computers, Internet etc) as comprising a larger system than the 19th century mathematician(s) plus 19th century tools (chalk, pencil and paper). Thus, in the 21st century we can expect more sophisticated computations and deeper conjectures, theorems and proofs. This applies less strongly, if at all, to the development of concepts and models.

Tools

The most important tools developed since the early 20th century are

- The digital computer, allowing
  - Numerical computations
  - Symbolic computations using various software packages
  - TeX and LaTeX
- The Internet, allowing
  - Collaboration by email, Skype etc
  - Search engines such as Google
  - Online databases such as arXiv, OEIS, DLMF
  - Online journals and books (these are less effective than they should be, thanks to copyright restrictions)

Of course, these tools don't apply just to mathematics. They have changed our lives more generally, but outside the scope of this talk.

Case study

I have time for just one example. I'll choose an example from my own work, not because it is particularly notable, but because I know it and it illustrates most of the dot points on the previous slide. Thanks to my collaborators (in chronological order)

- Paul Zimmermann (France) [Z]
- Judy-anne Osborn (Australia) [O1]
- Will Orrick (USA) [O2]
- Warren Smith (USA) [S]
- Hideyuki Ohtsuka (Japan) [O3]
- Helmut Prodinger (South Africa) [P]
- Ole Warnaar (Australia) [W]
- Christian Krattenthaler (Austria) [K]

The length of the list illustrates that mathematics is no longer a solitary affair, but may involve collaboration of several people with differing specialties and abilities, located virtually anywhere with Internet access.

Outline of the case study

- Hadmard maxdet problem – how good is the Hadamard bound $n^{n/2}$ for arbitrary order n? [B, O1, O2, S, Z]
- Application of the "probabilistic method" of Erdős to obtain lower bounds for maximal determinants. [B, O1, S]
- 2D binomial sum needed for the proofs of the lower bound. Proof by induction. [B, O1, P]
- Various generalisations and analogies, all for dimensions ≤ 4. Ad hoc proofs. [B, O2, P]
- Generalisation to arbitrary dimensions, expressing the sum as a product of (ratios of) Gamma functions. Analogy with Macdonald-Mehta integrals. [B, W]
- Analogous sums in arbitrary dimensions, q-analogues. Proofs depending on enumeration of lattice paths, theory of root systems for finite reflection groups, etc. [B, K, W]

The Hadamard maxdet problem

How large can $\det(A)$ be for an $n \times n$ matrix A whose elements lie in $[-1, 1]$? We can assume that the elements are in $\{-1, 1\}$.

Hadamard showed that $|\det(A)| \leq n^{n/2}$ (the Hadamard bound).

The bound is attainable for $n = 1, 2, 4$ and many multiples of four (conjecturally all multiples of four – this is the *Hadamard conjecture*, true at least for $n < 668$).

If n is not a multiple of four, how close can we get to the Hadamard bound? For example, can we always find a matrix A such that $\det(A)/n^{n/2} \geq 1/2$? The answer is yes if $n \leq 120$, by extensive computations [O2, Rokicki et al].

The computations suggest a conjecture: there is a positive constant c such that, for each positive order n, there exists a $\{\pm 1\}$-matrix A of order n such that $\det(A)/n^{n/2} \geq c$.

Assuming the Hadamard conjecture, we can prove this (with $c = (\pi e/2)^{-3/2} \approx 0.1133$) [B, O1, S].

Finding maximal determinants

Our point today is not the mathematical details, but what tools we used and how our research proceeded in a serendipitous manner.

Previous work on the maxdet problem, including papers by [B, O1, O2, Z], constructed large-determinant matrices by (deterministically) adjoining a border of one or more rows and columns to a Hadamard matrix, or by removing some rows and columns from a Hadamard matrix.

Warren Smith saw our work on arXiv (who waits for papers to appear in journals nowadays?) and suggested trying the *probabilistic method* of Erdős.

A naive application of the probabilistic method fails (by an easy result of Turan), but we found a better way, by (randomly) adjoining a border to a Hadamard matrix (no time for details).

A curious binomial sum

To apply the probabilistic method, we [B, O1, S] needed an explicit formula (or at least a good upper bound) for the 2-fold binomial sum

$$S(2,n) := \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} |k^2 - \ell^2| \binom{2n}{n+k} \binom{2n}{n+\ell}.$$

We can assume that $k, \ell \in [-n, n]$ as otherwise the product of binomial coefficients vanishes.

By numerical experiment we found, and eventually proved (by induction on n), that

$$S(2,n) = 2n^2 \binom{2n}{n}^2.$$

Note on binomial sums

Evaluating binomial sums has gone out of fashion, since the method of Wilf and Zeilberger (WZ) can evaluate most common binomial sums algorithmically (and rigorously, although not always transparently).

However, our sum $S(2,n)$ has the unusual feature of an absolute value $|k^2 - \ell^2|$ appearing inside the summation, making it unclear how to apply the WZ algorithm.

Also, as we'll discuss, we were able to generalise the 2-fold sum $S(2,n)$ to an r-fold sum $S(r,n)$, for arbitrary positive $r \in Z$, but the WZ algorithm only applies for fixed (small) r.

What next?

We [B, O1] put our proof for $S(2,n)$ up on arXiv. Almost immediately, we got emails from people we had never met.

- Hideyuki Ohtsuka sent us a long list of similar binomial sums (without proofs). We were able to prove most of these, but in a rather ad hoc manner [B, O1, O3, P].
- Ohtsuka conjectured an analogous triple-sum identity:

$$S(3,n) := \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |\Delta(k^2,\ell^2,m^2)| \binom{2n}{n+k}\binom{2n}{n+\ell}\binom{2n}{n+m}$$
$$= 3n^3(n-1)\binom{2n}{n}^2 2^{2n-1},$$

where $\Delta(x,y,z) := (y-x)(z-y)(z-x)$ and $n \geq 2$. This was proved by Helmut Prodinger, who put his (rather complicated) inductive proof up on arXiv.

What next?

- Ole Warnaar saw the arXiv papers on 2-fold and 3-fold sums, and suggested generalising to r-fold sums. These may be regarded as discrete analogues of r-fold integrals due to Selberg, Mehta, Dyson and Macdonald. These integrals are known to people working in the areas of statistical mechanics and mathematical physics.

For example,

$$S_\alpha(r,n) := \sum_{k_1,\ldots,k_r\in\mathbb{Z}} |\Delta(k_1^\alpha,\ldots,k_r^\alpha)| \prod_{j=1}^{r} \binom{2n}{n+k_j}.$$

Here α, r, n are non-negative integer parameters and Δ is a Vandermonde determinant.

What next?

The integrals can be expressed as products of Gamma functions. Proceeding numerically (using Magma, Maple etc), B and W found ten cases where the r-fold sums could also be expressed as products of Gamma functions. For example,

$$S_2(r,n) = \sum_{k_1,\ldots,k_r\in\mathbb{Z}} |\Delta(k_1^2,\ldots,k_r^2)| \prod_{j=1}^{r} \binom{2n}{n+k_j}$$
$$= \prod_{j=0}^{r-1} \frac{(2n)!}{(n-j)!} \frac{\Gamma(\frac{j+3}{2})}{\Gamma(\frac{3}{2})} \frac{\Gamma(n-j+\frac{3}{2})}{\Gamma(n-\frac{j}{2}+\frac{3}{2})} \frac{\Gamma(\frac{j+1}{2})}{\Gamma(n-\frac{j-1}{2})}.$$

This generalises the cases $r = 2$ and $r = 3$ found earlier by [B, O1, O3, P]. But how to prove it (and the other nine cases)?

What next?

Using his knowledge of root systems for finite reflection groups, Ole Warnaar was able to prove some of our conjectured identities. However, we were stuck on others.

Knowing that Christian Krattenthaler was an expert in this area, we enlisted his help (via email).

Eventually, the three of us [B, K, W] proved all ten conjectures. In some cases we used formulas derived from enumeration of non-intersecting lattice paths.

In other cases we deduced the results by letting $q \to 1$ in q-series generalisations, which could be proved using known techniques.

Note that q-series identities occur in statistical mechanics, e.g. Rodney Baxter's solution of the "hard hexagon model" uses the famous *Rogers-Ramanujan identities*. Hence, they are familiar to people working in mathematical physics.

Remarks

We don't know why there are only ten cases where the binomial sums can be expressed as Gamma products.

Indeed, we don't have a rigorous proof of this, although numerical evidence is convincing.

In three of the ten cases we don't know if a q-generalisation exists. It is difficult to rule out because an identity can have many different q-generalisations.

The fact that I have listed these loose ends illustrates the point that mathematicians like generality and dislike exceptions.

Unfortunately, the mathematical world is not always as nice as we would like it to be.

For example, consider the classification of finite simple groups. It would be much neater if the 26 (or 27, with the Tits group) "sporadic" simple groups did not exist.

References

Other relevant papers can be found by working back from these.

R. P. Brent, J. H. Osborn and W. D. Smith,
Probabilistic lower bounds on maximal determinants of binary matrices, *Australasian J. Combinatorics* 66 (2016), 350–364. Also arXiv:1501.06235.

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