AN OPTIMAL SECANT METHOD FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS

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We describe an efficient algorithm for finding an approximate solution of a system of nonlinear equations. The algorithm uses function, but not derivative, evaluations.

Suppose that x_0, x'_0 are distinct approximations to a zero x^* of the system $f(x) = 0$ of n nonlinear equations in n unknowns. Let k be a positive integer chosen as described below. The algorithm S_k generates sequences (x_i) and (x'_i) with limit x^*, provided the Jacobian of f is nonsingular at x^*, satisfies a Lipschitz condition, and the approximations x_0 and x'_0 are sufficiently close to x^*. If x_i and x'_i have been generated, then x_{i+1} and x'_{i+1} are found in the following way: if $f(x_i) = 0$, then $x_{i+1} = x'_{i+1} = x_i$, otherwise

A. The unique orthogonal matrix $Q_i = \pm(I - 2u_i u_i^T)$ such that $x'_i = x_i + h_i Q_i e_1$ is found. Here $h_i = \|x_i - x'_i\|_2$, u_i is a unit vector, and e_j $(j = 1, \ldots, n)$ is the j-th coordinate vector.

B. The matrix A_i whose j-th column is $A_i e_j = [f(x_i + h_i Q_i e_j) - f(x_i)]/h_i$ for $j = 1, \ldots, n$ is found. A function evaluation may be saved by making use of the previously computed value of $f(x'_i)$.

C. Let $y_{i,0} = x_i$, $J_i = A_i Q_i^T$, and compute $y_{i,j} = y_{i,j-1} - J_i^{-1} f(y_{i,j-1})$ for $j = 1, \ldots, k$.

D. Let $x_{i+1} = y_{i,k}$ and $x'_{i+1} = y_{i,k-1}$.

Method S_k requires n+k-1 evaluations of f for each iteration of steps A to D after the first, and the order of convergence is $(k + (k^2+4)^{1/2})/2$. Hence, k is chosen to maximize the efficiency $E(S_k)$ given by

$$E(S_k) = \frac{\log((k + (k^2 + 4)^{1/2})/2)}{n + k - 1} .$$